

The combinatorics and dynamics of a discrete k winners take all network model

We consider a modification of the classical perceptron model of neural networks inspired by neuroscience research on hippocampus. In this model, those k neurons fire whose weighted inputs are maximal. The model tries to mimic the dynamics of the neural activity in the hippocampus, where only a few percentages of the neurons fire at the same time, however, all the neurons eventually fire in a longer time frame. Neuroscientists conjecture that this dynamics is caused by the structured connections of excitatory neurons (pyramidal cells) and the random connections of inhibitory neurons (basket cells). The structured connections of the pyramidal cells are responsible for the periodic dynamics of the firing activity, while the inhibitory neurons are responsible to prevent the system from blowing up (all or almost all neurons firing the same time). In the simplified model, this inhibition is modeled by letting only the k neurons with the highest weighted input fire.

The main goal of the research is to characterize what kinds of connections of the pyramidal cells are necessary to obtain a periodic firing dynamics.

The model

The model consists of a directed, edge weighted graph, and a positive integer number k . Let $G(V,E)$ denote the directed graph, and let $w: E \rightarrow \mathbb{R}^+$ denote the weight function. Furthermore, let the weight function be such that for any $I, I' \subseteq E$, $\sum_{e \in I} w(e) \neq \sum_{e \in I'} w(e)$. The in-neighbors of a vertex v are those vertices u for which there is an edge e going from u to v . Edge e is also denoted by (u,v) . Similarly, the out-neighbors of a vertex u are those vertices v for which there is an edge going from u to v . Let $N^-(v)$ denote the set of in-neighbors of vertex v , and let $N^+(u)$ denote the set of out-neighbors of vertex u .

The vertices of G model the neurons; the directed edges model the connections between the neurons. There are two types of neurons; those who are “active” and those who are not active. Modeling the neural activity starts with a set of active neurons, and this set changes in each time step in the following way. The active neurons “fire”, they send signal to all their out-neighbors. If A denotes the set of active neurons, we define the score function s in the following way:

$$s(v) := \sum_{u \in A \cap N^-(v)} w((u,v))$$

The set of active neurons in the next step will be the set of the k neurons with the largest score.

We are interested in the dynamics of this model, namely, the temporal behavior of the set of active neurons. It depends on the graph G , the size of the active neurons, and the starting set of the active neurons. It is easy to see that whatever the G , k and the starting set are, the set of active

neurons are either converges to a fixed set of neurons, also called fixpoint, or the dynamics converges to a limit cycle (defined below). For the same G and k , several fixpoints and limit cycles might exist. It depends on the starting set which fixpoint or limit cycle the dynamics converges to. Therefore the central question is what the fixpoints and limit cycles of a given graph G are.

The following list of exercises gives some picture how we are going to infer this model.

Exercises

Exercise 1. Let A_t denote the set of active nodes at a time t . Prove if $A_{t+p} = A_t$ then $A_{t+2p} = A_t$.

Exercise 2. Let n denote the set of vertices in G . Let p be the smallest positive integer for which a t exists such that $A_{t+p} = A_t$. Prove that $p \leq \binom{n}{k}$, particularly, such p and t always exist. What can we say about $t+p$?

If $p = 1$, then A_t is called a fixpoint, and then for any $t' > t$, $A_{t'} = A_t$. If $p > 1$, then the series of sets of active neurons $A_t, A_{t+1}, \dots, A_{t+p-1}$ is called limit cycle.

Exercise 3. In the following exercises, we are going to infer the simplest case, the case when $k = 1$. Let G be an edge weighted directed graph, and let G' be a graph derived from G where for each vertex v , all the outgoing edges of v are deleted but the edge with maximal weight. Prove that the dynamics of the active vertex on G and G' is the same. What can we say about the in-degrees and out-degrees of G' (the number of in-neighbors and the number of out-neighbors)?

As you can see, when $k = 1$, the problem naturally reduces to directed graphs with constraints on the degrees of their vertices. If weights are randomly assigned to a complete graph G , then the derived G' is uniformly distributed on all possible graphs with the prescribed constraints on their degrees. The consequence is that calculating probabilities and expectations needs only enumerative combinatorics. The following exercises show the way how to do that.

Exercise 4. Let G be a directed graph in which each vertex has exactly one outgoing edge (in the remaining exercises, G will be always such). How many such G exist with n vertices if

- a) loops are allowed
- b) loops are not allowed?

Consider the dynamics of the active vertex on G (k is still 1). We define a vertex v is a limit vertex if a positive integer p exists such that $A_{1+p} = \{v\}$ when $A_1 = \{v\}$. A vertex is transient if it is not a limit vertex. How many graphs exist without transient vertices if

- a) loops are allowed
- b) loops are not allowed?

In the following exercises always consider the two cases (loops are/are not allowed).

Exercise 5. Consider the graphs G without transient vertices. How many such G exist with 1 limit cycle, 2 limit cycles, etc. l limit cycles? What is the probability that there are l limit cycles in a random G without transient vertices? How do these probabilities converge when the number of vertices tends to infinite?

Exercise 6. Consider the graphs without transient vertices. How many such G exist in which the longest limit cycle has length M ? How many such G exist in which the shortest limit cycle has length m ? Calculate also the probabilities and limits as in the previous exercise.

Exercise 7. Consider a graph G (recall that still each vertex has exactly 1 outgoing edge), and consider the G' obtained from G by deleting all limit vertices. Characterize G' . If G' is obtained from a G , we say that G is the inverse image of G' . Prove the following: if G'_1 and G'_2 both have m number of vertices, and both obtained by deleting the limit vertices, then both have the same number of inverse images with n vertices. Calculate the number of inverse images as a function of m and n .

Exercise 8. Calculate the probability that a random graph with n vertices has m transient vertices.

Exercise 9. Calculate the probability that a random graph G with n vertices

- a) has l limit cycles,
- b) the longest limit cycle has length M .
- c) the shortest limit cycle has length m .

Calculate how the probabilities converge when n tends to infinite.

Our research will continue with the case when $k > 1$, and also when $k = 1$, but the edge weighted graph to start is not the complete one, but something else. For example: hypercube, toroid grid, random regular graph, etc.