Packing sparse degree sequences

Research proposal, 2016 Fall

August 16, 2016

Problem description

A degree sequence $D = d_1, d_2, \dots d_n$ is a series on non-negative integers. A degree sequence is graphical if there is a vertex labeled graph G in which the degrees of the vertices are exactly D. Such graph G is called a realization of D. The color degree matrix problem also known as edge disjoint realization, edge packing or graph factorization problem is the following: given a $c \times n$ degree matrix $D = \{\{d_{1,1}, d_{1,2}, \dots d_{1,n}\}, \{d_{2,1}, d_{2,2}, d_{2,n}\}, \dots \{d_{c,1}, d_{c,2}, d_{c,}\}\}$, in which each row of the matrix is a degree sequence, decide if there is an ensemble of edge disjoint realizations of the degree sequences. Such set of edge disjoint graphs is called a realization of the degree matrix. A realization can also be presented as an edge colored simple graph, in which the edges with a given color form a realization of the degree sequence in a given row of the color degree matrix. Above the existence problem, we are also interested in the connectivity problem: given a color degree matrix D, what is the necessary and sufficient set of perturbations such that any realizations of D can be transformed into any other realization of D via realizations of D using the set of perturbations.

The existence problem in general is a hard computational problem for any $c \geq 2$. However, it is easy for some special cases. One special case is when the degree matrix is very sparse, the total sum of the degrees is less than or equal to $2n - 2$, where n is the number of vertices. In that case, necessary and sufficient conditions exists if such a realization of the degree matrix exists that is a colored forest. Further examples are the outcome the 2016 summer research project that can be summarized in the following theorems:

Theorem 1. *Let $D = d_1, d_2, \dots d_n$ and $F = f_1, f_2, \dots f_n$ be two tree degree sequences, that is, for both degree sequences, the sum of the degrees is $2n - 2$ and each degree is at least 1. Then there exist edge disjoint tree realizations of D and F iff $D + F (= d_1 + f_1, d_2 + f_2, \dots d_n + f_n)$ is graphical.*

Theorem 2. *Let $D = d_1, d_2, \dots d_n$ and $F = f_1, f_2, \dots f_n$ be two tree degree sequences, such that* $\min_i\{d_i + f_i\} \leq 3$. *Then D and F have edge disjoint caterpillar realizations.*

Note that a caterpillar is a tree in which the non-leaf vertices form a path. Theorem 2. implicitly says that the sum of tree degree sequences is always graphical if there are no common leaves.

The proposed research project is to solve the existence (and possibly the connectivity) problem for other sparse degree matrices.

Assignment for the first week

Read Chapter 14 from this electronic note:
`http://www.renyi.hu/~miklosi/AlgorithmsOfBioinformatics.pdf`.

This chapter gives the definition of degree sequences and also provides a randomization algorithm (the swap Markov chain) that generates a random network with prescribed degree sequence. Please, also solve the following exercises:

1. Prove that a degree sequence $D = d_1, d_2, d_n$ is graphical and there is a forest realizing it iff the sum of the degrees, $\sum_{i=1}^{n} d_i$ is even and less than or equal to $2s - 2$, where s is the number of non-zero degrees in the degree sequence.

2. A tree is called a caterpillar if the vertices with degree more than one form a path. Prove if a degree sequence has a tree realization then it also has a caterpillar realization.

3. Prove that any caterpillar realization of a degree sequence D can be transformed into any other caterpillar realization of D by swaps via caterpillar realizations.

4. Prove that any tree realization of a degree sequence D can be transformed into any other tree realization of D by swaps via tree realizations.

5. Prove that any forest realization of a degree sequence D can be transformed into any other forest realization of D by swaps via forest realizations.

6. The Erdős-Gallai theorem says that a degree sequence $D = d_1 \geq d_2 \geq \ldots \geq d_n$ is graphical iff the sum of the degrees is even and for all $k = 1, \ldots n - 1$

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{j=k+1}^{n} \min\{k, d_j\} \tag{1}$$

   Prove that inequality 1 holds for any $k \geq 4$ if D is the sum of two tree degree sequences.

7. Prove that the sum of two tree degree sequences is always graphical if the minimum degree is at least 3. Do not use Theorem 2.