Notations:

$G_1 \Delta G_2$ denotes the symmetric difference of G_1 and G_2, $| G_1 \Delta G_2 |$ denotes the number of edges in $G_1 \Delta G_2$.

$G*s$ is the graph we get by applying the swap s on G.

We say that a swap s acting on G affects edge e if e is deleted from G by the swap or e is added to G by the swap.

The (total) degree of a vertex v is the number of edges of v. It is denoted by $d(v)$. If v is a vertex of an edge colored graph, then the number of edges of v having color c is denoted by $d_c(v)$. The color spectrum sequence of the edge-colored graph $G(\{v_1, v_2 \ldots v_n\}, \mathrm{E})$, edges colored by c_1, c_2, … c_k is

$$\left(\left(d_{c_1}(v_1), d_{c_2}(v_1), \ldots\ d_{c_k}(v_1)\right), \left(d_{c_1}(v_2), d_{c_2}(v_2), \ldots\ d_{c_k}(v_2)\right), \ldots \left(d_{c_1}(v_n), d_{c_2}(v_n), \ldots\ d_{c_k}(v_n)\right)\right)$$

Exercise 1. Let G_1 and G_2 be two realizations of the same degree sequence. Assume that $G_1 \Delta G_2$ is a cycle with $2n$ edges. Show that there exists a series of swaps s_1, s_2, … s_{n-1} such that

$G_1*s_1*s_2*\ldots*s_{n-1} = G_2$.

(Hint: you do *not* have to use the Havel-Hakimi theorem to solve this exercise)

Exercise 2. Decide whether or not the following claim is true. If it is true, prove it, if it is not true, give a counterexample.

Claim: Let G_1 and G_2 be two realizations of the same *regular* degree sequence and let $e \in G_1 \Delta G_2$. Then there exists a swap s affecting e such that $| G_1*s \Delta G_2 | \leq | G_1 \Delta G_2 |$

Exercise 3. Let $G(U,V,E)$ be an edge-colored bipartite graph, and let the color spectrum on U be regular, namely, for any color c and any two vertices $u_1, u_2 \in U$, $d_c(u_1) = d_c(u_2)$. Furthermore, assume that there is a vertex $v \in V$ that has differently colored edges or its total degree is neither 0 nor $|U|$. Prove that there exist a $G' \neq G$ such that G' and G have the same color spectrum sequence.