Packing caterpillars without common leaves

Research proposal, 2018 Summer

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Problem description

A degree sequence $D = d_1, d_2, \ldots d_n$ is a series on non-negative integers. A degree sequence is graphical if there is a vertex labeled graph G in which the degrees of the vertices are exactly D. Such graph G is called a realization of D. The color degree matrix problem also known as edge disjoint realization, edge packing or graph factorization problem is the following: given a $c \times n$ degree matrix $D = \{\{d_{1,1}, d_{1,2}, \ldots d_{1,n}\}, \{d_{2,1}, d_{2,2}, d_{2,n}\}, \ldots \{d_{c,1}, d_{c,2}, d_{c,n}\}\}$, in which each row of the matrix is a degree sequence, decide if there is an ensemble of edge disjoint realizations of the degree sequences. Such set of edge disjoint graphs is called a realization of the degree matrix. A realization can also be presented as an edge colored simple graph, in which the edges with a given color form a realization of the degree sequence in a given row of the color degree matrix.

The existence problem in general is a hard computational problem for any $c \geq 2$. However, it is easy for some special cases. One special case is when the degree matrix is very sparse, the total sum of the degrees is at most $2n - 1$, where n is the number of vertices. In that case, necessary and sufficient conditions exists for realizing a colored degree matrix with a colored forest [3]. Further examples are the outcomes of previous research projects. One of the theorems of the 2016 Summer research project is the following [1]:

Theorem 1. *Let $D = d_1, d_2, \ldots d_n$ and $F = f_1, f_2, \ldots f_n$ be two tree degree sequences, such that $\min_i\{d_i + f_i\} \leq 3$. Then D and F have edge disjoint caterpillar realizations.*

A tree degree sequence is a degree sequence of positive integers that sum to $2n - 2$. A caterpillar is a tree in which the non-leaf vertices form a path. Theorem 1. implicitly says that the sum of tree degree sequences is always graphical if there are no common leaves. Kundu proved that two degree sequences have edge disjoint realizations if and only if their sum is graphical [4]. This theorem cannot be extended to 3 or more tree degree sequences. However, he showed that 3 tree degree sequences have edge disjoint realizations if the sum of the degrees is at least 5 for any vertex [5]. Clearly, this condition holds if the trees do not have common leaves. The outcome of the 2016 Fall research project is the following theorem [2].

Theorem 2. *Let $\mathcal{D} = D_1, D_2, D_3, D_4$ be tree degree sequences such that each vertex is a leaf in at most one of the trees. Then $\mathcal{D}$ has edge disjoint tree realizations.*

We have computer aided proof that similar theorem holds for 5 tree degree sequences and we conjecture that it is true for arbitrary number of tree degree sequences. In 2017 Summer we proved the following theorem:

Theorem 3. *Let $\mathcal{D} = D_1, D_2, D_3$ be tree degree sequences such that each vertex is a leaf in at most one of the trees. Then $\mathcal{D}$ has edge disjoint caterpillar realizations.*

The proposed research project is to prove or disprove the following conjecture.

Conjecture 1. *For any positive integer k, the following holds. Let $\mathcal{D} = D_1, D_2, \ldots, D_k$ be tree degree sequences such that each vertex is a leaf in at most one of the trees. Then $\mathcal{D}$ has edge disjoint caterpillar realizations.*

Observe that it is stronger to conjecture edge disjoint caterpillar realizations than to conjecture edge disjoint (arbitrary) tree realizations. However, we believe that it is easier to prove this stronger conjecture. The key here is to find *rainbow matchings* in certain graphs (see their definition in the following exercises), and it seems easier to find rainbow matchings in a set of paths than in a set of trees. To prove Conjecture 1, rainbow matchings should be found in a set of edge disjoint paths with a forbidden vertex.

Assignment for the first week

Please, solve the following exercises:

1. A tree is called a caterpillar if the vertices with degree more than one form a path. Prove if a degree sequence has a tree realization then it also has a caterpillar realization.
2. The Erdős-Gallai theorem says that a degree sequence $D = d_1 \geq d_2 \geq \ldots \geq d_n$ is graphical iff the sum of the degrees is even and for all $k = 1, \ldots n-1$

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{j=k+1}^{n} \min\{k, d_j\} \tag{1}$$

   Prove that inequality 1 holds for any $k \geq 4$ if D is the sum of two tree degree sequences.
3. Prove that the sum of two tree degree sequences is always graphical if the minimum degree is at least 3. Do not use Theorem 1.
4. Prove that the sum of three tree degree sequences is always graphical if each vertex is a leaf in at most one tree.
5. Prove that the sum of k tree degree sequences is always graphical if each vertex is a leaf in at most one tree.
6. Let $\mathcal{D} = D_1, D_2, \ldots, D_k$ be tree degree sequences on n vertices such that each vertex is a leaf in at most one of the trees. Show that $k \geq 2n$.
7. Show that K_4, the complete graph on 4 vertices, can be decomposed into two edge-disjoint Hamiltonian paths. Show that K_6 can be decomposed into three edge-disjoint Hamiltonian paths, and in general, K_{2n} can be decomposed into n edge-disjoint Hamiltonian paths.
8. A matching in a graph is a set of edges not sharing vertices. The size of a matching is the number of edges in it. A *rainbow matching* in an edge colored graph is a matching, such that each edge has different color. Prove the following. If a graph G is edge-colored with k colors and contains a matching of size $2k-1$ for each color, then it contains a rainbow matching of size k.
9. Prove that any tree with t internal nodes contains a matching of size $\lceil \frac{t+1}{2} \rceil$.
10. Let $\mathcal{D} = D_1, D_2, D_3$ be tree degree sequences without common leaves. Prove that $\mathcal{D}$ has edge disjoint tree realizations. Do not use the Kundu theorem mentioned above.
11. Let $\mathcal{D} = D_1, D_2, D_3$ be tree degree sequences without common leaves. Prove that $\mathcal{D}$ has edge disjoint caterpillar realizations.

References

[1] Bérzci, K., Changshuo, L., Király, Z., Miklós, I. (2017) Packing tree degree sequences I. submitted. https://arxiv.org/abs/1704.07112.

[2] Gollakota, A., Hardt, W., Miklós, I. (2017) Packing tree degree sequences II. submitted. https://arxiv.org/abs/1704.03148

[3] Hillebrand, A., McDiarmid, C. (2016) Colour degree matrices of graphs with at most one cycle, Discrete Appl. Math., 209:144–152.

[4] Kundu, S. (1974) Disjoint Representation of Tree Realizable Sequences. SIAM Journal on Applied Mathematics, 26(1):103–107.

[5] Kundu, S. (1975) Disjoint representation of three tree realizable sequences, SIAM J. of Appl. Math., 28:290–302.