Packing sparse degree sequences

Research proposal, 2017 Spring

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Problem description

A degree sequence $D = d_1, d_2, \ldots d_n$ is a series on non-negative integers. A degree sequence is graphical if there is a vertex labeled graph G in which the degrees of the vertices are exactly D. Such graph G is called a realization of D. The color degree matrix problem also known as edge disjoint realization, edge packing or graph factorization problem is the following: given a $c \times n$ degree matrix $D = \{\{d_{1,1}, d_{1,2}, \ldots d_{1,n}\}, \{d_{2,1}, d_{2,2}, d_{2,n}\}, \ldots \{d_{c,1}, d_{c,2}, d_{c,}\}\}$, in which each row of the matrix is a degree sequence, decide if there is an ensemble of edge disjoint realizations of the degree sequences. Such set of edge disjoint graphs is called a realization of the degree matrix. A realization can also be presented as an edge colored simple graph, in which the edges with a given color form a realization of the degree sequence in a given row of the color degree matrix. Above the existence problem, we are also interested in the connectivity problem: given a color degree matrix D, what is the necessary and sufficient set of perturbations such that any realizations of D can be transformed into any other realization of D via realizations of D using the set of perturbations.

The existence problem in general is a hard computational problem for any $c \geq 2$. However, it is easy for some special cases. One special case is when the degree matrix is very sparse, the total sum of the degrees is less than or equal to $2n - 2$, where n is the number of vertices. In that case, necessary and sufficient conditions exists if such a realization of the degree matrix exists that is a colored forest. Further examples are the outcome the 2016 summer research project that can be summarized in the following theorems:

Theorem 1. *Let $D = d_1, d_2, \ldots d_n$ and $F = f_1, f_2, \ldots f_n$ be two tree degree sequences, such that $\min_i\{d_i + f_i\} \leq 3$. Then D and F have edge disjoint caterpillar realizations.*

(A caterpillar is a tree in which the non-leaf vertices form a path.)

Theorem 2. *Let $D_1, D_2, \ldots, D_m$ be tree degree sequences with $D_i = d_{i,1}, d_{i,2}, \ldots, d_{i,n}$ such that each vertex is a leaf in all except at most one of them. Then $D_1, D_2, \ldots, D_m$ have edge disjoint realizations if and only if $\max_{i,j}\{d_{i,j}\} \leq n - m$.*

The manuscript is downloadable from `http://www.renyi.hu/~miklosi/2017SpringRES/bckm.pdf` Theorem 1. implicitly says that the sum of tree degree sequences is always graphical if there are no common leaves. Kundu proved that two degree sequences have edge disjoint realizations if and only if their sum is graphical. This theorem cannot be extended to 3 or more tree degree sequences. However, he showed that 3 tree degree sequences have edge disjoint realizations if the sum of the degrees is at least 5 for any vertex.

The outcome of the 2016 Fall research project is the following theorem.

Theorem 3. *Let $\mathcal{D} = D_1, D_2, D_3, D_4$ be tree degree sequences such that each vertex is a leaf in at most one of the trees. Then $\mathcal{D}$ has edge disjoint tree realizations.*

We have computer aided proof that similar theorem holds for 5 tree degree sequences and we conjecture that it is true for arbitrary number of tree degree sequences.

The proposed research project is to solve the existence (and possibly the connectivity) problem for other sparse degree matrices. There are many open problems, a list of examples are

(a) When will 3 forest degree sequences have forest realizations?

(b) What are the necessary and sufficient conditions for 3 tree degree sequences have edge disjoint tree realizations? (Note that we have only sufficient conditions.)

(c) What are the necessary and sufficient conditions for 2 (3) degree sequences have edge disjoint caterpillar realizations? (Note that we have only sufficient conditions and only for 2 sequences.)

Assignment for the first week

Please, solve the following exercises:

1. Prove that a degree sequence $D = d_1, d_2, d_n$ is graphical and there is a forest realizing it iff the sum of the degrees, $\sum_{i=1}^{n} d_i$ is even and less than or equal to $2s - 2$, where s is the number of non-zero degrees in the degree sequence.

2. A tree is called a caterpillar if the vertices with degree more than one form a path. Prove if a degree sequence has a tree realization then it also has a caterpillar realization.

3. The Erdős-Gallai theorem says that a degree sequence $D = d_1 \geq d_2 \geq \ldots \geq d_n$ is graphical iff the sum of the degrees is even and for all $k = 1, \ldots n - 1$

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{j=k+1}^{n} \min\{k, d_j\} \tag{1}$$

   Prove that inequality 1 holds for any $k \geq 4$ if D is the sum of two tree degree sequences.

4. Prove that the sum of two tree degree sequences is always graphical if the minimum degree is at least 3. Do not use Theorem 1.

5. A matching in a graph is a set of edges not sharing vertices. The size of a matching is the number of edges in it. A *rainbow matching* in an edge colored graph is a matching, such that each edge has different color. Prove the following. If a graph G is edge-colored with k colors and contains a matching of size $2k - 1$ for each color, then it contains a rainbow matching of size k.

6. Prove that any tree with t internal nodes contains a matching of size $\lceil \frac{t+1}{2} \rceil$.

7. Let $\mathcal{D} = D_1, D_2, D_3$ be tree degree sequences without common leaves. Prove that $\mathcal{D}$ has edge disjoint tree realizations. Do not use the Kundu theorem mentioned above.