Trying to pack even more trees and caterpillars

Research proposal, 2017 Fall

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Problem description

A degree sequence $D = d_1, d_2, \ldots d_n$ is a series on non-negative integers. A degree sequence is graphical if there is a vertex labeled graph G in which the degrees of the vertices are exactly D. Such graph G is called a realization of D. The color degree matrix problem also known as edge disjoint realization, edge packing or graph factorization problem is the following: given a $c \times n$ degree matrix $D = \{\{d_{1,1}, d_{1,2}, \ldots d_{1,n}\}, \{d_{2,1}, d_{2,2}, d_{2,n}\}, \ldots \{d_{c,1}, d_{c,2}, d_{c,}\}\}$, in which each row of the matrix is a degree sequence, decide if there is an ensemble of edge disjoint realizations of the degree sequences. Such set of edge disjoint graphs is called a realization of the degree matrix. A realization can also be presented as an edge colored simple graph, in which the edges with a given color form a realization of the degree sequence in a given row of the color degree matrix.

The existence problem in general is a hard computational problem for any $c \geq 2$. However, it is easy for some special cases. One special case is when the degree sequences are *tree degree sequences*, that is, all degrees are positive and $\sum_{i=1}^{n} d_i = 2n - 2$. It is known when two tree degree sequences can be packed together:

Theorem 1. *[4] Let D and F be two tree degree sequences. Then D and F have edge disjoint tree realizations if and only if $D + F$ is graphical.*

A caterpillar is a tree in which the non-leaf vertices form a path. Two tree degree sequences might not have edge disjoint caterpillar realizations even if their sum is graphical. However, we have the following theorem.

Theorem 2. *[3] Let D and F be two tree degree sequences such that $d_1 + f_1$ is the largest summed degree. Let $S := \{i | \min\{d_i, f_i\} = 1\}$. Then D and F have edge disjoint caterpillar realizations if and only if*

(a) $D + F$ is graphical,

(b) $d_1 + f_1 \leq |S| + 4$.

We do not know necessary and sufficient conditions when 3 tree degree sequences can be packed together. Kundu showed that 3 tree degree sequences have edge disjoint realizations if the sum of the degrees is at least 5 for any vertex [5]. Clearly, this condition holds if the trees do not have common leaves. The outcome of the 2016 Fall research project is the following theorem:

Theorem 3. *[2] Let $\mathcal{D} = D_1, D_2, D_3, D_4$ be tree degree sequences such that each vertex is a leaf in at most one of the trees. Then $\mathcal{D}$ has edge disjoint tree realizations.*

We have computer aided proof that similar theorem holds for 5 tree degree sequences and we conjecture that it is true for arbitrary number of tree degree sequences. The outcome of the 2017 Summer research (above the already mentioned how to pack two caterpillars theorem) is that three tree degree sequences have edge disjoint caterpillar realizations if they do not have common leaves.

Another special case is when there are no common high degrees. In that special case we also know the necessary and sufficient condition to have edge disjoint realizations.

Theorem 4. *[1] Let $D_1, D_2, \ldots, D_m$ be tree degree sequences with $D_i = d_{i,1}, d_{i,2}, \ldots, d_{i,n}$ such that each vertex is a leaf in all except at most one of them. Then $D_1, D_2, \ldots, D_m$ have edge disjoint realizations if and only if* $\max_{i,j}\{d_{i,j}\} \leq n - m$.

The proposed research project is to explore further when tree degree sequences have edge disjoint tree and/or caterpillar realizations. There are many open problems, a list of examples are

(a) What are the necessary and sufficient conditions for 3 tree degree sequences to have edge disjoint tree realizations? (Note that we have only sufficient conditions.)

(b) What are the necessary and sufficient conditions for 3 degree sequences to have edge disjoint caterpillar realizations?

(c) Can we find other sufficient conditions similar to those in Theorems 4 and 3 for a set of tree sequences to have edge disjoint tree or caterpillar realizations?

Assignment for the first week

Please, solve the following exercises:

1. A tree is called a caterpillar if the vertices with degree more than one form a path. Prove if a degree sequence has a tree realization then it also has a caterpillar realization.

2. What is the shortest tree degree sequence that has a non-caterpillar realization?

3. Prove the $\Longrightarrow$ direction of Theorem 2. That is, show if two tree degree sequences have edge disjoint caterpillar realizations, then their sum is graphical and the largest summed degree is at most the number of vertices with at least one leaf plus 4.

4. The Erdős-Gallai theorem says that a degree sequence $D = d_1 \geq d_2 \geq \ldots \geq d_n$ is graphical iff the sum of the degrees is even and for all $k = 1, \ldots n-1$

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{j=k+1}^{n} \min\{k, d_j\} \tag{1}$$

   Prove that inequality 1 holds for any $k \geq 4$ if D is the sum of two tree degree sequences.

5. Prove that the sum of two tree degree sequences is always graphical if the minimum degree is at least 3. Do not use Theorem 1.

6. A matching in a graph is a set of edges not sharing vertices. The size of a matching is the number of edges in it. A *rainbow matching* in an edge colored graph is a matching, such that each edge has different color. Prove the following. If a graph G is edge-colored with k colors and contains a matching of size $2k-1$ for each color, then it contains a rainbow matching of size k.

7. Prove that any tree with t internal nodes contains a matching of size $\lceil \frac{t+1}{2} \rceil$.

8. Let $\mathcal{D} = D_1, D_2, D_3$ be tree degree sequences without common leaves. Prove that $\mathcal{D}$ has edge disjoint tree realizations. Do not use the Kundu theorem mentioned above.

References

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[3] Miklós, I., Wei, Z. (2017) Packing caterpillars. Manuscript in preparation.

[4] Kundu, S. (1974) Disjoint Representation of Tree Realizable Sequences. SIAM Journal on Applied Mathematics, 26(1):103–107.

[5] Kundu, S. (1975) Disjoint representation of three tree realizable sequences, SIAM J. of Appl. Math., 28:290–302.