On Paul Turán's Influence on Graph Theory

M. Simonovits
EÖTVÖS LORAND UNIVERSITY, BUDAPEST

One of those few people from whom I learned a great deal in mathematics, Paul Turán, died on the 26th of September, 1976, after being ill for several years. In this paper, written in his memory, I will endeavor to give a picture of his influence on graph theory as well as his views on mathematics.

Today most mathematicians specialize in one narrow field of mathematics. Perhaps the most evident reason for this is that only the best can excel in more than one branch. Turán was one of these. His ideas and results deeply influenced the development of graph theory. However, his main work, his most important results, concern other topics of mathematics, primarily number theory, interpolation and approximation theory, the theory of polynomials and algebraic equations, complex analysis, and Fourier analysis. And one additional item should be mentioned: Turán discovered (created) a new method in analysis, called the method of power sums, which gave results interesting in themselves and applicable in many branches of mathematics, for instance, number theory, numerical analysis, and stability problems in the theory of differential equations. Hence the picture given here is just a fragment of the whole picture, reflecting Paul Turán's personality in some cases but not in other ones. The deep knowledge and great proof power of Turán, for example, is reflected more in his theorems and proofs from analysis and number theory. On the other hand, I hope that the survey given here on Turán's graph theoretical work and on the theory developed around his work will clearly show that Turán had a very good eye for finding interesting mathematical problems and could create starting points for mathematical theories.

Journal of Graph Theory, Vol. 1 (1977) 102–116

The reader may have read Paul Turán's "Note of Welcome" [17] published in the first issue of this journal. This note and the preceding paper by Erdös on Turán are very interesting. I tried to avoid overlapping them wherever it was possible, but in many cases it was unavoidable. I hope the reader will excuse me for this.

1. DETERMINANTS OF MATRICES WHOSE ENTRIES ARE ± 1

First I would like to mention some joint work by P. Turán and Gy. Szekeres [2]. Let $A = (a_{ij})$ be an $n \times n$ matrix with $a_{ij} = \pm 1$. How large can $\det A$ be? A theorem of Hadamard yields a trivial upper bound:

$$\det A \leq n^{n/2}.$$

This upper bound is attained if and only if A is an orthogonal matrix, that is, A is an Hadamard matrix. Sylvester [41] proved the existence of Hadamard matrices for $n = 2^k$; Paley [36] did the same for $n = p+1$, where p is a prime of the form $4t-1$. Others attempted to provide lower bounds for the maximum of $\det A$. Szekeres and Turán showed that the square sum of all possible such determinants is $(n!)2^{n^2}$, i.e., the average of $\det^2 A$ is $n!$, while the average of the fourth powers is $(n!)^2 f(n)$, where $f(1) = 1$, $f(2) = 2$, and

$$f(n) = f(n-1) + \frac{2}{n} f(n-2).$$

These assertions imply that the maximum of $\det A$ is at least $(n!)^{1/2}$ and $(n!)^{1/2} f(n)^{1/4}$, respectively. Why is this result extremely interesting for me? Nowadays in extremal graph theory (and in many other cases, e.g., in information theory) we obtain lower bounds by using so-called probabilistic arguments. What Szekeres and Turán did was the same only much earlier: namely, they obtained a lower bound by calculating the average and standard deviation (?!) of the square of the determinant of a random ± 1 matrix. Later Turán developed this method and extended the above given results in [3], [8].

(Though it is not graph theory, let me remark that in [1] Turán proved the famous Hardy–Ramanujan theorem in a very elegant way, by using a probabilistic argument. Without really noticing it, he calculated the average and the standard deviation of the number of prime factors of an integer $k \in [1, n]$ and applied the Chebychev inequality, thus giving a much shorter proof, showing the essence of the Hardy–Ramanujan theorem much better and probably providing the first example of applications of probabilistic methods in investigating the value distribution of number-theoretic functions.)

2. TURÁN'S GRAPH THEOREM

Erdös and Szekeres [33] established a theorem from geometry in 1935, and without knowing it, rediscovered and proved Ramsey's theorem. Turán changed the original formulation of the problem by asking [4], [5]: For given n, let G^n denote a graph with n vertices (loops and multiple edges excluded). Ramsey's problem asks that if G^n does not contain a set of t independent vertices, how large a complete graph K_p must occur in G^n? We may replace the condition that G^n does not contain t independent vertices by other conditions and try to ensure that the graph contains a large complete graph K_p. This is how Turán arrived at the question: How large can $e(G^n)$ be if G^n contains no K_p, where $e(G)$ denotes the number of edges of G? Turán proved the following theorem: Let $K_d(n_1, \ldots, n_d)$ be the complete d-chromatic graph with n_i vertices in its ith class: the $n = n_1 + \cdots + n_d$ vertices are divided into d classes $C_1, \ldots, C_d$ where C_i has n_i vertices and two vertices are joined if and only if they belong to different classes. Let us define the n_i's so that $|n_i - (n/d)| \leq 1$; then $K_d(n_1, \ldots, n_d)$ is determined uniquely up to an isomorphism. This graph will be denoted by $T^{n,d}$.

TURÁN'S THEOREM. For given n and p any graph having more edges than $T^{n,p-1}$ or having exactly as many edges as $T^{n,p-1}$ but being different from it must contain a complete p-graph K_p as a subgraph, while $T^{n,p-1}$ does not contain K_p.

As Turán remarks, from this one can easily verify that the maximum number of edges G^n can have without containing a K_p is

$$\left(1-\frac{1}{p-1}\right)\left(\frac{n^2}{2}-\frac{r^2}{2}\right)+\binom{r}{2} \quad \text{if} \quad 0 \leq r \leq p-2$$

$$n \equiv r \pmod{p-1}.$$

A lesser known, but equally useful form of Turán's theorem can be obtained by switching to the complementary graph $\bar{G}^n = H^n$: If H^n has no p independent vertices, then $e(H^n) \geq e(\bar{T}^{n,p-1})$ and the equality implies that $H^n = \bar{T}^{n,p-1}$.

3. THE THEORY DEVELOPING AROUND TURÁN'S THEOREM

Since 1941 a wide theory has developed around Turán's theorem and the framework of this paper is too narrow to give a survey on this theory. Beyond any doubt, Turán's most significant influence on graph theory was the creation of an "initial point" for this theory, not only by finding a single theorem and proving it, but by putting it in an appropriate surrounding. I shall return to this point later. Here, instead of giving a survey, I shall attempt to illustrate a few chapters of this theory by selecting one or two theorems from them. I shall select those theorems which are either the most significant or the easiest to explain.

(a) Hypergraph Problem

For given r, p, and n let us consider the r-uniform hypergraphs in which no loops or multiple hyperedges are allowed. How many hyperedges must an r-uniform hypergraph have if it has n vertices and no p independent vertices, where $x_1, \dots, x_p$ is said to be independent if no $(x_{i_1}, \dots, x_{i_r})$ belongs to the hyperedge set of the hypergraph? Obviously, this is a direct generalization of Turán's problem on the complementary graph, given above. This was the original form, as Turán posed the question in his paper. The question is still unsolved, though many people have attempted to solve it. The simplest unsolved problem here is the following one: Let $r=3$, $p=4$ and let $T^{n,3,3}$ be the 3-uniform hypergraph with the vertices $x_{i,j}(i=1,2,3;\ j=1,2,\dots,[n/3]$ or $[n/3]+1)$ and with the edges $(x_{i,j}, x_{i,j'}, x_{i,j''})$ for $i=1,2,3$ and $(x_{i,j}\ x_{i,j'}, x_{i+1,j''})$ where $x_{4,j''}=x_{1,j''}$ by definition and we take these hyperedges for all the (j, j', j'') when the three vertices are different. Prove, that if H^n is a 3-uniform hypergraph with fewer triples then $T^{n,3,3}$, then it contains 4 independent vertices. (It is easy to see that $T^{n,3,3}$ does not contain 4 independent vertices.)

(b) General Problem

Let $\mathscr{L}$ be a finite or infinite family of graphs and let $ex(n, \mathscr{L}) = \max\{e(G^n)\colon G^n$ has no subgraph $L \in \mathscr{L}\}$. Further, let $Ex(n, \mathscr{L})$ denote the family of extremal graphs, i.e., $Ex(n, \mathscr{L})$ consists of those graphs S^n having $ex(n, \mathscr{L})$ edges and not containing any $L \in \mathscr{L}$. The general problem is stated: Given a family $\mathscr{L}$, determine $ex(n, \mathscr{L})$ and $Ex(n, \mathscr{L})$!

It is useful to introduce the following notation: If $\chi(G)$ denotes the chromatic number of G, let

$$p(\mathscr{L}) = \min\{\chi(L)\colon L \in \mathscr{L}\}. \tag{1}$$

Turán asked many questions in connection with his theorem, some of them in his paper, others in letters to Erdös. One of them led to the Erdös–Gallai theorem [28]: Let P^t be a path of t vertices. Then

$$ex(n, P^t) \leq \frac{t-2}{2} n, \tag{2}$$

and the equality holds if and only if n is divisible by $t-1$, when there exists exactly one extremal graph $S^n \in Ex(n, P^t)$ consisting of vertex-disjoint complete $(p-1)$-graphs. (Later Faudree and Schelp [34] proved that $Ex(n, P^t)$ always contains an S^n consisting of vertex-disjoint complete graphs, each of which but at most one is a K_{t-1}.) A nice unsolved question of Erdös and V. T. Sós is whether (2) remains valid if P^t is replaced by any other fixed tree of t vertices.)

(c) Degenerated Problems

If $p(\mathcal{L}) = 2$, we shall call the problem degenerated. When there exists a forest or a tree in $\mathcal{L}$, the problem could be called very degenerated: a necessary and sufficient condition for $ex(n, \mathcal{L}) = O(n)$ is that the family $\mathcal{L}$ contain a tree or a forest.

Another nice result concerning the degenerated case is a theorem of T. Kövári, V. T. Sós, and Turán [7]: If $p \leq q$, then

$$ex(n, K_2(p, q)) \leq \tfrac{1}{2}\sqrt[p]{q-1}\, n^{2-(1/p)} + o(n^{2-(1/p)}). \tag{3}$$

The inequality (3) is sharp for $p = q = 2$, and it is sharp but for the value of the multiplicative constant for $p = 2, 3, q \geq p$. I am unaware of any counterexample to the following conjecture: (3) is always sharp. (Independently from Kövári and the Turáns, Erdös also proved (3).)

An easy consequence of (3) is that if $p(\mathcal{L}) = 2$, then $ex(n, \mathcal{L}) = o(n^2)$: there exists an $L \in \mathcal{L}$ with $\chi(L) = 2$ and a $K_2(p, q) \supseteq L$; hence

$$ex(n, \mathcal{L}) \leq ex(n, K_2(p, q)) = O(n^{2-(1/p)}).$$

(d) Asymptotic Results in the General Case

Turán's theorem reflects the general situation much better than one would think: Given a family $\mathcal{L}$ and $p = p(\mathcal{L})$, then $ex(n, \mathcal{L})$ and $Ex(n, \mathcal{L})$

behave nearly in the same way as $ex(n, K_p)$ and $Ex(n, K_p)$. More precisely:

$$\frac{ex(n, \mathscr{L})}{\binom{n}{2}} \to 1 - \frac{1}{p-1} \quad \text{as} \quad n \to \infty. \tag{4}$$

(This theorem of Erdös and Simonovits [31] is a trivial consequence of the Erdös–Stone theorem [32] asserting (4) for $\mathscr{L} = \{K_p(r, r, \ldots, r)\}$.) Further, as Erdös and Simonovits independently proved [27, 28, 40], if S^n is an extremal graph for $\mathscr{L}$, then we may omit and add $o(n^2)$ edges to it so that the newly obtained graph is $T^{n,p-1}$, which is the extremal graph for K_p. Here I would like to point out that the most important conclusion of these theorems is that the maximum number of edges and the extremal graphs do not really depend on $\mathscr{L}$; they depend first of all on $p(\mathscr{L})$, which contains only very poor information on $\mathscr{L}$ itself. A further interesting conclusion is that for any $\mathscr{L}$ we can find an $L \in \mathscr{L}$ such that $ex(n, \mathscr{L}) - ex(n, L) = o(n^2)$; there is not much difference whether we consider just one prohibited graph or a family of such graphs. (Observe that for the degenerated case, these assertions follow from $ex(n, \mathscr{L}) = o(n^2)$.)

(e) Problem of Oversaturated Graphs

For the sake of simplicity, we restrict our consideration to the case where $\mathscr{L} = \{L\}$. By definition, if $e(G^n) = ex(n, L) + 1$, then G^n contains an L. It is surprising that, generally, $e(G^n) = ex(n, L) + 1$ ensures much more than just one $L \subseteq G^n$. To illustrate this, I will mention two theorems. First let $L = K_3$. As Rademacher proved (unpublished, 1941), if $e(G^n) = ex(n, K_3) + 1$, then G^n contains at least $[n/2]$ copies of K_3. Erdös generalized this result [24] by showing that there exists a constant $c > 0$ such that if $1 \le k \le cn$ and $e(G^n) = ex(n, K_3) + k$ (where $ex(n, K_3) = [n^2/4]$), then G^n contains at least $k[n/2]$ copies of K_3. One can easily check that this theorem is sharp, but for the value of c; namely, if we add k independent edges to the larger class of $T^{n,2}$, we obtain a G^n with $[n^2/4] + k$ edges and $k[n/2]$ triangles. Many further results of this type were proved later by Erdös, Moser, Moon, Bollobás, Lovász, and Simonovits. Let me mention just one theorem of G. Dirac [22]: Let $e(G^n) \ge ex(n, K_p)$ but $G^n \ne T^{n,p-1}$ and let $t \in [p, 2p-2]$. Then G^n contains a G^t with at least $\binom{t}{2} - (t-p)$ edges, i.e., a K_t from which $(t-p)$ edges have been deleted. (For $t = p$, we return to Turán's theorem.)

(f) Problem of Multigraphs and Directed Graphs

Here I formulate only the directed graph problem, which includes the multigraph problem. Let r be fixed and consider digraphs in which for any two vertices at most r arcs of the same orientation can join them, so that the number of arcs joining them in the two directions is at most $2r$. The problem is obvious: For a given family $\vec{\mathscr{L}}$ of digraphs what is the maximum number of arcs which a digraph $\vec{D}^n$ can possess without containing an $\vec{L} \in \vec{\mathscr{L}}$? The concepts $ex(n, \vec{\mathscr{L}})$ and $Ex(n, \vec{\mathscr{L}})$ are defined in the obvious way. Brown and Harary [21] started the systematic investigation of such problems. Some general theorems were proved for $r = 1$ by W. G. Brown, P. Erdös, and M. Simonovits, showing that the situation is much more complicated than for (ordinary) graphs.

(g) Extremal Hypergraph Problems Again

The extremal hypergraph problems are generally much more complicated than the others. Here I would like to mention a theorem of Erdös, which is a direct generalization of (3) and often very useful in the theory of graphs [25].

Let $K^{(r)}(p, \ldots, p)$ denote the hypergraph whose vertices are divided into r p-tuples $C_1, \ldots, C_r$ and whose hyperedges are obtained by choosing one vertex from each C_i in all the possible p^r ways. Then

$$ex(n, K^{(r)}(p, \ldots, p)) = O(n^{r-(1/p^{r-1})}).$$

(h) Other Problems

Many other problems occur in this extensive theory, which I choose not to mention. References are given at the end of this paper, though this list is certainly incomplete. Let me mention just a few topics not considered above. The theory of random graphs has a strong connection with extremal graph problems [29]. The problem of topological subgraphs is motivated partly by Turán's theorem, partly by Kuratowski's theorem. Nice applications of finite geometrical constructions can be found in [19] and [30]. Finally let me mention what I call "problems with parameters," i.e., cases where the prohibited graph depends on some parameters as well [18]. As a matter of fact, the Erdös–Stone theorem is such a theorem; it asserts that if, e.g., $p=2$, $e(G^n)\geq cn^2$, then

$$G^n \supseteq K_2([c' \log n], [c' \log n]),$$

where c is arbitrarily fixed and c' depends on c, $(c, c'>0)$.

4. SOME APPLICATIONS OF TURÁN'S THEOREM INITIATED BY P. TURÁN

Turán's theorem can be applied to graph theory in many ways. However, what may be surprising is that there are many situations in which Turán's theorem helps in proving theorems from other branches of mathematics as well. Here I mention applications of Turán's theorem in potential theory, geometry, and the theory of conformal mappings which were initiated by Turán.

As I mentioned, graph theory was not the main field of Turán. During 1954–1969 he did not publish graph-theoretic papers. However, in 1968 he lectured in Budapest and then in Calgary, and published his results in the Calgary proceedings [9]. The same volume contains another paper on applications of graph theory to potential theory by V. T. Sós [40], inspired by the work of Turán. She gave upper bounds for some potentials, estimated by Turán from below. Soon a fast-developing theory emerged from this topic, mostly contained in a sequence of three papers written by P. Erdős, A. Meir, V. T. Sós, and P. Turán [10–12]. Let us see how graph theory was applied to potential theory, geometry, and theory of complex capacity.

Let M be a metric space and F a family of finite subsets of M with the properties that:

(i) There exists a constant C such that each $A\in F$ has diameter at most C.
(ii) If $A'\subseteq A\in F$, then $A'\subset F$.
(iii) For any $A\in F$, $P\in A$, and $\epsilon>0$, there exists a P' such that $A\cup\{P'\}\in F$ and $\overline{PP'}<\epsilon$.

(The typical examples are (a) all the finite subsets of a bounded domain $D\subseteq M$ or (b) all the finite subsets of a bounded and closed domain $D\subseteq M$ with diameter at most C.)

I. We are interested in the distribution of distance $d(P_i, P_j)$ for an n-element set $(P_1,\dots,P_n)\in F$. In characterizing these distributions, we find a useful tool to be the notion of the "packing constants" defined as follows:

$$d_k = \sup_{(P_1,\ldots,P_k)\in F} \min_{1\le i<j\le k} d(P_i, P_j).$$

Clearly, $d_i \ge d_{i+1}$. If M is a subset of the m-dimensional Euclidean space R^m, then $d_k \to 0$ as well. Let $i_1 = 1$ and $i_2, \ldots, i_j$ be defined by

$$d_{i_j} < d_{i_j+1} = d_{i_j+2} = \cdots = d_{i_{j+1}} < d_{i_{j+1}+1} = \cdots.$$

Now we can formulate Turán's theorem on the distance distribution:

THEOREM. For each $k \ge 2$ and $n \ge i_2$ the number of distances $d(P_i, P_j) \le d_{i_{k+1}} (i \ne j)$ for any set $(P_1, \ldots, P_k) \in F$ is at least

$$\frac{n^2}{2i_k} - \frac{n}{2}.$$

This theorem is sharp in a very strong sense. As a matter of fact, the theorem remains valid under more general conditions; (ii) and (iii) are needed to ensure the sharpness.

This theorem is interesting in itself and has consequences for very ancient geometrical problems as well [10], [13]. Namely, Newton and Gregory had a discussion on whether there exist 13 pairwise nonintersecting unit spheres in R^3 which are tangent to a 14th unit sphere. This problem is equivalent to the following question: Let H be the property that each P_i from $(P_1, \ldots, P_n)$ is on the unit sphere of R^3. Then which of the following two relations is valid: $d_{12} = d_{13}$ or $d_{12} < d_{13}$? As we have seen, this is strongly dependent on the possible distance distribution on the surface of the unit sphere in R^3.

II. Perhaps the following application of the above theorem is even more surprising. Let $f(r)$ be a decreasing function, $r_{x,y}$ be the distance of x and y in R^m and $D \subseteq R^m$ be a closed subset in R^m. If μ is a mass distribution ($=$ measure) on D, the generalized potential of this mass distribution is expressed by

$$I(f) = \iint_{D\times D} f(r_{x,y})\, d\mu_x\, d\mu_y. \tag{5}$$

(Note that $f(r) = r^{m-2}$ is generally used in the classical physics for $m \geq 3$, while $f(r) = \log(1/r)$ is used for the plane.) Turán gave a lower bound on $I(f)$ by using the above theorem on the distance distribution: If d_k is the kth packing constant of D (more precisely, of the family of finite subsets of D), and if $f(r)$ is monotone decreasing and bounded from below in $(0, d_2)$, then

$$I(f) \geq |D| \sum_k \frac{f(d_k)}{k^2 - k}, \tag{6}$$

and d_k is its kth packing constant, then the divergence of $\sum_k (1/k^2) \log d_k$ where $|D|$ is the measure of D. Another nice application of this distance distribution theorem is the following theorem: If $D \subseteq R^2$ is a compact set and d_k is its kth packing constant, then the divergence of $\sum_k (1/k^2) \log d_k$ implies that the so-called complex capacity of D is 0 and this condition is sharp; an example of Erdös shows that it is insufficient to assume the divergence of $\sum_k (1/k^{2-c}) \log d_k$ (for any fixed $c > 0$). (Here the complex capacity is an important tool to measure whether a set in the plane is small or not in a certain sense.)

I have finished the list of the interesting results Turán and his co-authors have established by using Turán's graph theorem. Perhaps I shall mention just one result from another family of applications, due to G. Katona [35]: Let ξ and μ be vector-valued independent random variables with the same distribution; then

$$P(|\xi + \mu| \geq x) \geq \tfrac{1}{2} P(|\xi| \geq x)^2. \tag{7}$$

Probably most people wishing to verify this inequality would not even imagine that graph theory yields the solution.

Finishing this part of the paper, I would like to emphasize once again that in the preceding two sections I made no attempt to give a systematic account of what Turán and others did in graph theory and its applications; the theorems above were primarily illustrations.

5. SOME FINAL THOUGHTS

Here I will try to provide answers for the following two exciting questions:

(1) What was Turán's *ars mathematica*; what were his most important views guiding him in his mathematical work?

(2) What was the secret of Turán's great ability to create initial points for interesting mathematical theories?

Let me begin with the second question. Above we have seen two cases where Turán created starting points for completely new theories. His theorem led to the theory of extremal graphs and his applications of his theorem to the distance distribution of points in a metric space again yielded a vast field to investigate, with many results of a completely new type. In the introduction I have mentioned his proof of the Hardy–Ramanujan theorem and that this proof was one starting point for investigating number theoretic functions by probabilistic methods. Also I mentioned one of his most important contributions to mathematics: the creation of the so called "method of power sums," so useful in many different fields of mathematics. These and other examples show that Turán had some particular ability of creating new, interesting theories. How and why? I think the answer to this question is that one of the most characteristic features of Turán was that he was constantly searching for new phenomena: problems, theorems different in type from those already existing. He searched for phenomena which were interesting and reflected a possible new area of mathematics. To find beautiful and interesting theorems was important for Turán. To prove them in their most general form was never important for him.

On the occasion of Turán's 50th birthday, Rényi wrote a thorough paper on Turán's mathematical work [37]. In this paper Rényi formulated his opinion on the question above as follows (translated by myself):

> In spite of his great knowledge, which should be called encyclopedic, when scrutinizing a problem, he never was influenced by how this problem *usually* was approached. He never made a step on the beaten track without checking, whether this path was the only or most expedient one. Clearly, if one proceeds like this, he will often discover new paths, some of which will grow into the highways of the science in time.

(Of course, I wrote about the way of finding the problems, and Rényi wrote about solving them; but there is really no difference.)

Let me give just a few more problems found by Turán which became important later. The problem of the crossing number of a $K_2(p, q)$ asks: Embedding a graph G, say $G = K_2(p, q)$, into a plane, how can we find the minimum number of intersections of the different edges, among all embeddings? As Turán explains in [17], he arrived at this problem during the war, transporting bricks from certain places to others by trams. They had no problems while pushing these trams on the rails; however, at the intersections of different rails, the trams often went off the rails, causing additional work. This is how he found the question: How should the rails be built so that the number of intersections is minimum? (Today, in the age of IC and printed circuits this problem becomes really important: to produce inexpensive electronic circuits one must solve crossing number problems.)

A famous problem of Erdös and Turán (connected with a theorem of van der Waerden) is the following one: Is it true, that if $a_1, \ldots, a_n, \ldots$ is a sequence of integers, $A(n)$ is the number of a_i's in $(0, n)$ and $\liminf A(n)/n > 0$, then $a_1, \ldots, a_n, \ldots$ contains, for any k, an arithmetic progression of k terms. This longstanding and famous conjecture was finally proved by E. Szemerédi. Just to give a last example of Turán's "good questions," I mention that in a private conversation with L. Babai, who was investigating the automorphism groups of graphs, Turán asked him whether he could characterize the groups which are automorphism groups of planar graphs. As a result, L. Babai worked almost two years on this problem and finally developed a nice new theory on the deeper connection of the structure of graphs and their automorphism groups.

I think these examples give a "new proof" of what Rényi wrote on Turán. Of course, Turán had some luck in finding his sufficiently general, though simple and elegant, graph theorem, and it was also fortunate for him that he was surrounded by the members of König's graph school, who could reflect on his theorem and further questions. Evidently, Erdös had enormous influence on this theory. However, it was not a matter of luck, it was necessary, that a mathematician and personality like Turán created starting points, new paths, and had a very deep influence on many parts of mathematics.

Though it may seem a secondary thing at first sight, let me mention that I enjoyed Turán's lectures very much as well as talks with him, because he had the exceptional ability to make things interesting: theorems and problems, which thought initially uninteresting, could become exciting after Turán's explanations. He told us how one problem is related to others, which interesting results could be obtained from its solution and how one arrived at this question in the first place.

Let me return to the first question. I think, answering the second one, I have partially answered the first one as well. Perhaps I would like to emphasize one final thing: *Turán loved mathematics passionately.* Of course, any real mathematician loves mathematics, but for Turán mathematics was much more: *Mathematics was one of the most important things in his life.* In the most difficult periods of his life mathematics was his help; he could forget about everything around himself and escape to mathematics.

References

Part I. Paul Turán's Papers Quoted Here

[1] On a theorem of Hardy and Ramanujan. *J. London Math. Soc.* 9 (1934) 274–276.

[2] Egy szélsöértékfeladatról a determinánselméletben. *Math. és Természettudományi Értesitö* (1937) 796–806 (with Gy. Szekeres).

[3] Determinánsokra vonatkozó szélsöértékfeladatok. *Math. és Természettudományi Értesitö* (1940) 95–105.

[4] Egy gráfelméleti szélsöértékfeladatról. *Mat. és Fizikai Lapok* (1941) 436–452.

[5] On the theory of graphs. *Colloq. Math.* 3 (1954) 146–163.

[6] A kinai mathematika torténetének egy problémájáról. *Mat. Lapok* 5 (1954) 1-6.

[7] On a problem of K. Zarankiewicz. *Colloq. Math.* 3 (1954) 50–57 (with T. Kövári and V. T. Sós).

[8] On a problem in the theory of determinants. *Acta Sinica* (1955) 411–423.

[9] Applications of graph theory to geometry and potential theory. *Combinatorial Structures and Their Applications.* Gordon and Breach, New York (1970) 423–434.

[10] On some applications of graph theory I. *Discrete Math.* 2 (1972) 207–228 (with P. Erdös, A. Meir and V. T. Sós).

[11] On some applications of graph theory II. *Studies in Pure Mathematics.* Academic, London (1971) 89–100 (with P. Erdös, A. Meir and V. T. Sós).

[12] On some applications of graph theory III. *Can. Math. Bull.* 15 (1972) 27–32 (with P. Erdös, A. Meir and V. T. Sós).

[13] Megjegyzés az egységgömb felületének pakolási állandóiról. *Mat. Lapok* 21 (1970) 39–44.

[14] A general inequality of potential theory. *Proceedings of the NRL Conference on Classical Function Theory* (1970) 137–141.

[15] On some connections between combinatorics and group theory. *Combinatorial Theory and Its Applications, Coll. Math. Soc. J. Bólyai,* Balatonfüred (1969) 1055–1082.

[16] On some applications of graph theory to analysis. *Proceedings of the International Conference on Constructive Theory of Functions,* Varna (1970).

[17] A note of welcome. *J. Graph Theory* 1 (1977) 7–9.

Part II. Other Papers Referenced Here

[18] B. Bollobás, P. Erdös, and M. Simonovits, On the structure of edge graphs II. *J. London Math. Soc.* 12 (1976) 219–224.

[19] W. G. Brown, On a graph that does not contain a Thomsen graph. *Can. Math. Bull.* 9 (1966) 281–285.

[20] W. G. Brown, P. Erdös, and M. Simonovits, Extremal problems for directed graphs. *J. Combinatorial Theory* 15B (1973) 77–93.

[21] W. G. Brown and F. Harary, Extremal digraphs. *Combinatorial Theory and Its Applications, Coll. Math. Soc. J. Bólyai* (1969) 135–198.

[22] G. A. Dirac, Extensions of Turán's theorem on graphs. *Acta Math. Acad. Sci. Hung.* 14 (1963) 417–422.

[23] P. Erdös, *The Art of Counting.* MIT Press, Cambridge (1973).

[24] P. Erdös, On a theorem of Rademacher–Turán. *Illinois J. Math.* 6 (1962) 122–127.

[25] P. Erdös, On extremal problems on graphs and generalized graphs. *Israel J. Math.* 2 (1964) 183–190.

[26] P. Erdös, Some recent results on extremal problems in graph theory. *Theory of Graphs, International Symposium*, Rome (1966) 118–123.

[27] P. Erdös, On some new inequalities concerning extremal properties of graphs. *Theory of Graphs, Proceedings of Colloquium*, Tihany (Hungary) (1966) 77–81.

[28] P. Erdös and T. Gallai, On maximal paths and circuits in graphs. *Acta Math. Acad. Sci. Hung.* 10 (1959) 337–356.

[29] P. Erdös and A. Rényi, On the evolution of random graphs. *Publ. Math. Inst. Hung. Acad. Sci.* 5 (1960) 17–61.

[30] P. Erdös, A. Renyi, and V. T. Sós, On a problem of graph theory. *Studia Sci. Math. Hung.* 1 (1966) 215–235.

[31] P. Erdös and M. Simonovits, A limit theorem in graph theory. *Studia Sci. Math. Hung.* 1 (1966) 51–57.

[32] P. Erdös and A. H. Stone, *On the structure of linear graphs. Bull. Am. Math. Soc.* 52 (1946) 1089–1091.

[33] P. Erdös and Gy. Szekeres, A combinatorial problem in geometry. *Compositio Math.* 2 (1935) 463–470.

[34] R. J. Faudree and R. H. Schelp, Ramsey type results, *Coll. Math. Soc. János Bolyai, 10. Infinite and Finite Sets*, Keszthely (Hungary), 1973, 657–666.

[35] Gy. Katona, Gráfok, vektorok és valószinüségszámitási egyenlötlenségek. *Mat. Lapok* 20 (1969) 123–127.

[36] R. E. A. C. Paley, On orthogonal matrices. *J. Math. Phys.* (1933) 311–320.

[37] A. Rényi, Turán Pál matematikai munkásságáról. *Mat. Lapok* 11 (1960) 229–263.

[38] M. Simonovits, A method for solving extremal problems in graph theory, stability problems. *Theory of Graphs Proceedings of the Colloquium*, Tihany (Hungary) (1966) 279–319.

[39] M. Simonovits, Extremal graph problems with symmetrical extremal graphs, additional chromatic conditions. *Discrete Math.* 7 (1974) 349–376.

[40] V. T. Sós, On extremal problems in graph theory. *Combinatorial Structures and Their Applications*. Gordon and Breach, New York (1970) 407–410.

[41] J. J. Sylvester, Thoughts on inverse orthogonal matrices. *Philos. Mag.* 24 (1867) 461–475.

[42] A. A. Zykov, On some properties of linear complexes. *Mat. Sb.* 24 (1949) 188. *Am. Math. Soc. Translations* N79.