PAIRWISE GENERATING AND COVERING SPORADIC SIMPLE GROUPS

PETRA E. HOLMES AND ATTILA MARÓTI

ABSTRACT. Let G be a non-cyclic finite group that can be generated by two elements. A subset S of G is said to be a pairwise generating set for G if every distinct pair of elements in S generates G. The maximal size of a pairwise generating set for G is denoted by $\omega(G)$. The minimal number of proper subgroups of G whose union is G is denoted by $\sigma(G)$. This is an upper bound for $\omega(G)$. In this paper we give lower bounds for $\omega(G)$ and upper bounds for $\sigma(G)$ whenever G is a sporadic simple group.

1. INTRODUCTION

Dixon [8] showed that two randomly chosen elements of an alternating group A_n generate A_n with probability tending to 1 as n tends to infinity. He conjectured that a similar result holds for all finite simple groups. Kantor and Lubotzky [12] confirmed this conjecture for classical (and small rank exceptional) groups. The proof of Dixon's conjecture was completed by Liebeck and Shalev in [14], where the large rank exceptional groups of Lie type were dealt with.

Let G be a finite simple group, let $m(G)$ be the smallest index of a proper subgroup of G, and let $P(G)$ be the probability that two randomly chosen elements of G generate G. Liebeck and Shalev [15] proved that there exist constants c_1, $c_2 > 0$ such that

$$1 - (c_1/m(G)) < P(G) < 1 - (c_2/m(G))$$

for all non-abelian finite simple groups G. Moreover, we have

$$\liminf m(G)(1 - P(G)) = 1 \text{ and } \limsup m(G)(1 - P(G)) = 3$$

where the limits are taken as G ranges over all non-abelian finite simple groups. This result has an interesting consequence (Corollary 1.7 of [15]) which we will state below.

Let G be a non-cyclic finite group that can be generated by two elements. We define $\omega(G)$ to be the largest integer m so that there exists a subset S of G of size m with the property that any two distinct elements of S generate G. We say that a subset S of G pairwise generates G if any two distinct elements of S generate G. We will also say that S is a pairwise generating set for G.

In Corollary 1.7 of [15] Liebeck and Shalev observed that their above-mentioned result, together with Turán's theorem [20] of extremal graph theory, imply that there exists a constant $c > 0$ so that $c \cdot m(G) \leq \omega(G)$ for any non-abelian finite simple group G.

No group is the union of two proper subgroups. Scorza [18] showed that a group G is the union of three proper subgroups if and only if the Klein four group is a

[1]The first author was supported by a Royal Society Dorothy Hodgkin fellowship and the EPSRC grant EP/C523229/1(P).

[2]The research of the second author was supported by a Marie Curie International Reintegration Grant within the 7th European Community Framework Programme and partially by grants OTKA T049841 and OTKA NK72523.

Date: 16th of October 2009.

factor group of G. By a result of Neumann [17], if G is the union of finitely many proper subgroups then the intersection of all these subgroups is a subgroup of finite index in G.

The above results lead to the following definitions. For a non-cyclic finite group G a set of proper subgroups of G whose union is G is called a covering for G. A covering consisting of the smallest possible number of subgroups is called a minimal covering. Let $\sigma(G)$ denote the size of a minimal covering for G. Obviously, $\omega(G) \leq \sigma(G)$ for a non-cyclic finite group that can be generated by two elements. This upper bound for $\omega(G)$ is often exact.

Cohn [5] was the first mathematician to study the invariant σ systematically. He described all groups G with $\sigma(G) = 4$, 5, and 6. By answering two of Cohn's questions, Tomkinson [19] showed that there is no group G with $\sigma(G) = 7$ and that for every finite solvable group G we have $\sigma(G) = q + 1$ where q is the minimal size of a chief factor of G that has more than one complement. Recently Bhargava [1] proved that for every positive integer n there exists a finite (possibly empty) set of finite groups $S(n)$ such that $\sigma(G) = n$ if and only if G has a factor group in $S(n)$ but does not have a factor group in $S(m)$ for $m < n$. Even more recently, answering a question of Tomkinson [19], Detomi and Lucchini [7] showed that there is no group G with $\sigma(G) = 11$.

Interestingly, the exact values of $\omega(G)$ and $\sigma(G)$ can be computed for many finite groups G. Let S_n and A_n be the symmetric and alternating groups on n letters respectively. Blackburn [2] showed that if n is a sufficiently large odd integer, then $\omega(\mathrm{S}_n) = \sigma(\mathrm{S}_n) = 2^{n-1}$, and that if n is a sufficiently large integer congruent to 2 modulo 4, then $\omega(\mathrm{A}_n) = \sigma(\mathrm{A}_n) = 2^{n-2}$. Later Britnell, Evseev, Guralnick, Holmes, Maróti [4] showed that if G is any of the groups $\mathrm{(P)GL}(n,q)$, $\mathrm{(P)SL}(n,q)$ and if $n \geq 12$, then

$$\omega(G) = \frac{1}{b}\prod_{\substack{i=1 \\ b\nmid i}}^{n-1}(q^n - q^i) + [N(b)/2]$$

where b is the smallest prime factor of n, $N(b)$ is the number of proper subspaces of an n-dimensional vector space over the field of q elements whose dimensions are not divisible by b, and $[x]$ denotes the integer part of x. There is also a more complicated formula for $\sigma(G)$ for G any of the groups $\mathrm{(P)GL}(n,q)$, $\mathrm{(P)SL}(n,q)$ for $n \geq 12$.

The methods so far developed to calculate the exact value of $\omega(G)$ for a finite simple group G are efficient only when the order of G is 'large'. However, we are also interested in developing techniques that work for 'small' finite simple groups.

In this paper we deal with the 26 sporadic simple groups. We provide lower bounds for $\omega(G)$ and upper bounds for $\sigma(G)$ whenever G is a sporadic simple group.

Theorem 1.1. *Let G be a sporadic simple group. An ad hoc, greedy, or König method (see Table and Sections 2 and 3) was used to find the lower bound for $\omega(G)$ as in the third entry of the given row of the Table. The given subgroups in the fourth entry of the row of the Table are representatives of the conjugacy classes of proper subgroups of G all of whose union is G. The cardinality of this covering is at most the number given in the last entry of the row of the Table.*

In [11] the exact values of $\sigma(G)$ were found for the groups M_{11}, M_{22}, M_{23}, Ly, and O'N. Here we calculate the exact values of $\sigma(G)$ for the groups $G \cong \mathrm{Fi}_{22}$, HS, Ru, HN, He, and M_{24}.

Theorem 1.2. *The exact value of the invariant $\sigma(G)$ is known and is given in Section 3 for G any of the groups* M_{11}, M_{22}, M_{23}, M_{24}, Fi_{22}, He, Ru, HS, HN, O'N, *and* Ly.

G	method used	lower bound for $\omega(G)$	the set consisting of all conjugates of these subgroups is a covering for G	upper bound for $\sigma(G)$
M_{11}	ad hoc	23	$\mathrm{L}_2(11)$, M_{10}	23
M_{12}	ad hoc	131	M_{11}, $\mathrm{A}_6.2^2$, $\mathrm{L}_2(11)$	222
M_{22}	greedy	732	M_{21}, $\mathrm{L}_2(11)$, $2^4{:}\mathrm{A}_6$	771
M_{23}	König	41079	$2^4{:}\mathrm{A}_7$ or $\mathrm{L}_3(4){:}2$, A_8, $23{:}11$	41079
M_{24}	König	2145	M_{23}, $\mathrm{L}_3(4){:}\mathrm{S}_3$, $\mathrm{M}_{12}{:}2$	3336
J_1	greedy	4813	$19{:}6$, $2^3{:}7{:}3$, $\mathrm{D}_6 \times \mathrm{D}_{10}$, $\mathrm{L}_2(11)$	5777
J_2	greedy	380	$3{:}\mathrm{S}_6$, $2^{1+4}{:}\mathrm{A}_5$, $\mathrm{U}_3(3)$, $2^{2+4}(3 \times \mathrm{S}_3)$	1220
J_3	König	23648	$\mathrm{L}_2(16){:}2$, $\mathrm{L}_2(19)$, $(3 \times \mathrm{A}_6){:}2_2$	44100
J_4	greedy	2.510122×10^{17}	$29{:}28$, $43{:}14$, $2^{11}{:}\mathrm{M}_{24}$, $2^{1+12}.3.\mathrm{M}_{22}{:}2$, $2^{3+12}.(\mathrm{S}_5 \times \mathrm{L}_3(2))$, $2^{10}{:}\mathrm{L}_5(2)$, $\mathrm{U}_3(11){:}2$	2.510127×10^{17}
Fi_{22}	greedy	149276	$2.\mathrm{U}_6(2)$, $\mathrm{O}_8^+(2).\mathrm{S}_3$, $2^{10}{:}\mathrm{M}_{22}$, $\mathrm{O}_7(3)$	221521
Fi_{23}	greedy	8768674848	$\mathrm{O}_8^+(3).\mathrm{S}_3$, $2^{11}.\mathrm{M}_{23}$, $\mathrm{A}_{12}.2$, $2^2.\mathrm{U}_6(2).2$, $\mathrm{S}_8(2)$, $2.\mathrm{Fi}_{22}$	8875303987
Fi_{24}'	greedy	3.091639×10^{21}	Fi_{23}, $(3 \times \mathrm{O}_8^+(3){:}3){:}2$, $\mathrm{O}_{10}^-(2)$, $29{:}14$, $3^7.\mathrm{O}_7(3)$, $\mathrm{N}_{\mathrm{Fi}_{24}'}(3A)$	3.091640×10^{21}
Co_3	greedy	265413	M_{23}, $3^5{:}(\mathrm{M}_{11} \times 2)$, $\mathrm{U}_3(5){:}\mathrm{S}_3$, $\mathrm{M^cL}{:}2$	833452
Co_2	greedy	4327363	$\mathrm{U}_6(2).2$, $2^{1+8}{:}\mathrm{S}_6(2)$, $\mathrm{M^cL}$, M_{23}, $\mathrm{HS}{:}2$	4730457
Co_1	König	46490622576	$(\mathrm{A}_4 \times \mathrm{G}_2(4)){:}2$, $3.\mathrm{Suz}.2$, $2^{1+8}\mathrm{O}_8^+(2)$, $(A_5 \times \mathrm{J}_2){:}2$, $2^{11}{:}\mathrm{M}_{24}$	58033605710
Suz	greedy	194928	$\mathrm{G}_2(4)$, $2^{1+6}\mathrm{U}_4(2)$, $\mathrm{U}_5(2)$, $\mathrm{J}_2{:}2$	540333
$\mathrm{M^cL}$	greedy	13245	M_{22}, $2.\mathrm{A}_8$, $\mathrm{U}_4(3)$	24575
He	greedy	212937	$\mathrm{S}_4(4){:}2$, $2^2.\mathrm{L}_3(4).\mathrm{S}_3$, $2^{1+6}\mathrm{L}_3(2)$, $3{:}\mathrm{S}_7$	464373
Ru	greedy	12970337	$\mathrm{L}_2(29)$, $(2^2 \times \mathrm{S}_2(8)){:}3$, $2^{1+4+6}{:}\mathrm{S}_5$	12992175
Th	König	103423277855	$2^5.\mathrm{L}_5(2)$, $2^{1+8}.\mathrm{A}_9$, $\mathrm{U}_3(8){:}6$, $(3 \times \mathrm{G}_2(3)){:}2$, $3^9.2.\mathrm{S}_4$	103614133000
HS	greedy	1247	M_{22}, S_8, $\mathrm{U}_3(5).2$	1376
HN	König	162639021	$\mathrm{U}_3(8).3$, $2.\mathrm{HS}.2$, A_{12} $2^{1+8}.(A_5 \times A_5).2$, $5^{1+4}{:}2^{1+4}.5.4$	229758831
O'N	greedy	20141165	$4_2.\mathrm{L}_3(4){:}2_1$, J_1, $\mathrm{L}_2(31)$	36450855
Ly	greedy	1.128456×10^{15}	$2.\mathrm{A}_{11}$, $3.\mathrm{M^cL}{:}2$, $37{:}18$, $67{:}22$, $\mathrm{G}_2(5)$	1.128457×10^{15}
B	ad hoc	3.8434×10^{30}	$\mathrm{N_B}(2A)$, $\mathrm{N_B}(2B)$, $\mathrm{N_B}(3A)$, $\mathrm{N_B}(3B)$, $\mathrm{N_B}(5A)$ $\mathrm{N_B}(5B)$, $\mathrm{N_B}(47A)$, $2^9.2^{16}.\mathrm{S}_8(2)$, Th	3.8437×10^{30}
$\mathbb{M}$	ad hoc	1.2×10^{49}	$\mathrm{N}_{\mathbb{M}}(2A)$, $\mathrm{N}_{\mathbb{M}}(2B)$, $\mathrm{N}_{\mathbb{M}}(3A)$, $\mathrm{N}_{\mathbb{M}}(3B)$, $\mathrm{N}_{\mathbb{M}}(3C)$ $\mathrm{N}_{\mathbb{M}}(5A)$, $\mathrm{N}_{\mathbb{M}}(5B)$, $\mathrm{N}_{\mathbb{M}}(7A)$, $\mathrm{N}_{\mathbb{M}}(7B)$, $\mathrm{N}_{\mathbb{M}}(13A)$ $\mathrm{N}_{\mathbb{M}}(13B)$, $\mathrm{N}_{\mathbb{M}}(3^8)$, $\mathrm{L}_2(59)$, $\mathrm{L}_2(71)$	1.5×10^{49}

2. Preliminaries

Unless specified otherwise, all information used in this section is given in the Atlas [6] and the Gap character table library [9]. All computations with group elements are performed using Magma [3].

In this section we describe two different methods to compute lower bounds for $\omega(G)$.

2.1. Cyclic subgroups.

Proposition 2.1. *Let G be a finite group that can be generated by two elements. Let $H_1, \ldots, H_n$ be representatives of the conjugacy classes of maximal subgroups of*

G. Let C be a conjugacy class of cyclic subgroups. Suppose $c \in C$ is contained in exactly k_i conjugates of H_i, $1 \leq i \leq n$. Then a pairwise generating set always exists which consists of generators of members of C and has size the integer part of

$$\frac{|C|}{1+\sum_{i=1}^{n} k_i \left(\frac{|C|k_i}{(G:H_i)}-1\right)}.$$

Proof. Choose $c \in C$ and set $S = \{g\}$, where $c = \langle g \rangle$. For a fixed index i ($1 \leq i \leq n$) there are k_i conjugates of H_i that contain c, and each of these conjugates contain $|C|k_i/(G{:}H_i)$ members of C, including c itself. Hence there are at most

$$1+\sum_{i=1}^{n} k_i \left(\frac{|C|k_i}{(G{:}H_i)}-1\right)$$

members of C that generate a proper subgroup of G with c. We can always choose a generator from a member of C to add to S when

$$|S|\left(1+\sum_{i=1}^{n} k_i \left(\frac{|C|k_i}{(G{:}H_i)}-1\right)\right) < |C|.$$

This gives the result. □

A corollary of Proposition 2.1 is the following.

Corollary 2.1. *Suppose that a pairwise generating set S contains elements from m_i conjugates of H_i, $1 \leq i \leq n$. Then the number of generators of members of C that can be added to S is at least the integer part of*

$$\frac{|C|-\sum_{i=1}^{n} m_i \frac{|C|k_i}{(G:H_i)}}{1+\sum_{i=1}^{n} k_i \left(\frac{|C|k_i}{(G:H_i)}-1\right)}.$$

Let G and C be given as in the statement of Proposition 2.1. We use Corollary 2.1 to define functions $f(C)$ and $f(\mathcal{C})$ where $\mathcal{C}$ is a union of some conjugacy classes of G. Let $f(C)$ be the integer part of

$$\frac{|C|}{1+\sum_{i=1}^{n} k_i \left(\frac{|C|k_i}{(G:H_i)}-1\right)}.$$

For each conjugacy class of cyclic subgroups D fix a conjugacy class of generators of members of D and denote this subset of G by $\bar{D}$. For a union of some conjugacy classes $\mathcal{C}$ of G define $f'(\mathcal{C})$ to be the size of a pairwise generating set obtained in the following manner.

(1) Take a pairwise generating set S, of size at least $f(D)$, which is a subset of $\bar{D} \subseteq \mathcal{C}$, where $f(D) \geq f(D')$ for any conjugacy class of cyclic subgroups D' with $\bar{D'} \subseteq \mathcal{C}$.
(2) Let D be a conjugacy class of cyclic subgroups with $\bar{D} \cap S = \emptyset$, $\bar{D} \subseteq \mathcal{C}$, and $f(D) \geq f(D')$ for all conjugacy classes of cyclic subgroups D' such that $\bar{D'} \cap S = \emptyset$ and $\bar{D'} \subseteq \mathcal{C}$. Include in S at least as many members of $\bar{D}$ as given by Corollary 2.1.
(3) Repeat Step 2 until all conjugacy classes of cyclic subgroups have been considered. Let $f'(\mathcal{C})$ be the size of S.

Note that $f'(\mathcal{C})$ is not well defined. The invariant $f'(\mathcal{C})$ depends on the choices of $\bar{D}$'s for the various D's, it depends on the numbers of elements taken into S in Steps (1) and (2), and it depends on which of the two distinct conjugacy classes of subgroups C or D is considered first in case $f(C) = f(D)$. Consequently, we define $f(\mathcal{C})$ to be the maximum value of all possible values of $f'(\mathcal{C})$.

2.2. **König's Theorem.** A bipartite graph $\Gamma = (X, Y, E)$ is a graph with vertex set $X \cup Y$, edge set E with the property that every edge connects a vertex of X with a vertex of Y. A matching is a set of edges such that no pair of edges meet at a common vertex. A maximum matching is a matching of largest possible size. A covering K is a set of vertices of Γ so that every edge in Γ has an endpoint in K. A minimum covering is a covering of least possible size.

Theorem 2.1 (König, [13])**.** *Suppose that $\Gamma = (X, Y, E)$ is a bipartite graph. Then the number of edges in a maximum matching equals the number of vertices in a minimum covering.*

Theorem 2.1 (also Hall's Marriage Theorem) has the following consequence.

Theorem 2.2. *Let $\mathcal{C}$ be a conjugacy class in G with the property that there are exactly two conjugacy classes of maximal subgroups of G containing elements from $\mathcal{C}$. Let H_1 and H_2 be representatives of these conjugacy classes of maximal subgroups. Suppose also that every element of $\mathcal{C}$ is contained in a unique conjugate of H_1 and also in a unique conjugate of H_2. Let k be the minimum of $(G{:}H_1)$ and $(G{:}H_2)$. Then there exists a pairwise generating set for G consisting of k members of $\mathcal{C}$.*

Proof. One may define a bipartite graph whose vertices are the conjugates of H_1 and H_2 with an edge between two vertices if and only if a member of $\mathcal{C}$ is contained in their intersection. The number of vertices in a minimum covering of this graph is clearly k. Hence, by Theorem 2.1, the number of edges in a maximum matching of the graph also equals k. □

3. The groups

The notations are that of the Atlas [6] and the Gap character table library [9].

Lemma 3.1. $\omega(\mathrm{M}_{11}) = \sigma(\mathrm{M}_{11}) = 23$.

Proof. The set consisting of all conjugates of the subgroups $\mathrm{L}_2(11)$ and M_{10} is a covering for M_{11}. Since the size of this covering is 23, we have $\sigma(\mathrm{M}_{11}) \leq 23$. (In fact, in [11] it is shown that $\sigma(\mathrm{M}_{11}) = 23$.) To prove the lemma it is sufficient to show that $\omega(\mathrm{M}_{11}) \geq 23$. Each element of order 11 in M_{11} is contained in exactly one copy of $L_2(11)$ and no other maximal subgroup of M_{11}. Similarly, each element of order 8 in M_{11} is in exactly one copy of M_{10} and no other maximal subgroup. Hence we may choose one element of order 11 from each conjugate of $L_2(11)$ and one element of order 8 from each conjugate of M_{10} to obtain a pairwise generating set for M_{11} of size 23. □

Lemma 3.2. $131 \leq \omega(\mathrm{M}_{12}) \leq \sigma(\mathrm{M}_{12}) \leq 222$.

Proof. The set consisting of all conjugates of the subgroups M_{11}, $\mathrm{A}_6.2^2$, and $\mathrm{L}_2(11)$ is a covering for M_{12}. Since the size of this covering is 222, we have $\sigma(\mathrm{M}_{12}) \leq 222$. We use the computer package Magma [3] to perform a random search for a pairwise generating set. We find a pairwise generating set for M_{12} of size 131. □

Lemma 3.3. $732 \leq f(\mathrm{M}_{22}) \leq \omega(\mathrm{M}_{22}) \leq \sigma(\mathrm{M}_{22}) = 771$.

Proof. By [11], the set consisting of all conjugates of the subgroups M_{21}, $\mathrm{L}_2(11)$, and $2^4{:}\mathrm{A}_6$ is a minimal covering for M_{22}. Since the size of this covering is 771, we have $\sigma(\mathrm{M}_{22}) = 771$. By Magma [3] we find that $732 \leq f(\mathrm{M}_{22})$. □

Lemma 3.4. $\omega(\mathrm{M}_{23}) = \sigma(\mathrm{M}_{23}) = 41079$.

Proof. By [11], the set consisting of all conjugates of the subgroups $2^4{:}\mathrm{A}_7$, A_8, and 23:11 is a minimal covering for M_{23}. Since the size of this covering is 41079, we have $\sigma(\mathrm{M}_{23}) = 41079$. An element of the class $14B$ is in one of the 253 conjugates of $2^4.\mathrm{A}_7$ and in one of the 253 conjugates of $L_3(4).2$. By Theorem 2.2, we get a pairwise generating set of 253 elements of $14B$. Similarly we get a pairwise generating set of 506 elements of class $15B$, as an element of this class is contained in one of the 506 conjugates of A_8 and in one of the 1771 conjugates of $2^4.(3 \times \mathrm{A}_5)$. An element of class $23A$ is in one conjugate of the maximal subgroup 23:11 and no other maximal subgroup, so we can adjoin 40320 elements of class $23A$ to our pairwise generating set. This gives $41079 \leq \omega(\mathrm{M}_{23})$. □

Lemma 3.5. $2145 \leq \omega(\mathrm{M}_{24}) \leq \sigma(\mathrm{M}_{24}) = 3336$.

Proof. The set consisting of all conjugates of the subgroups M_{23}, $\mathrm{L}_3(4){:}\mathrm{S}_3$, and $\mathrm{M}_{12}{:}2$ is a covering for M_{24}. Since the size of this covering is 3336, we have $\sigma(\mathrm{M}_{24}) \leq 3336$.

An element of class $23A$ is in one of the 24 conjugates of M_{23} and one of the 40320 conjugates of $\mathrm{L}_2(23)$. If we use all conjugates of M_{23} then we would also cover the classes $8A$, $11A$, $14A$, and $15A$. The only class that we could cover by using $\mathrm{L}_2(23)$ instead would be $12B$, but a cheaper way to cover the two classes $23A$ and $12B$ would be to use all conjugates of M_{23} and all 1288 conjugates of $\mathrm{M}_{12}{:}2$. So any minimal covering of M_{24} must include all 24 conjugates of M_{23}.

The classes whose elements generate maximal cyclic subgroups and which are not covered by the conjugates of the maximal subgroup M_{23} are $10A$, $12A$, $12B$, and $21A$. An element of the class $21A$ is in one of the 2024 conjugates of $\mathrm{L}_3(4){:}\mathrm{S}_3$ and one of the 3795 conjugates of $2^6{:}(\mathrm{L}_3(2) \times \mathrm{S}_3)$. If we use the latter conjugacy class of subgroups, then we would also cover the classes $12A$ and $12B$. But fewer subgroups are needed if we choose all conjugates of $\mathrm{L}_3(4){:}\mathrm{S}_3$ and all conjugates of $\mathrm{M}_{12}{:}2$. This proves that we must include all conjugates of $\mathrm{L}_3(4){:}\mathrm{S}_3$ in our covering.

The classes whose elements generate maximal cyclic subgroups and which are not covered by the conjugates of M_{23} and $\mathrm{L}_3(4){:}\mathrm{S}_3$ are $10A$, $12A$, and $12B$. The most efficient way to cover $12B$ is to use the 1288 conjugates of $\mathrm{M}_{12}{:}2$. This gives us all the remaining group elements, so this choice is optimal. This yields $\sigma(\mathrm{M}_{24}) = 3336$, as claimed.

By Theorem 2.2, we get a pairwise generating set of size 2024 consisting of elements of class $21A$. As said above, an element of the class $21A$ is in one conjugate of $\mathrm{L}_3(4){:}\mathrm{S}_3$ and in one conjugate of $2^6{:}(\mathrm{L}_3(2) \times \mathrm{S}_3)$ (and in no other maximal subgroup of M_{24}). The conjugacy classes of M_{24} not intersecting either subgroup $\mathrm{L}_3(4){:}\mathrm{S}_3$ and $2^6{:}(\mathrm{L}_3(2) \times \mathrm{S}_3)$ are $10A$, $23A$, and $11A$. We find that $f(10A \cup 23A \cup 11A)$ is 121. This gives the lower bound of 2145 for $\omega(\mathrm{M}_{24})$. □

Lemma 3.6. $4813 \leq f(\mathrm{J}_1) \leq \omega(\mathrm{J}_1) \leq \sigma(\mathrm{J}_1) \leq 5777$.

Proof. The set consisting of all conjugates of the subgroups 19:6, $2^3{:}7{:}3$, $\mathrm{D}_6 \times \mathrm{D}_{10}$, and $\mathrm{L}_2(11)$ is a covering for J_1. Since the size of this covering is 5777, we have $\sigma(\mathrm{J}_1) \leq 5777$. By Magma [3] we find that $4813 \leq f(\mathrm{J}_1)$. □

Lemma 3.7. $380 \leq f(\mathrm{J}_2) \leq \omega(\mathrm{J}_2) \leq \sigma(\mathrm{J}_2) \leq 1220$.

Proof. The set consisting of all conjugates of the subgroups $3{:}\mathrm{S}_6$, $2^{1+4}{:}\mathrm{A}_5$, $\mathrm{U}_3(3)$, and $2^{2+4}.(3 \times \mathrm{S}_3)$ is a covering for J_2. Since the size of this covering is 1220, we have $\sigma(\mathrm{J}_2) \leq 1220$. By Magma [3] we find that $380 \leq f(\mathrm{J}_2)$. □

Lemma 3.8. $23648 \leq \omega(\mathrm{J}_3) \leq \sigma(\mathrm{J}_3) \leq 44100$.

Proof. The set consisting of all conjugates of the subgroups $\mathrm{L}_2(16){:}2$, $\mathrm{L}_2(19)$, and $(3 \times \mathrm{A}_6){:}2_2$ is a covering for J_3. Since the size of this covering is 44100, we have

$\sigma(\mathrm{J}_3) \leq 44100$. An element of class $19A$ is in the 14688 conjugates of both classes of maximal subgroups isomorphic to $\mathrm{L}_2(19)$. By Theorem 2.2, we get a pairwise generating set consisting of 14688 elements of class $19A$. The conjugacy classes not in either subgroup isomorphic to $\mathrm{L}_2(19)$ are $12A$, $15A$, $17A$, and $8A$. We find that $f(12A \cup 15A \cup 17A \cup 8A) = 8960$. This gives the lower bound of 23648 for $\omega(\mathrm{J}_3)$. □

Lemma 3.9. $2.510122 \times 10^{17} \leq f(\mathrm{J}_4) \leq \omega(\mathrm{J}_4) \leq \sigma(\mathrm{J}_4) \leq 2.510127 \times 10^{17}$.

Proof. The set consisting of all conjugates of the subgroups 29:28, 43:14, 2^{11}:M_{24}, and $2^{1+12}.3.\mathrm{M}_{22}$:2 is a covering for J_4. Since the size of this covering is at most 2.510127×10^{17}, we have $\sigma(\mathrm{J}_4) \leq 2.510127 \times 10^{17}$. By Magma [3] we find that $2.510122 \times 10^{17} \leq f(\mathrm{J}_4)$. □

Lemma 3.10. $149276 \leq f(\mathrm{Fi}_{22}) \leq \omega(\mathrm{Fi}_{22}) \leq \sigma(\mathrm{Fi}_{22}) = 221521$.

Proof. The set consisting of all conjugates of the subgroups $2.\mathrm{U}_6(2)$, $\mathrm{O}_8^+(2).\mathrm{S}_3$, 2^{10}:M_{22}, and $\mathrm{O}_7(3)$ is a covering for Fi_{22}. Since the size of this covering is 221521, we have $\sigma(\mathrm{Fi}_{22}) \leq 221521$.

Each element of class $22B$ is in one of the 3510 conjugates of $2\mathrm{U}_6(2)$ and nothing else, so the covering must contain this conjugacy class of subgroups.

An element of class $21A$ is in one subgroup in each class of S_{10}, one of the 1647360 conjugates of $\mathrm{S}_3 \times \mathrm{U}_4(3).2$ and one of the 61776 conjugates of $\mathrm{O}_8^+(2).3.2$. Using one of the conjugacy classes of S_{10} would cover $21A$ and $9C$ in 17791488 subgroups. A cheaper way is to use all conjugates of $\mathrm{O}_8^+(2).3.2$ and one class of 14080 conjugates of $\mathrm{O}_7(3)$. Using $\mathrm{S}_3 \times \mathrm{U}_4(3).2$ would not give us any conjugacy classes not available in $\mathrm{O}_8^+(2).3.2$. So the covering contains all conjugates of $\mathrm{O}_8^+(2).3.2$.

The remaining classes are $13B$ and $16B$. The optimal way of covering $16B$ is to use all 142155 conjugates of 2^{10}:M_{22}. The only other choice would be to use conjugates of the Tits group, as a $16B$ element is in four of these. This would also cover $13B$. But the cheapest way to cover these two conjugacy classes is to use all 142155 conjugates of 2^{10}:M_{22} and one conjugacy class of subgroups isomorphic to $\mathrm{O}_7(3)$. (The index of $\mathrm{O}_7(3)$ in Fi_{22} is 14080.) This completes the covering.

By Magma [3] we find that $149276 \leq f(\mathrm{Fi}_{22})$. □

Lemma 3.11. $8768674848 \leq f(\mathrm{Fi}_{23}) \leq \omega(\mathrm{Fi}_{23}) \leq \sigma(\mathrm{Fi}_{23}) \leq 8875303987$.

Proof. The set consisting of all conjugates of the subgroups $\mathrm{O}_8^+(3).\mathrm{S}_3$, $2^{11}.\mathrm{M}_{23}$, $\mathrm{A}_{12}.2$, $2^2.\mathrm{U}_6(2).2$, $\mathrm{S}_8(2)$, and $2.\mathrm{Fi}_{22}$ is a covering for Fi_{23}. Since the size of this covering is 8875303987, we have $\sigma(\mathrm{Fi}_{23}) \leq 8875303987$. By Magma [3] we find that $8768674848 \leq f(\mathrm{Fi}_{23})$. □

Lemma 3.12. $3.091639 \times 10^{21} \leq f(\mathrm{Fi}_{24}') \leq \omega(\mathrm{Fi}_{24}') \leq \sigma(\mathrm{Fi}_{24}') \leq 3.091640 \times 10^{21}$.

Proof. The set consisting of all conjugates of the subgroups Fi_{23}, $(3 \times \mathrm{O}_8^+(3)$:3):2, $\mathrm{O}_{10}^-(2)$, 29:14, $3^7.\mathrm{O}_7(3)$, and $\mathrm{N}_{\mathrm{Fi}_{24}'}(3A)$ is a covering for Fi_{24}'. Since the size of this covering is at most 3.091640×10^{21}, we have $\sigma(\mathrm{Fi}_{24}') \leq 3.091640 \times 10^{21}$. By Magma [3] we find that $3.091639 \times 10^{21} \leq f(\mathrm{Fi}_{24}')$. □

Lemma 3.13. $265413 \leq f(\mathrm{Co}_3) \leq \omega(\mathrm{Co}_3) \leq \sigma(\mathrm{Co}_3) \leq 833452$.

Proof. The set consisting of all conjugates of the subgroups M_{23}, 3^5:$(\mathrm{M}_{11} \times 2)$, $\mathrm{U}_3(5)$:S_3, and $\mathrm{M^cL}$:2 is a covering for Co_3. Since the size of this covering is 833452, we have $\sigma(\mathrm{Co}_3) \leq 833452$. By Magma [3] we find that $265413 \leq f(\mathrm{Co}_3)$. □

Lemma 3.14. $4327363 \leq f(\mathrm{Co}_2) \leq \omega(\mathrm{Co}_2) \leq \sigma(\mathrm{Co}_2) \leq 4730457$.

Proof. The set consisting of all conjugates of the subgroups $\mathrm{U}_6(2).2$, 2^{1+8}:$\mathrm{S}_6(2)$, $\mathrm{M^cL}$, M_{23}, and HS:2 is a covering for Co_2. Since the size of this covering is 4730457, we have $\sigma(\mathrm{Co}_2) \leq 4730457$. By Magma [3] we find that $4327363 \leq f(\mathrm{Co}_2)$. □

Lemma 3.15. $46490622576 \leq \omega(\mathrm{Co}_1) \leq \sigma(\mathrm{Co}_1) \leq 58033605710$.

Proof. The set consisting of all conjugates of the subgroups $(\mathrm{A}_4 \times \mathrm{G}_2(4)){:}2$, 3.Suz.2, $2^{1+8}\mathrm{O}_8^+(2)$, $(A_5 \times \mathrm{J}_2){:}2$, and $2^{11}{:}\mathrm{M}_{24}$ is a covering for Co_1. Since the size of this covering is 58033605710, we have $\sigma(\mathrm{Co}_1) \leq 58033605710$.

An element of the class $39B$ is in the 688564800 conjugates of $(\mathrm{A}_4 \times \mathrm{G}_2(4)){:}2$ and is in the 1545600 conjugates of 3.Suz.2. By Theorem 2.2 we get a pairwise generating set consisting of 688564800 elements of class $39B$. The conjugacy classes not in either subgroup are $35A$, $36A$, $21C$, $23B$, $20C$, $28A$, $24F$, $30D$, $20B$, $30E$, and $12I$. We find that

$$f(35A \cup 36A \cup 21C \cup 23B \cup 20C \cup 28A \cup 24F \cup 30D \cup 20B \cup 30E \cup 12I) = 45802057776.$$

This gives the lower bound of 46490622576 for $\omega(\mathrm{Co}_1)$. □

Lemma 3.16. $194928 \leq f(\mathrm{Suz}) \leq \omega(\mathrm{Suz}) \leq \sigma(\mathrm{Suz}) \leq 540333$.

Proof. The set consisting of all conjugates of the subgroups $\mathrm{G}_2(4)$, $2^{1+6}\mathrm{U}_4(2)$, $\mathrm{U}_5(2)$, and $\mathrm{J}_2{:}2$ is a covering for Suz. Since the size of this covering is 540333, we have $\sigma(\mathrm{Suz}) \leq 540333$. By MAGMA [3] we find that $194928 \leq f(\mathrm{Suz})$. □

Lemma 3.17. $13245 \leq f(\mathrm{M^cL}) \leq \omega(\mathrm{M^cL}) \leq \sigma(\mathrm{M^cL}) \leq 24575$.

Proof. The set consisting of all conjugates of the subgroups M_{22}, $2.\mathrm{A}_8$, and $\mathrm{U}_4(3)$ is a covering for $\mathrm{M^cL}$. Since the size of this covering is 24575, we have $\sigma(\mathrm{M^cL}) \leq 24575$. By MAGMA [3] we find that $13245 \leq f(\mathrm{M^cL})$. □

Lemma 3.18. $212937 \leq f(\mathrm{He}) \leq \omega(\mathrm{He}) \leq \sigma(\mathrm{He}) = 464373$.

Proof. The set consisting of all conjugates of the subgroups $\mathrm{S}_4(4){:}2$, $2^2.\mathrm{L}_3(4).\mathrm{S}_3$, $2^{1+6}\mathrm{L}_3(2)$, and $3{:}\mathrm{S}_7$ is a covering for He. Since the size of this covering is 464373, we have $\sigma(\mathrm{He}) \leq 464373$.

An element of class $17B$ is in one of the 2058 conjugates of $\mathrm{S}_4(4).2$ and no other subgroup. This implies that a minimal covering must contain all 2058 conjugates of this subgroup.

The remaining classes are $12B$, $14D$, $21B$, $21D$ and $28B$. An element of class $14D$ is in one of the 187425 conjugates of $2^{1+6}\mathrm{L}_3(2)$, one of the 625800 conjugates of $7^{1+2}{:}(\mathrm{S}_3 \times 3)$ and one of the 244800 conjugates of $7^2{:}\mathrm{SL}_2(7)$. Using $2^{1+6}.\mathrm{L}_3(2)$ would give us $12B$, but $7^{1+2}(\mathrm{S}_3 \times 3)$ would give $21B$ and $21D$, while $7^2{:}\mathrm{SL}_2(7)$ does not give any of the other conjugacy classes. We note that the set of all conjugates of $2^{1+6}.\mathrm{L}_3(2)$, $3^{\cdot}\mathrm{S}_7$, and $2^2\mathrm{L}_3(3).\mathrm{S}_3$ is a covering for He of size 462315. This is smaller than 652800, the index of $7^{1+2}(\mathrm{S}_3 \times 3)$, so it must be best to include all conjugates of $2^{1+6}.\mathrm{L}_3(2)$ in the covering at this point.

A $28B$-element is in one of the 8330 conjugates of $2^2\mathrm{L}_3(3).\mathrm{S}_3$ and one conjugate each of $7{:}3 \times \mathrm{L}_3(2)$ and $\mathrm{S}_4 \times \mathrm{L}_3(2)$. The latter two subgroups both have index greater than 462315, so we use the conjugates of $2^2\mathrm{L}_3(3).\mathrm{S}_3$. The only other class is $21B$. The best way to cover this class is to use all conjugates of $3^{\cdot}\mathrm{S}_7$. This completes the covering and gives $\sigma(\mathrm{He}) = 464373$.

By MAGMA [3] we find that $212937 \leq f(\mathrm{He})$. □

Lemma 3.19. $12970337 \leq f(\mathrm{Ru}) \leq \omega(\mathrm{Ru}) \leq \sigma(\mathrm{Ru}) = 12992175$.

Proof. The set consisting of all conjugates of the subgroups $\mathrm{L}_2(29)$, $(2^2 \times \mathrm{S}_2(8)){:}3$, and $2^{1+4+6}{:}\mathrm{S}_5$ is a covering for Ru. Since the size of this covering is 12992175, we have $\sigma(\mathrm{Ru}) \leq 12992175$.

First we note that an element of class $29A$ is only in one conjugate of the maximal subgroup $\mathrm{L}_2(29)$, so all conjugates of this subgroup must be in a covering.

The remaining classes are $26C$ and $24B$. First consider $26C$. This is in one of the 417600 conjugates of $(2^2 \times \mathrm{Sz}(8)){:}3$ and two of the 4677120 conjugates of

$\mathrm{L}_2(25).2^2$. The latter has the advantage of covering some elements of $24B$. But we would need at least half the conjugates of $\mathrm{L}_2(25).2^2$ to cover $26C$, so a better way to cover $26C$ and $24B$ would be to use all conjugates of $(2^2 \times \mathrm{Sz}(8))$:3 and all 593775 conjugates of 2^{1+4+6}:S_5. So we put all conjugates of $(2^2 \times \mathrm{Sz}(8))$:3 into the covering.

The most efficient way to cover $24B$ is to use all conjugates of 2^{1+4+6}:S_5, and this completes the covering. This proves $\sigma(\mathrm{Ru}) = 12992175$.

By MAGMA [3] we find that $12970337 \leq f(\mathrm{Ru})$. □

Lemma 3.20. $103423277855 \leq \omega(\mathrm{Th}) \leq \sigma(\mathrm{Th}) \leq 103614133000$.

Proof. The set consisting of all conjugates of the subgroups $2^5.\mathrm{L}_5(2)$, $2^{1+8}.\mathrm{A}_9$, $\mathrm{U}_3(8)$:6, $(3 \times \mathrm{G}_2(3))$:2, and $3^9.2.\mathrm{S}_4$ is a covering for Th. Since the size of this covering is 103614133000, we have $\sigma(\mathrm{Th}) \leq 103614133000$.

An element of the class $27A$ is in the 2×96049408000 conjugates of the two conjugacy classes of $3^9.2.\mathrm{S}_4$. By Theorem 2.2 we get a pairwise generating set of 96049408000 elements of class $27A$. An element of class $39B$ is only in the subgroup $(3 \times \mathrm{G}_2(3))$:2, of which there are 3562272000 conjugates. Conjugacy classes not in either subgroup are $19A$, $20A$, $21A$, $28A$, $30B$, $31B$. We find that $f(19A \cup 20A \cup 30B \cup 28A \cup 31B \cup 21A)$ is 3811597855. This gives a lower bound of 103423277855 for $\omega(\mathrm{Th})$. □

Lemma 3.21. $1247 \leq f(\mathrm{HS}) \leq \omega(\mathrm{HS}) \leq \sigma(\mathrm{HS}) = 1376$.

Proof. The set consisting of all conjugates of the subgroups M_{22}, S_8, and $\mathrm{U}_3(5).2$ is a covering for HS. Since the size of this covering is 1376, we have $\sigma(\mathrm{HS}) \leq 1376$.

An element of class $15A$ is in one conjugate of each of the 1100 maximal subgroups conjugate to S_8, and one of the 5775 maximal subgroups conjugate to 5:4 $\times$ A_5. If we use all conjugates of S_8 then we also cover the classes $6A$, $7A$, $8A$, $10B$ and $12A$, but if we were to use 5:4 $\times$ A_5 then we could cover $20A$. Considering $20A$ and $15A$ alone, we see that this would not be the most efficient choice for covering those classes, as it could be done by using S_8 and the 176 conjugates of $\mathrm{U}_5(2)$. So we include all conjugates of S_8 in our covering.

Next we look at $11A$. This is in one conjugate of each class of M_{11} and one conjugate of M_{22}. Using any of those subgroups would also cover $5C$, but either choice of class of M_{11}'s would give a conjugacy class of elements of order 8. This would use 5600 subgroups to cover $11A$ and one class of elements of order 8, but a cheaper method would be to use all 100 conjugates of M_{22} and all 176 conjugates of $\mathrm{U}_3(5).2$. So the conjugates of M_{22} go into the covering.

This leaves $20A$, $8B$, and $8C$. Using either class of $\mathrm{U}_3(5).2$ is an optimal way to cover $20A$, and would complete the covering. This gives $\sigma(\mathrm{HS}) = 1376$.

By MAGMA [3] we find that $1247 \leq f(\mathrm{HS})$. □

Lemma 3.22. $162639021 \leq \omega(\mathrm{HN}) \leq \sigma(\mathrm{HN}) = 229758831$.

Proof. The set consisting of all conjugates of the subgroups $\mathrm{U}_3(8).3$, 2.HS.2, A_{12}, $2^{1+8}(A_5 \times A_5).2$, and 5^{1+4}:$2^{1+4}.5.4$ is a covering for HN. Since the size of this covering is 229758831, we have $\sigma(\mathrm{HN}) \leq 229758831$.

The only maximal subgroup containing an element of class $19B$ is $\mathrm{U}_3(8).3$ There are 16500000 conjugates of this subgroup. The only maximal subgroup containing an element of class $22A$ is 2.HS.2, and there are 1539000 of these. There are two maximal subgroups containing an element of class $35B$. These are $(\mathrm{D}_{10} \times \mathrm{U}_3(5)).2$ and A_{12}. There are 1140000 conjugates of the latter and 108345600 of the former. An element of class $25B$ is in both of the 5-normalizers, 5^{1+4}:$2^{1+4}.5.4$ and $5^{2+1+2}.4.\mathrm{A}_5$. There are 136515456 conjugates of the former maximal subgroup of HN and there are 364041216 of the latter. By Theorem 2.2, we may have 1140000 elements of class $35B$ and 136515456 elements of class $25B$ in a pairwise generating set. This way, in total, we get a pairwise generating set of size $16500000+1539000+1140000+136515456$. This gives our lower bound for $\omega(\mathrm{HN})$.

The above observations prove that any minimal covering must contain all conjugates of $\mathrm{U}_3(8).3$ and $2^{\cdot}\mathrm{HS}.2$. The only remaining conjugacy class contained in a conjugate of $(\mathrm{D}_{10} \times \mathrm{U}_3(5)).2$ is $35B$, so the best way to cover $35B$ is to use all conjugates of A_{12}. This only leaves $20E$, $25B$ and $30C$.

Take the class $30C$. An element of this class is contained in $2^{1+8}(\mathrm{A}_5 \times \mathrm{A}_5).2$, $5^{2+1+2}4^{\cdot}\mathrm{A}_5$, and a 3-centralizer. If we use the involution centralizer, $2^{1+8}(\mathrm{A}_5 \times \mathrm{A}_5).2$, then we also get $20E$, but if we use the 5-normalizer, $5^{2+1+2}4^{\cdot}\mathrm{A}_5$, then we get $25B$. The 5-normalizer, $5^{2+1+2}4^{\cdot}\mathrm{A}_5$ has index 364041216, and an element of class $20E$ is contained in three conjugates of it, so this would complete the covering with 364041216 groups. But this is not optimal, as using the involution centralizer, $2^{1+8}(\mathrm{A}_5\times\mathrm{A}_5).2$ means that we only need the 136515456 conjugates of $5^{1+4}{:}2^{1+4}5.4$ to complete the covering. This gives our formula for $\sigma(\mathrm{HN})$. □

Lemma 3.23. $20141165 \leq f(\mathrm{O'N}) \leq \omega(\mathrm{O'N}) \leq \sigma(\mathrm{O'N}) = 36450855$.

Proof. By [11], the set consisting of all conjugates of the subgroups $4_2.\mathrm{L}_3(4){:}2_1$, J_1, and $\mathrm{L}_2(31)$ is a minimal covering for O'N. Since the size of this minimal covering is 36450855, we have $\sigma(\mathrm{O'N}) = 36450855$. By Magma [3] we find that $20141165 \leq f(\mathrm{O'N})$. □

Lemma 3.24. $1.128456 \times 10^{15} \leq f(\mathrm{Ly}) \leq \omega(\mathrm{Ly}) \leq \sigma(\mathrm{Ly}) = 112845655268156$.

Proof. By [11], the set consisting of all conjugates of the subgroups $2.\mathrm{A}_{11}$, $3.\mathrm{M^cL}{:}2$, $37{:}18$, $67{:}22$, and $\mathrm{G}_2(5)$ is a minimal covering for Ly. Since the size of this covering is 112845655268156, we have $\sigma(\mathrm{Ly}) = 112845655268156$. By Magma [3] we find that $1.128456 \times 10^{15} \leq f(\mathrm{Ly})$. □

Lemma 3.25. $3.8434 \times 10^{30} \leq \omega(\mathrm{B}) \leq \sigma(\mathrm{B}) \leq 3.8437 \times 10^{30}$.

Proof. The power maps between conjugacy classes of B show that every maximal cyclic group contains an element of class $2A$, $2B$, $2C$, $2D$, $3A$, $3B$, $5A$, $5B$, $31A$, or $47A$.

The $2C$ centralizer is $(2^2 \times \mathrm{F}_4(2)){:}2$ where the 2^2 has two elements of class $2A$ and one of class $2C$. So any element that powers up to a $2C$ involution is in the subgroup $2^2\times\mathrm{F}_4(2)$ and hence can be found in a $2A$ centralizer, and we do not need to include any $2C$ centralizers in a covering if it already includes all conjugates of the $2A$ centralizer.

The $2D$ centralizer $2^{26\cdot}\mathrm{O}_8^+(2)$ is a subgroup of the maximal subgroup $2^{9+16\cdot}\mathrm{S}_8(2)$ and the $31A$ centralizer is contained in Th. So there is a covering consisting all conjugates of the maximal subgroups $N_{\mathrm{B}}(2A)$, $N_{\mathrm{B}}(2B)$, $N_{\mathrm{B}}(3A)$, $N_{\mathrm{B}}(3B)$, $N_{\mathrm{B}}(5A)$, $N_{\mathrm{B}}(5B)$, $N_{\mathrm{B}}(47A)$, $2^{9+16\cdot}\mathrm{S}_8(2)$, and Th. Since the size of this covering is at most 3.8437×10^{30}, we have $\sigma(\mathrm{B}) \leq 3.8437 \times 10^{30}$.

Taking one element from each cyclic group of order 47 in B gives the lower bound of 3.8434×10^{30} for $\omega(\mathrm{B})$. □

Lemma 3.26. $1.2 \times 10^{49} \leq \omega(\mathbb{M}) \leq \sigma(\mathbb{M}) \leq 1.5 \times 10^{49}$.

Proof. The power maps between conjugacy classes of $\mathbb{M}$ show that every maximal cyclic group contains an element of class $2A$, $2B$, $3A$, $3B$, $3C$, $5A$, $5B$, $7A$, $7B$, $13A$, $13B$, $41A$, $59A$ or $71A$.

We know from [10] that the centralizer of an element in $59A$ is contained in a maximal subgroup $\mathrm{L}_2(59)$, while the centralizer of an element in $71A$ is contained in a maximal subgroup $\mathrm{L}_2(71)$. Similarly, the centralizer of an element in $41A$ is contained in a maximal subgroup $N_{\mathbb{M}}(3^8)$. This gives a covering consisting of all conjugates of $N_{\mathbb{M}}(2A)$, $N_{\mathbb{M}}(2B)$, $N_{\mathbb{M}}(3A)$, $N_{\mathbb{M}}(3B)$, $N_{\mathbb{M}}(3C)$, $N_{\mathbb{M}}(5A)$, $N_{\mathbb{M}}(5B)$, $N_{\mathbb{M}}(7A)$, $N_{\mathbb{M}}(7B)$, $N_{\mathbb{M}}(13A)$, $N_{\mathbb{M}}(13B)$, $N_{\mathbb{M}}(3^8)$, $\mathrm{L}_2(59)$ and $\mathrm{L}_2(71)$. Since the size of this covering is at most 1.5×10^{49}, we have $\sigma(\mathbb{M}) \leq 1.5 \times 10^{49}$.

A pairwise generating set can be found by taking an element of order 71 from each conjugate of $\mathrm{L}_2(71)$ and an element of order 59 from each conjugate of $\mathrm{L}_2(59)$. This gives a lower bound of 1.2×10^{49} for $\omega(\mathbb{M})$. □

Acknowledgements

The authors thank both anonymous referees for their careful reading and helpful remarks.

References

[1] Bhargava, M. Finiteness criteria for coverings of groups by finitely many subgroups or cosets. *Int. Electronic J. of Alg.* Vol. **2** (2007) 83-89.
[2] Blackburn, S. Sets of permutations that generate the symmetric group pairwise. *J. Combin. Theory Ser. A* **113** (2006), no. 7, 1572-1581.
[3] Bosma, W. and Cannon, J.J. *Handbook of Magma functions.* School of Mathematics and Statistics, University of Sydney, Sydney, 1995.
[4] Britnell, J. R.; Evseev, A; Guralnick, R. M.; Holmes, P. E.; Maróti, A. Sets of elements that pairwise generate a linear group. *J. Combin. Theory Ser. A.* **115** (2008), no. 3, 442-465.
[5] Cohn, J. H. E. On n-sum groups. *Math. Scand.* **75**, (1994), no. 1, 44-58.
[6] Conway, J. H; Curtis, R. T; Norton, S. P; Parker, R. A; Wilson, R. A. Atlas of finite groups. Maximal subgroups and ordinary characters for simple groups. *Oxford University Press*, Eynsham, (1985).
[7] Detomi, E.; Lucchini, A. On the structure of primitive n-sum groups. *Cubo* **10**, No. 3, (2008), 195-210.
[8] Dixon, J. D. The probability of generating the symmetric group. *Math. Z.* **110** (1969), 199–205.
[9] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.4*; 2005, (`http://www.gap-system.org`).
[10] Holmes, P. E. Computing in the Monster, PhD thesis, *University of Birmingham,* 2001.
[11] Holmes, P. E. Subgroup coverings of some sporadic groups. *J. Combin. Theory, Ser. A* **113** (2006), no. 6, 1204-1213.
[12] Kantor, W. M.; Lubotzky, A. The probability of generating a finite classical group. *Geom. Dedicata* **36** (1990), 67–87.
[13] König, D. Graphen und Matrizen. *Mat. Fiz. Lapok* **38** (1931), 116-119.
[14] Liebeck, M. W.; Shalev, A. The probability of generating a finite simple group. *Geom. Dedicata* **56** (1995), 103–113.
[15] Liebeck, M. W.; Shalev, A. Simple groups, probabilistic methods, and a conjecture of Kantor and Lubotzky. *J. Algebra* **184** (1996), no. 1, 31–57.
[16] Maróti, A. Covering the symmetric groups with proper subgroups. *J. Combin. Theory Ser. A* **110** (2005), no. 1, 97–111.
[17] Neumann, B. H. Groups covered by permutable subsets. *J. London Math. Soc.* **29**, (1954), 236–248.
[18] Scorza, G. I gruppi che possono pensarsi come somma di tre loro sottogruppi. *Boll. Un. Mat. Ital.* **5**, (1926), 216-218.
[19] Tomkinson, M. J. Groups as the union of proper subgroups. *Math. Scand.* **81**, (1997), 191-198.
[20] Turán, P. An extremal problem in graph theory. *Mat. Fiz. Lapok* **48**, (1941), 436-452.

Petra E. Holmes, DPMMS, Centre for Mathematical Sciences, University of Cambridge, Cambridge CB3 0BW, UK
E-mail address: P.E.Holmes@dpmms.cam.ac.uk

Attila Maróti, MTA Rényi Alfréd Matematikai Kutatóintézet, Budapest, Hungary.
E-mail address: maroti@renyi.hu