A proof for a generalized Nakayama conjecture *

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Abstract

In a recent paper Külshammer, Olsson, and Robinson proved a deep generalization of the Nakayama conjecture for symmetric groups. We provide a similar but a shorter and relatively elementary proof of their result. Our method enables us to obtain a more general H-analogue of the Nakayama conjecture where H is a set of positive integers.

1 Introduction

Let G be a finite group, let p be a prime, and let F be an algebraically closed field of characteristic p. The group algebra FG may be written as the direct sum of minimal two-sided ideals called p-blocks. Each complex irreducible character χ of G is associated with a unique p-block B. We say that χ is in the p-block B. In the special case of the symmetric group $G = S_n$, the complex irreducible characters are naturally labelled by partitions of n. In 1940 Nakayama [5] conjectured that two irreducible characters χ_λ and χ_μ are in the same p-block of S_n if and only if the partitions λ and μ have the same p-core. Nakayama's conjecture was proved in 1947 by Brauer and Robinson [1]. Since then, several different proofs were published the shortest of which is probably the work of Meier and Tappe [3].

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Recently, Külshammer, Olsson, and Robinson [2] developed a d-analogue of part of the p-modular representation theory of a finite group where $d \geq 2$ is an integer not necessarily a prime. Let G be a finite group. Let $\mathcal{C}$ be the union of a set of conjugacy classes of G, and let $\mathrm{Irr}(G)$ be the set of complex irreducible characters of G. Külshammer, Olsson, and Robinson defined a $\mathcal{C}$-block to be a non-empty subset B of $\mathrm{Irr}(G)$ which is minimal subject to the following condition. If $\chi \in B$, $\psi \in \mathrm{Irr}(G)$, and if there exists a natural number k and a sequence $\chi = \chi_0, \ldots, \chi_k = \psi$ so that for all $0 \leq i < k$ the truncated inner product of χ_i and χ_{i+1} across $\mathcal{C}$, that is, the inner product

$$\langle \chi_i, \chi_{i+1} \rangle_{\mathcal{C}} := \frac{1}{|G|} \sum_{g \in \mathcal{C}} \chi_i(g) \overline{\chi_{i+1}(g)}, \tag{1}$$

is not 0, then $\psi \in B$. It is a basic fact of p-modular representation theory, that if $\mathcal{C}$ is the set of all elements of G with orders not divisible by a prime p, then the $\mathcal{C}$-blocks of G are precisely the subsets of $\mathrm{Irr}(G)$ corresponding to the usual p-blocks of G. One of the main results of [2] is the following beautiful generalization of the Nakayama conjecture. Let $G = S_n$, and view S_n as a permutation group of degree n. Let $d \geq 1$ be an arbitrary integer. We say that $B \subseteq \mathrm{Irr}(S_n)$ is a combinatorial d-block of S_n if B consists of all irreducible characters of S_n that are labelled by partitions with the same d-core. (We say that the 1-core of any partition is the empty partition.) Let $\mathcal{C}$ be the set of all elements of S_n which have no cycle (of their disjoint cycle decompositions) of length divisible by $d \geq 2$. In [2] it is proved that if $d \geq 2$, then $\mathcal{C}$-blocks and combinatorial d-blocks for S_n are the same. If d is prime, then this gives the original Nakayama conjecture.

In this paper we generalize the Nakayama conjecture and the Külshammer, Olsson, Robinson result even further. Let H be an arbitrary (finite or infinite) set of positive integers. Following [2] we say that a permutation (of finite order) is H-regular if for all $h \in H$ no cycle (of its disjoint cycle decomposition) has length equal to h. Let $\mathcal{C}$ be the set of H-regular elements of S_n, and define the

H-blocks of S_n to be the $\mathcal{C}$-blocks. Our main result is the following.

Theorem 1.1. *Let H be an arbitrary set of positive integers, and let d be the greatest common divisor of the elements in H. With the notations above, the following are true.*

(i) Combinatorial d-blocks are unions of H-blocks for S_n for all $n \geq 1$.

(ii) If $H = \{1\}$, or if $d \geq 2$ and $d \in H$, then combinatorial d-blocks are the same as H-blocks for S_n for all $n \geq 1$.

(iii) If $d \geq 2$ and $d \notin H$, then the conclusion of (ii) is false for infinitely many positive integers n.

Notice that if $d \geq 2$ and H is the set of all integers divisible by d, then part (ii) reduces to the Külshammer, Olsson, Robinson result. In particular, if d is a prime, then we obtain the Nakayama conjecture.

Let us conclude with a few words on the proof of Theorem 1.1. We need many ideas of [2], most importantly the 'perfect isometry' and the Osima rule. However, our proof is shorter and more general than the Külshammer, Olsson, Robinson argument. We do not use decomposition numbers, u-numbers, nor ladders. The new tools are induction on $|H|$ and the Inclusion-Exclusion Principle.

2 The Inclusion-Exclusion Principle

In this section we only deal with the situation when $H = \{1\}$. By the Inclusion-Exclusion Principle, the proportion of H-regular, that is, fixed-point-free permutations in the symmetric group S_n is

$$\langle \chi_{(n)}, \chi_{(n)} \rangle_H = \sum_{i=0}^{n} (-1)^i \frac{1}{i!} \langle \chi_{(n-i)}, \chi_{(n-i)} \rangle \tag{2}$$

where the first inner product denotes the truncated inner product across H-regular elements in S_n (see (1) for the definition), the other inner products are the ordinary inner products in the groups S_{n-i} ($n > i$), and $\langle \chi_{(0)}, \chi_{(0)} \rangle = 1$.

Notice that the summands on the right-hand-side of (2) have alternating signs, but the sum itself is not 0. Also, formula (2) can easily be generalized in the following way. Let χ_λ be an arbitrary irreducible character of S_n. Then we have

$$\langle \chi_\lambda, \chi_{(n)} \rangle_H = \sum_{i=0}^{n} (-1)^i \frac{1}{i!} \langle Res_{S_{n-i}}(\chi_\lambda), \chi_{(n-i)} \rangle \tag{3}$$

where $Res_{S_{n-i}}(\chi_\lambda)$ denotes the restriction of the character χ_λ to the group $S_1 \times \ldots \times S_1 \times S_{n-i}$. The summands on the right-hand-side of (3) again have alternating signs, but can we conclude that the sum is not 0? The answer is yes. Let us give a proof.

For convenience, for each $0 \leq i \leq n-1$, let K_i denote the non-negative integer $\langle Res_{S_{n-i}}(\chi_\lambda), \chi_{(n-i)} \rangle$. Notice that $K_n = K_{n-1}$ holds. Put $K_{n+1} = 0$. By the branching rule, K_i (for $i < n$) is the number of ways we can 'get down' to the partition $(n-i)$ by removing i removable boxes from the Ferrers diagram of the partition λ. The ordering of the boxes to be removed is important. Consider the set $\{1, \ldots, i\}$. Let us put i in the first box to be removed from λ, the number $i-1$ in the second, and so on. We get a diagram (not necessarily a partition) full of numbers such that the numbers increase in each row and each column. We only note that K_i is the degree of the relevant skew character, and also the number of standard tableaux of shape $\lambda - (n-i)$.

In this paragraph we show the inequality $K_{i+1} \leq (i+1)K_i$ for all integers $0 \leq i \leq n$. If $i = n$, then the estimate is clear. Otherwise, fill in the boxes of the diagram $\lambda - (n-i-1)$ with the numbers $\{1, \ldots, i+1\}$ such that first pick an arbitrary number, put it in the $(n-i)$-th position of the first row, and then put the remaining numbers in the diagram so that we get a standard tableau of shape $\lambda - (n-i-1)$. The number of such arrangements is exactly $(i+1)K_i$, and this proves our inequality. (Note that the estimate can be sharp, for example if the largest part of λ is sufficiently large.)

Let us now return to equation (3). The first few terms of the sum on the right-hand-side are 0's, and the rest are not. Let the first non-zero term be the

m-th. Then the right-hand-side of (3) may be written in one of the following two forms

$$\left(\frac{(-1)^m}{m!}K_m + \frac{(-1)^{m+1}}{(m+1)!}K_{m+1}\right) + \ldots + \left(\frac{(-1)^{n-1}}{(n-1)!}K_{n-1} + \frac{(-1)^n}{n!}K_n\right); \quad (4)$$

or

$$\left(\frac{(-1)^m}{m!}K_m + \frac{(-1)^{m+1}}{(m+1)!}K_{m+1}\right) + \ldots + \left(\frac{(-1)^{n-2}}{(n-2)!}K_{n-2} + \right.$$
$$\left. + \frac{(-1)^{n-1}}{(n-1)!}K_{n-1}\right) + \left(\frac{(-1)^n}{n!}K_n + K_{n+1}\right). \qquad (5)$$

We claim that in every term in parentheses in (4) and (5), the first summand is not smaller than the second in absolute value. Indeed, for all $i \geq m$ we have $K_i/i! \geq K_{i+1}/(i+1)!$ by our previous observation. This means that in both cases the terms in parentheses are either all non-negative or all non-positive. Hence it is sufficient to see that the last terms in parentheses in (4) and (5) are non-zero. But this is obvious in both cases since $K_{n-1} = K_n \neq 0$ and $K_{n+1} = 0$. This proves that the expression (3) is not 0.

We showed part (ii) and hence part (i) of Theorem 1.1 in case $H = \{1\}$.

3 The Murnaghan-Nakayama rule

In the previous section we generalized equation (2) to get formula (3). In this section we will make further generalizations. Let H be an arbitrary set of positive integers. Let χ_λ, χ_μ be two complex irreducible characters of S_n. We will present a formula for the truncated inner product of χ_λ and χ_μ across H-regular elements of S_n, which we denote by $\langle \chi_\lambda, \chi_\mu \rangle_H$.

For non-negative integers h and i with $hi \leq n$, and for a partition δ of $n-hi$, let $\mathcal{P}^{h,i}_{\lambda,\delta}$ be the set of paths P in the lattice of partitions, obtained by removing

i hooks of length h to go from λ to δ. Each path P has a sign σ_P, defined as $(-1)^{t(P)}$, where $t(P)$ denotes the sum of the leg lengths of the hooks in P. Define $m_{\lambda\delta}^{h,i}$ to be $\sum \sigma_P$ where the sum is over all paths P in $\mathcal{P}_{\lambda,\delta}^{h,i}$. We note that $m_{\lambda\delta}^{h,i} \neq 0$ implies that λ and δ have the same d-core whenever h is divisible by d, since the removal of a hook of length $h = dr$ from a partition may always be obtained by removing r hooks each of length d (see Pages 69-70 of [4]).

The Murnaghan-Nakayama rule states that whenever $x = y_i \cdot z$ is a permutation of n points so that y_i is a product of i disjoint cycles each of length h and z is a permutation of the $n - hi$ points fixed by y, then we have

$$\chi_\lambda(x) = \sum m_{\lambda\delta}^{h,i} \chi_\delta(z), \tag{6}$$

where the sum is over all partitions δ of $n - hi$, the corresponding characters are irreducible characters of S_{n-hi} for $hi < n$, and where $\chi_\emptyset = 1$ for $hi = n$.

Let H be an arbitrary set of positive integers, and suppose that $h \notin H$. Since the number of y_i's as above is $\binom{n}{hi}(hi)!/(h^i i!)$, the Inclusion-Exclusion Principle gives us

$$\sum \chi_\lambda(x)\chi_\mu(x^{-1}) = \sum_{i=0}^{[n/h]} (-1)^i \binom{n}{hi} \frac{(hi)!}{h^i i!} \sum \chi_\lambda(y_i z)\chi_\mu((y_i z)^{-1}) \tag{7}$$

where the first sum is over all $H \cup \{h\}$-regular elements x of S_n and the third is over all H-regular elements z of S_{n-hi}. By substituting the Murnaghan-Nakayama rules (see (6)) for $\chi_\lambda(y_i z)$ and $\chi_\mu((y_i z)^{-1})$ into the right-hand-side of (7) we obtain

$$\sum \chi_\lambda(x)\chi_\mu(x^{-1}) = \sum_{i=0}^{[n/h]} (-1)^i \binom{n}{hi} \frac{(hi)!}{h^i i!} \sum_{\delta,\delta'} m_{\lambda\delta}^{h,i} m_{\mu\delta'}^{h,i} \sum \chi_\delta(z)\chi_{\delta'}(z^{-1}) \tag{8}$$

where the third sum is over all partitions δ and δ' of $n - hi$. After dividing both sides of (8) by $n!$ we get

Theorem 3.1. *For every subset H of the positive integers and for every positive integer $h \notin H$, we have*

$$\langle \chi_\lambda, \chi_\mu \rangle_{H \cup \{h\}} = \sum_{i=0}^{[n/h]} (-1)^i \frac{1}{h^i i!} \sum_{\delta, \delta'} m_{\lambda\delta}^{h,i} m_{\mu\delta'}^{h,i} \langle \chi_\delta, \chi_{\delta'} \rangle_H, \tag{9}$$

where the second sum is over partitions δ, δ' of $n - hi$.

A consequence of Theorem 3.1 is the following. If H is a subset of the set of positive integers such that $d \geq 2$ is a divisor of all elements in H, then $\langle \chi_\lambda, \chi_\mu \rangle_H = 0$ in case λ and μ have different d-cores. Indeed, let H_n be the subset of H consisting of all elements of H no greater than n. It is clear that $\langle \alpha, \beta \rangle_H = \langle \alpha, \beta \rangle_{H_n}$ for all irreducible characters α and β of S_k whenever $k \leq n$. Now argue by induction on $|H_n|$. The claim is trivial for $|H_n| = 0$ (in which case the truncated inner product is the usual inner product). When $|H_n| > 0$, use Theorem 3.1 once, and apply the induction hypothesis for $|H_n| - 1$ on the right-hand-side of (9).

This completes the proof of part (i) of Theorem 1.1.

4 The Osima rule

In general it seems to be very difficult to determine directly whether the expression (9) is 0 or not. In fact, we can only answer the question by moving from S_n to the generalized symmetric group $Z_d \wr S_w$. The situation is similar to the following example taken from 'real life'. Imagine a huge but very thin carpet. We want to measure its width. One way of doing this is to roll it up and count its layers.

In this section we will derive an analogue of Theorem 3.1 for generalized symmetric groups.

Let $d \geq 2$ be an integer. There are $1-1$ correspondences between partitions of n with a fixed d-core and their d-quotients. Let us fix one. Let λ be a partition of n, and let β_λ be the d-quotient of λ. If there are w hooks of length

d to be removed from λ to go to its d-core, then w is called the d-weight of β_λ. The d-quotients serve as a natural index set for the irreducible characters of the generalized symmetric group $Z_d \wr S_w$. Let χ_{β_λ} be the irreducible character of $Z_d \wr S_w$ associated with β_λ. A hook in β_λ is a hook in one of its partitions and a hook-removal is defined correspondingly. For non-negative integers h and i with $hi \leq w$, define $\tilde{\mathcal{P}}^{h,i}_{\lambda,\delta}$ to be the set of paths $\tilde{P}$ of quotients obtained by removing i hooks each of length h to go from β_λ to β_δ where β_δ is the d-quotient of the partition δ under our fixed canonical correspondence. Each path $\tilde{P}$ has a sign $\sigma_{\tilde{P}}$, defined as $(-1)^{t(\tilde{P})}$, where $t(\tilde{P})$ denotes the sum of the leg lengths of the hooks in $\tilde{P}$. Define $\tilde{m}^{h,i}_{\lambda\delta}$ to be $\sum \sigma_{\tilde{P}}$ where the sum is over all paths $\tilde{P}$ in $\tilde{\mathcal{P}}^{h,i}_{\lambda,\delta}$. Let H be an arbitrary set of positive integers. We define an H-regular element of $Z_d \wr S_w$ to be an element $(a_1, \ldots, a_w)\sigma$ where $(a_1, \ldots, a_w)$ is in the base group ${Z_d}^w$ (which we consider to be the w-th power of the group of complex d-th roots of unity) and σ is a permutation of S_w, such that for all $h \in H$, the product of the a_j's corresponding to each h-cycle of σ is different from 1. The truncated inner product across H-regular elements in $Z_d \wr S_w$ is defined similarly as for symmetric groups (see (1)).

Let $x = (a_1, \ldots, a_w)\sigma_x$, $y_i = (b_1, \ldots, b_w)\sigma_{y_i}$, and $z = (c_1, \ldots, c_w)\sigma_z$ be elements of $Z_d \wr S_w$ where $(a_1, \ldots, a_w)$, $(b_1, \ldots, b_w)$, and $(c_1, \ldots, c_w)$ are elements of the base group and σ_x, σ_{y_i}, and σ_z are elements of S_w. Suppose that $x = y_i z$, $\sigma_x = \sigma_{y_i} \cdot \sigma_z$, and that σ_{y_i} and σ_z move a disjoint set of points in their permutation representation in S_w. Suppose also that σ_{y_i} is a product of $w - hi + i$ disjoint cycles i of which have length h so that the product of the b_j's corresponding to each cycle of σ_{y_i} (in the above decomposition of y_i) is 1. We are now in the position to state a formula of Osima (for details, we refer to Section 3 of [6] and page 539 of [2]). With the above notations and assumptions this is

$$\chi_{\beta_\lambda}(x) = \sum \tilde{m}^{h,i}_{\lambda\delta} \chi_{\beta_\delta}(z)$$

where the sum is over all quotients β_δ of weight $w - hi$, the corresponding characters are irreducible characters of $Z_d \wr S_{w-hi}$ for $hi < w$, and where $\chi_{\beta_\emptyset} = 1$ for $hi = w$.

We also have the following.

Theorem 4.1. *For every subset H of the positive integers and for every positive integer $h \notin H$, we have*

$$\langle \chi_{\beta_\lambda}, \chi_{\beta_\mu} \rangle_{H \cup \{h\}} = \sum_{i=0}^{[w/h]} (-1)^i \frac{1}{(hd)^i i!} \sum_{\beta_\delta, \beta_{\delta'}} \tilde{m}_{\lambda\delta}^{h,i} \tilde{m}_{\mu\delta'}^{h,i} \langle \chi_{\beta_\delta}, \chi_{\beta_{\delta'}} \rangle_H, \tag{10}$$

where the second sum is over quotients β_δ, $\beta_{\delta'}$ of weight $w - hi$.

Proof. The proof is similar to that of Theorem 3.1. For each i we count the number of possible y_i's, we write up the Inclusion-Exclusion Principle, and apply Osima's formula.

In particular, we see that

$$\sum \chi_{\beta_\lambda}(x) \chi_{\beta_\mu}(x^{-1}) =$$

$$= \sum_{i=0}^{[w/h]} (-1)^i \binom{w}{hi} \frac{(hi)!}{h^i i!} d^{(h-1)i} \sum_{\delta,\delta'} \tilde{m}_{\lambda\delta}^{h,i} \tilde{m}_{\mu\delta'}^{h,i} \sum \chi_{\beta_\delta}(z) \chi_{\beta_{\delta'}}(z^{-1})$$

holds where the first sum is over $H \cup \{h\}$-regular, the fourth over H-regular elements x and z of $Z_d \wr S_w$ and $Z_d \wr S_{w-hi}$, respectively, and the third is over all quotients β_δ and $\beta_{\delta'}$ of weight $w - hi$. After dividing both sides of the equation by $d^w w!$, we get the desired result.

□

5 The 'perfect isometry'

Let $d \geq 2$ be an arbitrary integer, and let H be a set of positive integers with the property that all elements of H are divisible by d. Define $H(d) =$

$\{h/d : h \in H\}$. As in the previous section, suppose that λ is a partition of n and the corresponding d-quotient β_λ has weight $w \leq [n/d]$. For each partition δ, the paths in $\mathcal{P}^{h,i}_{\lambda,\delta}$ are in bijective correspondence with the paths in $\tilde{\mathcal{P}}^{h/d,i}_{\lambda,\delta}$. Furthermore, the fundamental sign relation of Robinson and Osima states that if a path P of $\mathcal{P}^{h,i}_{\lambda,\delta}$ corresponds to a path $\tilde{P}$ of $\tilde{\mathcal{P}}^{h/d,i}_{\lambda,\delta}$, then we have $\sigma_\lambda\sigma_P = \sigma_\delta\sigma_{\tilde{P}}$, where σ_λ and σ_δ are certain signs (see Pages 61-63 of [4] and Pages 84-86 of [7]) associated with the partitions λ and δ. This result is proved as Proposition 3.3 in [4], and is implicitly mentioned without proof on Page 86 in [7].

With these notations we can state

Theorem 5.1. *If λ and μ are partitions of n with the same d-core, then*

$$\langle \chi_\lambda, \chi_\mu \rangle_H = \langle \sigma_\lambda \chi_{\beta_\lambda}, \sigma_\mu \chi_{\beta_\mu} \rangle_{H(d)}.$$

Proof. As at the end of Section 3, we may (and do) suppose, without loss of generality, that H consists only of elements no greater than n. Now proceed by induction on $|H|$. Case $|H| = 0$ is clear. Suppose that $|H| > 0$. By the Robinson-Osima fundamental sign relation and by the definitions above, we have $m^{h,i}_{\lambda\delta} = \sigma_\delta \cdot \tilde{m}^{h/d,i}_{\lambda\delta} \cdot \sigma_\lambda$ and $m^{h,i}_{\mu\delta'} = \sigma_{\delta'} \cdot \tilde{m}^{h/d,i}_{\mu\delta'} \cdot \sigma_\mu$. Apply these relations to the right-hand-side of the $\sigma_\lambda\sigma_\mu$-multiple of (10), then use the induction hypothesis for $|H| - 1$ to compare the right-hand-sides of (9) and (10).

□

By Theorem 5.1, to complete the proof of part (ii) of Theorem 1.1, it is sufficient to show that if $1 \in H(d)$, then for every irreducible character χ of $Z_d \wr S_w$, the truncated inner product across the set of $H(d)$-regular elements (of $Z_d \wr S_w$) of χ and the trivial character is not 0. This follows by a slight modification (generalization) of the (short, elementary, and self-contained) proof of Theorem 5.12 of [2]. To make this paper self-contained, we present a similar and a bit longer proof of a more general result. The reader is referred to Page 543 of [2] in case additional details are required.

We define the truncated inner product (across $H(d)$-regular elements) of two characters (of $Z_d \wr S_w$) to be the relevant linear combination of the truncated inner products of the relevant irreducible constituents.

Theorem 5.2. *Let $d \geq 2$, and suppose that $1 \in H(d)$. Let ψ be an arbitrary character of $Z_d \wr S_w$ lying over an S_w-stable linear character of the base group ${Z_d}^w$. Then for any irreducible character χ of $Z_d \wr S_w$, the algebraic integer*

$$\frac{d^w w! \cdot \langle \chi, \psi \rangle_{H(d)}}{\chi(1)}$$

is an integer, and it satisfies the congruence

$$\frac{d^w w! \cdot \langle \chi, \psi \rangle_{H(d)}}{\chi(1)} \equiv (-1)^w \psi(1) \pmod{d}. \tag{11}$$

Notice that if $\psi(1)$ is not divisible by d (for example, when ψ is the trivial character, or if ψ is the permutation character (of degree dw) minus the trivial character), then $\langle \chi, \psi \rangle_{H(d)} \neq 0$. So Theorem 5.2 would indeed complete the proof of part (ii) of Theorem 1.1.

Proof. For each partition λ of n we have d (essentially) different ways to define the d-quotient β_λ. In all cases σ_λ is the same sign. This gives us d (essentially) different 'perfect isometries' in Theorem 5.1. Hence if ψ and ψ' are two characters of $Z_d \wr S_w$ lying over an S_w-stable linear character of the base group ${Z_d}^w$ with $Res_{S_w}(\psi) = Res_{S_w}(\psi')$, then by Theorem 5.1, we have

$$\frac{d^w w! \cdot \langle \chi, \psi \rangle_{H(d)}}{\chi(1)} = \frac{d^w w! \cdot \langle \chi, \psi' \rangle_{H(d)}}{\chi(1)}.$$

This means that we may (and do) suppose that the character ψ in the statement of the theorem lies over the trivial character $\mathbf{1}$ of the base group.

Next we will show that if the congruence (11) holds for all w and for all irreducible characters χ lying over an S_w-stable linear character of the base group, then it holds for all irreducible characters χ of $Z_d \wr S_w$. Let $\chi = \chi_{\beta_\lambda}$ be an irreducible character of $Z_d \wr S_w$ labelled by the d-quotient $\beta_\lambda = (\lambda_0, \ldots, \lambda_{d-1})$.

For each $0 \le j \le d-1$, put $w_j := |\lambda_j|$ and define χ_j to be the character of $Z_d \wr S_{w_j}$ so that the character $\chi_1 \otimes \ldots \otimes \chi_w$ of $\prod_{j=1}^{d} Z_d \wr S_{w_j}$ induces to χ. Similarly, for each $0 \le j \le d-1$, let ψ_j be the character of $Z_d \wr S_{w_j}$ so that $\psi_1 \otimes \ldots \otimes \psi_w$ is the restriction of the character ψ to the subgroup $\prod_{j=1}^{d} Z_d \wr S_{w_j}$. Then by Frobenius reciprocity we have

$$\frac{\sum^{H(d)} \chi_{\beta_\lambda}(x)\overline{\psi(x)}}{\chi_{\beta_\lambda}(1)} = \prod_{j=1}^{d} \frac{\sum^{H(d)} \chi_j(x_j) \cdot \overline{\psi_j(x_j)}}{\chi_j(1)}, \tag{12}$$

where the first sum is over $H(d)$-regular elements x of $Z_d \wr S_w$ and the second is over $H(d)$-regular elements x_j of $Z_d \wr S_{w_j}$. By our assumption at the beginning of this paragraph, we may conclude that the right-hand-side of (12) is congruent to

$$\prod_{j=1}^{d} (-1)^{w_j} \psi_j(1) = (-1)^{w} \psi(1)$$

modulo d, which is exactly what we wanted.

So from now on we may (and do) suppose that χ is an irreducible character lying over an S_w-stable linear character of the base group, say over $\alpha \otimes \ldots \otimes \alpha$ where α is a linear character of Z_d. We wish to calculate

$$\frac{\sum^{H(d)} \chi(x)\overline{\psi(x)}}{\chi(1)} = \sum \frac{|K_\sigma| \cdot \chi(\sigma)}{\chi(1)} \overline{\psi(\sigma)} \sum_{(a_1,\ldots,a_w)} \alpha(a_1 \ldots a_w) \tag{13}$$

where the second sum is over representatives σ of all conjugacy classes K_σ of S_w and where the third sum is over all w-tuples $(a_1, \ldots, a_w)$ coming from $H(d)$-regular elements $x = (a_1, \ldots, a_w)\sigma$. It is easy to see by working with a cycle of σ at a time and by using the usual inner product of characters of Z_d that if σ has a cycle of length $t > 1$, then

$$\sum_{(a_1,\ldots,a_w)} \alpha(a_1 \ldots a_w)$$

is an integer divisible by d no matter if t is an element of $H(d)$ or not. Otherwise, if $\sigma = 1$, then this sum is an integer congruent to $(-1)^w$ modulo d. So the right-hand-side of (13) is indeed congruent to $(-1)^w \psi(1)$ modulo d. This completes the proof of the theorem.

$\square$

We have proved part (ii) of Theorem 1.1.

Finally, we turn to the proof of part (iii) of Theorem 1.1. Let H be an arbitrary set of positive integers, and let $d \geq 2$ be the greatest common divisor of the elements in H. Let the smallest element in H be rd where $r > 1$. Let λ be a d-core partition. Add $(r-1)d$ to the largest part of λ to get a partition λ' of the integer $n := |\lambda| + (r-1)d$. Let B be the combinatorial d-block of S_n containing $\chi_{\lambda'}$. By Theorem 5.1 and by our assumption on the minimality of r, for arbitrary irreducible characters χ_μ, χ_ν in B we have

$$\langle \chi_\mu, \chi_\nu \rangle_H = \langle \sigma_\mu \chi_{\beta_\mu}, \sigma_\nu \chi_{\beta_\nu} \rangle_{H(d)} = \langle \sigma_\mu \chi_{\beta_\mu}, \sigma_\nu \chi_{\beta_\nu} \rangle. \tag{14}$$

Since the right-hand-side of (14) is 0 whenever $\mu \neq \nu$, the set B is a union of $|Z_d \wr S_{r-1}|$ different H-blocks. There are infinitely many choices for the d-core partition λ, so part (iii) of Theorem 1.1 is established.

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