IRREDUCIBLE INDUCTION AND NILPOTENT SUBGROUPS IN FINITE GROUPS

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ABSTRACT. Suppose that G is a finite group and H is a nilpotent subgroup of G. If a character of H induces an irreducible character of G, then the generalized Fitting subgroup of G is nilpotent.

Dedicated to Kay Magaard

1. Introduction

Brauer's famous induction theorem asserts that every irreducible character of G is an integer linear combination of characters induced from nilpotent subgroups of G. When an irreducible character is induced from a character of a single nilpotent subgroup of G is a problem that has not been treated until now.

If γ is a character of H, a subgroup of a finite group G, it is not clear at all when to expect the induced character γ^G to be irreducible. The only case which is understood, using the Clifford correspondence, is when H happens to contain the stabilizer of an irreducible character of a normal subgroup N of G. In this case, $N\mathbf{C}_G(N) \subseteq H$, and in a well-defined sense, H is considered to be a *large* subgroup of G: the centralizer of the core of H in G is contained in H. But of course, irreducible induction of characters also occurs, we might say by *accident*, in other cases. Even more, some simple groups have irreducible characters that are induced from linear characters of *very easy* subgroups, which of course are core-free. For instance, $G = \mathrm{PSL}(2, p)$ with $p \equiv 3 \pmod 4$, has an irreducible character of degree $p+1$ which is induced from a linear character of the normalizer H of a Sylow p-subgroup of G. Here, H is the semidirect product of the cyclic group of order p by the cyclic group of order $(p-1)/2$. The key thing is that it does not matter how easy H is as long as it is not nilpotent.

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We write $\mathbf{F}^*(G)$ for the generalized Fitting subgroup of G, and recall its fundamental property that $\mathbf{C}_G(\mathbf{F}^*(G)) \subseteq \mathbf{F}^*(G)$. Also, $\mathbf{F}(G)$ is the Fitting subgroup of G.

Theorem A. *Let G be a finite group and let H be a maximal nilpotent subgroup of G. Suppose that $\gamma \in \mathrm{Irr}(H)$ is such that $\gamma^G \in \mathrm{Irr}(G)$. Then $\mathbf{F}^*(G) \subseteq H$. In particular $\mathbf{F}^*(G) = \mathbf{F}(G)$.*

Irreducible characters that are induced from Sylow subgroups were studied in [RS]. Their main result is easily seen to be a consequence of Theorem A.

Corollary B. *If G is a finite group, $P \in \mathrm{Syl}_p(G)$ and $\gamma \in \mathrm{Irr}(P)$ induces irreducibly to G, then $\mathbf{Z}(P) \lhd \lhd G$.*

Proof. Suppose that P is contained in a nilpotent subgroup $H = P \times K$ of G. Since γ^H is irreducible, $\mathbf{C}_H(P) \subseteq P$ by elementary character theory, and thus $H = P$. Therefore $\mathbf{F}^*(G) = \mathbf{O}_p(G)$ by Theorem A. Now $\mathbf{Z}(P) \subseteq \mathbf{C}_G(\mathbf{F}^*(G)) \subseteq \mathbf{F}^*(G)$, and we are done. □

We use a combination of several techniques to prove Theorem A. One of them is to prove that, in general, nilpotent subgroups are small in almost-simple groups. (This complements work in [V].) Some delicate character theory reductions are needed to bring this fact into the proof. The cases in which this does not happen are dealt using character theory of certain quasisimple groups. Perhaps it is worth to state here what we shall need and prove below.

Theorem C. *If Y is a nilpotent subgroup in an almost simple group X, then $|Y|^2 < |X|$.*

2. Almost simple groups

We begin with the proof of Theorem C. Note that during the proof we try to show the slightly stronger inequality $2|Y|^2 \leq |X|$. It turns out that in most cases even this stronger statement holds. Then we will be able to provide a very short list of groups where the inequality $2|Y|^2 \leq |X|$ fails (see Theorem 2.1).

Let X be an almost simple group with socle S and let $m(S)$ denote the largest possible size of a nilpotent subgroup in S. Let Y be a nilpotent subgroup in X.

Step 1. If $S \cong \mathrm{Alt}(k)$, then $2 \cdot |Y|^2 \leq |X|$, unless $k \in \{5, 6\}$ when $|Y|^2 < |X|$.

Assume first that $k \geq 9$. In this case we have $2 \cdot |Y|^2 \leq 2^{2k-1} < k!/2 \leq |X|$ by a result from [D] stating that a nilpotent permutation group of degree k has size at most 2^{k-1}.

By using [V, Theorem 2.1], the following table contains the value of $m(S)$ and $|\mathrm{Out}(S)|$ when $S = \mathrm{Alt}(k)$ for $k = 5, 6, 7, 8$.

$k =$	5	6	7	8
$m(\mathrm{Alt}(k))$	5	9	12	2^6
$\mathrm{Out}(\mathrm{Alt}(k))$	2	4	2	2

Now, $|Y| \leq m(S) \cdot |X : S| < \sqrt{|X|}$ holds for $k = 5, 6$, while $|Y| \leq m(S) \cdot |X : S| < \sqrt{|X|/2}$ holds for $k = 7, 8$.

Step 2. If S is a sporadic simple group or the Tits group, then $2 \cdot |Y|^2 \leq |X|$.

Let S be a sporadic simple group. The largest possible size of a nilpotent subgroup in S is equal to the size of a Sylow subgroup of S by [V, Section 2.4]. This in turn is less than $|S|^{1/2}/2$ by [At]. Since $|\mathrm{Out}(S)| \leq 2$, the claim follows.

Let S be the Tits group. If $X = S$, then $|Y|$ is at most the size of a Sylow subgroup in X, by the proof of [V, Theorem 2.2], and so $|Y| \leq 2^{11}$ by [At]. If $X = \mathrm{Aut}(S)$, then $|Y| \leq 2^{12}$, since the outer automorphism group of S has size 2, and again, the claim follows.

From now on, assume that S is a finite simple group of Lie type different from $\mathrm{Alt}(5)$ and $\mathrm{Alt}(6)$. Note that $2 \cdot |Y|^2 \leq |X|$ whenever $2 \cdot |\mathrm{Out}(S)| \cdot m(S)^2 \leq |S|$.

Step 3. If $m(S) > |S|_p$ where p is any natural characteristic for S, then $2 \cdot |Y|^2 \leq |X|$.

There are three possibilities for S according to [V, Table 3]: (1) $S \cong \mathrm{PSL}(2, 2^m)$ for some $m \geq 3$; (2) $S \cong \mathrm{PSL}(2, 2^m + 1)$ for some $m \geq 4$; and (3) $S \cong \mathrm{PSU}(3, 3)$. (Note that the third group in [V, Table 3] is solvable.)

Consider cases (1) and (2). Let q be the size of the field over which S is defined. Write d to satisfy $q = p^d$. Then $\mathrm{Out}(S) \cong C_{(2,q-1)} \times C_d$ where $(2, q-1)$ is the greatest common divisor of 2 and $q-1$, and $m(S) \leq q+1$ by [V, Table 3]. It is straightforward to check that $2 \cdot (2, q-1)^2 \cdot d \cdot (q+1) \leq q(q-1)$, establishing $2 \cdot |\mathrm{Out}(S)| \cdot m(S)^2 \leq |S|$.

In case $S \cong \mathrm{PSU}(3, 3)$ we have $m(S) = 32$, $|S| = 6048$, and $|\mathrm{Out}(S)| = 2$ by [At]. Thus $2 \cdot |\mathrm{Out}(S)| \cdot m(S)^2 \leq |S|$.

We may now assume that S is a finite simple group of Lie type of Lie rank ℓ defined over a field of size q in characteristic p and $m(S) = |S|_p$.

Step 4. $2 \cdot |Y|^2 \leq |X|$ unless possibly if $\ell = 1$ and $q < 2^{12}$, or $2 \leq \ell \leq 9$ and $q < 2^6$.

By the order formulas for $|S|$ and $|S|_p$ (see [KL, page 170]) and by $\prod_{i=1}^{\infty}(1 - 2^{-i}) > 2/7$ (see the proof of [B, Lemma 3.2]), we have

$$2 \cdot \min\{\ell + 1, q + 1\} \cdot \frac{|S|}{(|S|_p)^2} > \frac{7}{8} \cdot \prod_{i=1}^{\infty}(1 - 2^{-i}) \cdot q^{\ell} > \frac{1}{4} \cdot q^{\ell}.$$

Again by [KL, page 170], we have $2 \cdot |\mathrm{Out}(S)| \leq 8 \cdot \min\{\ell + 1, q + 3\} \cdot \log_p q$ unless $\ell = 4$, when $2 \cdot |\mathrm{Out}(S)| \leq 48 \cdot \log_p q$. These are smaller than

$$\frac{1}{8 \cdot \min\{\ell + 1, q + 1\}} \cdot q^{\ell} < \frac{|S|}{(|S|_p)^2},$$

unless $\ell = 1$ and $q < 2^{12}$, or $2 \leq \ell \leq 9$ and $q < 2^6$.

Step 5. If S is not isomorphic to any of the groups $\mathrm{Alt}(5)$, $\mathrm{Alt}(6)$, $\mathrm{PSL}(2, 7)$, $\mathrm{PSL}(3, 4)$, $\mathrm{PSp}(4, 3)$, $\mathrm{PSU}(4, 3)$, *then $2 \cdot |Y|^2 \leq |X|$.*

By a Gap [G] computation (using Step 4) together with [KL, page 170] we get that $2 \cdot |\mathrm{Out}(S)| \cdot (|S|_p)^2 < |S|$ unless S is isomorphic to $\mathrm{Alt}(5)$, $\mathrm{Alt}(6)$, $\mathrm{PSL}(2, 7)$, $\mathrm{PSL}(3, 4)$, $\mathrm{PSp}(4, 3)$, $\mathrm{PSU}(3, 5)$, $\mathrm{PSU}(3, 8)$, $\mathrm{PSU}(4, 3)$, $\mathrm{PSU}(6, 2)$, $\mathrm{P}\Omega^+(8, 2)$, or $\mathrm{P}\Omega^+(8, 3)$. If $S \cong$

$\mathrm{PSU}(3,5)$, $\mathrm{PSU}(3,8)$, $\mathrm{PSU}(6,2)$, $\mathrm{P}\Omega^+(8,2)$, or $\mathrm{P}\Omega^+(8,3)$, then a computation shows that $2 \cdot |Y|^2 \leq |X|$, using the fact that the outer automorphism group of S is not nilpotent.

Final Step.

We have $|\mathrm{Out}(S)| \cdot (|S|_p)^2 < |S|$ unless S is isomorphic to $\mathrm{PSL}(3,4)$, $\mathrm{PSU}(4,3)$, or $\mathrm{P}\Omega^+(8,3)$. In the latter case $2 \cdot |Y|^2 \leq |X|$ by Step 5.

Let $S \cong \mathrm{PSL}(3,4)$. The outer automorphism group of S is $C_2 \times \mathrm{Sym}(3)$, a non-nilpotent group. We have $|Y|^2 < |X|$ unless possibly if Y projects onto X/S and X/S is cyclic of order 6. In this exceptional case every element of order 6 in X has centralizer of order at most 54 by [At] and so $|Y| \leq 54$ giving $|Y|^2 < |X|$.

Finally, we need to check that $|Y|^2 < |X|$ holds for the case $S \cong \mathrm{PSU}(4,3)$. Using information from $\mathrm{Out}(S)$, the order of S, the fact that $m(S) = 3^6$, and that the sizes of the centralizers of elements of orders 10 or 14 in $\mathrm{Aut}(S)$ are too small (at most 56) by [At], one can prove that $|Y|^2 < |X|$ except possibly if $X = \mathrm{Aut}(\mathrm{PSU}(4,3))$, $YS = X$ and the set of prime divisors of $|Y|$ is $\{2,3\}$. Let us assume that this is the case. Then the nilpotency of Y guarantees that Y cannot contain a maximal unipotent subgroup of S. Therefore, if the Sylow 2-subgroup of Y is disjoint from S, then $|Y|^2 \leq (8 \cdot 3^5)^2 < 8 \cdot |\,\mathrm{PSU}(4,3)| = |X|$. Otherwise, let $Z = \{\alpha \in \mathbb{F}_9^\times \mid \alpha^4 = 1\} = \mathbf{Z}(\mathrm{SU}(4,3))$ and $Z < K < \mathrm{SU}(4,3)$ such that $|K : Z| = 2$ and K/Z is normal in $Y \cap \mathrm{PSU}(4,3)$. Let V be a 4 dimensional non-degenerate Hermitian space over $\mathbb{F}_9^\times$ with Hermitian product $(\ ,\ )$ and identify $\mathrm{SU}(4,3)$ with the special unitary group on V preserving $(\ ,\ )$. Let $x \in K \setminus Z$. Then x is diagonalisable with respect to a suitable basis of V. Since x^2 is a scalar transformation, all the eigenvalues of x are $\pm\gamma$ for some $\gamma \in \mathbb{F}_9^\times$. Let V_1 and V_2 be the eigenspaces corresponding to γ and $-\gamma$, respectively. We may assume that $\dim(V_1) \geq \dim(V_2)$, so either $\dim(V_1) = \dim(V_2) = 2$ or $\dim(V_1) = 3$ and $\dim(V_2) = 1$. If u is a non-singular eigenvector of x, then $0 \neq (u,u) = (x(u), x(u)) = \gamma^{3+1}(u,u)$, so $\gamma^4 = 1$. In that case $\det(x) = 1$ holds only if $\dim(V_1) = \dim(V_2) = 2$. Now, if $\dim(V_1) = 3$, then there must be a non-singular eigenvector of x in V_1, which leads to a contradiction. Thus, $\dim(V_1) = \dim(V_2) = 2$ and $\gamma^4 = 1$ must hold. If $v_1 \in V_1$ and $v_2 \in V_2$ are arbitrary, then $(v_1, v_2) = (x(v_1), x(v_2)) = (\gamma \cdot v_1, -\gamma \cdot v_2) = -\gamma^{3+1}(v_1, v_2) = -(v_1, v_2)$, so $v_1 \perp v_2$. Thus, we get that V_1 and V_2 are orthogonal complements to each other, so both V_1 and V_2 are non-degenerate subspaces. Let $Z < N < \mathrm{GU}(4,3) = \mathrm{GU}(V)$ with $N/Z = Y \cap \mathrm{GU}(4,3)$, so $|Y| = |N|/2$. Since N normalises K, it permutes the homogeneous components of K by Clifford theory, which are V_1 and V_2. It follows that $N \leq (\mathrm{GU}(V_1) \times \mathrm{GU}(V_2)) \rtimes C_2 \simeq \mathrm{GU}(2,3) \wr C_2$. Using that N is nilpotent, we have $|N| \leq 32^2 \cdot 2 = 2^{11}$, so $|Y|^2 \leq 2^{20} < 8 \cdot |\,\mathrm{PSU}(4,3)| = |X|$ follows.

The following is essentially a consequence of the proof of Theorem C.

Theorem 2.1. *Let Y be a nilpotent subgroup in an almost simple group X with socle S. Assume that $Y \cap S$ is a p-group for some prime p. Then $2 \cdot |Y|^2 \leq |X|$ except when*

$$S \in \mathrm{LIST} = \{\mathrm{Alt}(5), \mathrm{Alt}(6), \mathrm{PSL}(2,7), \mathrm{PSL}(3,4), \mathrm{PSU}(4,3)\}$$

and $Y \cap S$ is a Sylow p-subgroup in S. Moreover, if $S \cong \mathrm{Alt}(5)$ or $\mathrm{Alt}(6)$ and the inequality fails, then $p = 2$.

Proof. Let $S \cong \mathrm{PSp}(4,3)$. We have $|S| = 2^6 \cdot 3^4 \cdot 5$ and $|\mathrm{Out}(S)| = 2$ by [At]. Using these and the fact that the centralizer of a Sylow 3-subgroup in $\mathrm{Aut}(\mathrm{PSp}(4,3))$ has size 3, we get the inequality $2 \cdot |Y|^2 \leq |X|$. If $S \notin \{\mathrm{Alt}(5), \mathrm{Alt}(6), \mathrm{PSL}(2,7), \mathrm{PSL}(3,4), \mathrm{PSU}(4,3)\}$, then $2 \cdot |Y|^2 \leq |X|$ by Step 5.

Now assume that $S \in \{\mathrm{Alt}(5), \mathrm{Alt}(6), \mathrm{PSL}(2,7), \mathrm{PSL}(3,4), \mathrm{PSU}(4,3)\}$.

Let $X = \mathrm{Sym}(5)$. If $|Y| \leq 6$, then $2 \cdot |Y|^2 \leq |X|$ follows. Otherwise Y is a Sylow 2-subgroup of X and $2 \cdot |Y|^2 > |X|$. If $X = \mathrm{Alt}(5)$, then $2 \cdot |Y|^2 \leq |X|$.

Let $S = \mathrm{Alt}(6)$. If $Y \cap S$ is different from a Sylow 2-subgroup and different from a Sylow 3-subgroup of S, then $2 \cdot |Y|^2 \leq |X|$. Let $Y \cap S$ be a Sylow 3-subgroup of S. Then $|Y| = 9$ in case $X = \mathrm{Alt}(6)$ and $|Y| \leq 18$ otherwise. We conclude that $2 \cdot |Y|^2 \leq |X|$.

Finally, assume that $Y \cap S$ is not a Sylow p-subgroup of S where S is any of the groups $\mathrm{PSL}(2,7)$, $\mathrm{PSL}(3,4)$, $\mathrm{PSU}(4,3)$. Then $2 \cdot |Y|^2 \leq |X|$ by [At]. ☐

3. Quasisimple Groups

The goal of this section is to prove the following.

Theorem 3.1. *Suppose that G is a quasisimple group, with $S = G/\mathbf{Z}(G)$ in*

$$\mathrm{LIST} = \{\mathrm{Alt}(5), \mathrm{Alt}(6), \mathrm{PSL}(2,7), \mathrm{PSL}(3,4), \mathrm{PSU}(4,3)\}.$$

Let $H \geq \mathbf{Z}(G)$ be a nilpotent subgroup of G such that $H/\mathbf{Z}(G)$ is a Sylow p-subgroup of $G/\mathbf{Z}(G)$ for some prime p. If $G/\mathbf{Z}(G) \cong \mathrm{Alt}(5)$ or $\mathrm{Alt}(6)$, then assume in addition that $p = 2$. If $\gamma \in \mathrm{Irr}(H)$, then γ^G has at least two irreducible constituents with different degrees.

We will need the following technical lemma:

Lemma 3.2. *Let G be a finite group and let $H \geq \mathbf{Z}(G)$ be a nilpotent subgroup of G such that $H/\mathbf{Z}(G)$ is a Sylow p-subgroup of $G/\mathbf{Z}(G)$ for some prime p. Suppose that all irreducible constituents of γ^G are of the same degree D for some $\gamma \in \mathrm{Irr}(H)$. Then the following statements hold for any $g \in G \smallsetminus \mathbf{Z}(G)$.*

- (i) *$\gamma^G(g) = 0$ if $g \notin \cup_{x \in G} H^x$. In particular, $\gamma^G(g) = 0$ if the coset $g\mathbf{Z}(G)$ has order coprime to p in $G/\mathbf{Z}(G)$.*
- (ii) *Suppose there exist some $\alpha \in \mathbb{C}$ and an algebraic conjugate α^* of α such that $\chi(g) \in \{\alpha, \alpha^*\}$ for all $\chi \in \mathrm{Irr}(G)$ of degree D. If $\gamma^G(g) = 0$, then $\alpha + \alpha^* = 0$.*
- (iii) *Suppose $p \nmid |\mathbf{Z}(G)|$ so that $H = P \times \mathbf{Z}(G)$ for some $P \in \mathrm{Syl}_p(G)$. Then all irreducible characters of G that lie above both $\gamma|_P$ and $\gamma|_{\mathbf{Z}(G)}$ must have the same degree.*

Proof. (i) and (iii) are obvious.

For (ii), write $\gamma^G = \sum_{i=1}^k \chi_i$ with $\chi_i \in \mathrm{Irr}(G)$ of degree D. By the assumption, $\chi_i(g) = \alpha$ or α^*. Now if $\alpha = \alpha^*$, then $0 = \gamma^G(g) = k\alpha$ and so $\alpha = 0$. We may now assume that $\alpha \neq \alpha^*$ and that

$$\chi_1(g) = \ldots = \chi_j(g) = \alpha, \ \chi_{j+1}(g) = \ldots = \chi_k(g) = \alpha^*, \tag{3.1}$$

for some $1 \leq j \leq k-1$. Since the set of $\chi \in \mathrm{Irr}(G)$ of given degree D is stable under the action of $\Gamma := \mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q})$, the set $\{\alpha, \alpha^*\}$ is Γ-stable, and there is some $\sigma \in \Gamma$ that sends α to α^*, whence $\sigma(\alpha^*) = \alpha$. Now, by (3.1) we have

$$j\alpha + (k-j)\alpha^* = \gamma^G(g) = 0 = \sigma(0) = \sigma(\gamma^G(g)) = j\alpha^* + (k-j)\alpha,$$

whence $k(\alpha+\alpha^*) = (j\alpha + (k-j)\alpha^*) + (j\alpha^* + (k-j)\alpha) = 0$ and $\alpha+\alpha^* = 0$ as stated. $\square$

Proof of Theorem 3.1. Assume the contrary: all irreducible constituents of γ^G have the same degree D. Again write

$$\gamma^G = \sum_{i=1}^{k} \chi_i, \tag{3.2}$$

with $\chi_i \in \mathrm{Irr}(G)$ of degree D. Denoting $\lambda := \gamma|_{\mathbf{Z}(G)}$, we see that all χ_i in (3.2) lie above λ. Modding out the quasisimple group G by $\mathrm{Ker}(\lambda)$, we may therefore assume that all χ_i are faithful characters of G.

We will analyze all possible cases for $S = G/\mathbf{Z}(G)$. We will use the notation $b5 = (-1+\sqrt{5})/2$ and $b7 = (-1+\sqrt{-7})/2$ of [At], and freely use the character tables of G as listed in [At]. Also, $\mathbf{reg}_P$ will denote the regular character of $P \in \mathrm{Syl}_p(G)$.

Case 1: $S = \mathrm{Alt}(5)$. Then we have $p = 2$ by hypothesis. Taking $g \in G$ of order 5, we see that g fulfills the conditions of 3.2(ii): indeed,

$$\alpha = \begin{cases} j, & D \equiv j (\bmod 5) \text{ with } j \in \{0, \pm 1\}, \\ b5, & D \equiv 2 (\bmod 5), \\ -b5, & D \equiv 3 (\bmod 5). \end{cases} \tag{3.3}$$

By Lemma 3.2 applied to g we have that $\gamma^G(g) = 0$ and $\alpha + \alpha^* = 0$. As $b5 + b5^* = -1$, we conclude that $5|D$, and so in fact $D = 5$ and $\mathbf{Z}(G) = 1$. In this case, $\chi_i(h) = -1 \neq 0$ for an element $h \in G$ of order 3, contradicting Lemma 3.2 applied to h.

Case 2: $S = \mathrm{PSL}(2,7)$. Taking $g \in G$ of order 7, we see that g fulfills the conditions of 3.2(ii) with

$$\alpha = \begin{cases} j, & D \equiv j (\bmod 7) \text{ with } j \in \{0, \pm 1\}, \\ b7, & D \equiv 3 (\bmod 7), \\ -b7, & D \equiv 4 (\bmod 7). \end{cases} \tag{3.4}$$

Suppose first that $p \neq 7$. By Lemma 3.2 applied to g we have that $\gamma^G(g) = 0$ and $\alpha+\alpha^* = 0$. As $b7 + b7^* = -1$, we conclude that $7|D$, and so in fact $D = 7$ and $\mathbf{Z}(G) = 1$. In this case, $\chi_i(h) = -1 \neq 0$ for an element $h \in G$ of order 2, contradicting Lemma 3.2 applied to h.

Assume now that $p = 7$. Then $H = P \times \mathbf{Z}(G)$ with $P \in \mathrm{Syl}_p(G)$. If $\mathbf{Z}(G) = 1$, then there are characters in $\mathrm{Irr}(G)$ of degree 7 and degree 8, each containing $\mathbf{reg}_P$ on restriction to P. If $|\mathbf{Z}(G)| = 2$, then there exist $\theta_i \in \mathrm{Irr}(G)$, $i = 1,2,3$, with θ_1 of degree 8 containing $\mathbf{reg}_P$, θ_2 of degree 6 containing $\mathbf{reg}_P - 1_P$, and θ_3 of degree 4 containing 1_P. This is impossible by Lemma 3.2(iii).

Case 3: $S = \mathrm{Alt}(6)$. Then we have $p = 2$ by hypothesis. Taking $g \in G$ of order 5, we see that g fulfills the conditions of 3.2(ii) with α specified in (3.3). By Lemma 3.2 applied to g we have that $\gamma^G(g) = 0$ and $\alpha + \alpha^* = 0$. As $b5 + b5^* = -1$, we conclude that $5|D$; in particular, $|\mathbf{Z}(G)| \leq 3$.

Assume $\mathbf{Z}(G) = 1$, so that $D \in \{5, 10\}$. If $D = 10$, then $\chi_i(h) = 1 \neq 0$ for an element $h \in G$ of order 3, contradicting Lemma 3.2 applied to h. Suppose $D = 5$. Then $\chi_i \in \{\theta, \theta^*\}$

for all χ_i in (3.2), where $\theta(1) = \theta^*(1) = 5$ and

$$(\theta(x), \theta^*(x)) = (2, -1),\ (\theta(y), \theta^*(y)) = (-1, 2)$$

for some elements $x, y \in G$ of order 3. Without loss we may assume that

$$\chi_1 = \ldots = \chi_j = \theta,\ \chi_{j+1} = \ldots = \chi_k = \theta^*$$

for some $1 \leq j \leq k$. Applying Lemma 3.2 to x and to y, we obtain

$$2j + (k-j)(-1) = j(-1) + 2(k-j) = 0,$$

a contradiction since $k \geq 1$.

If $|\mathbf{Z}(G)| = 2$, then $D = 10$, and $\chi_i(h) = 1 \neq 0$ for an element $h \in G$ of order 3, contradicting Lemma 3.2 applied to h.

Assume now that $|\mathbf{Z}(G)| = 3$. Then $D = 15$ and $H = P \times \mathbf{Z}(G)$ with $P \in \mathrm{Syl}_2(G)$. However, a faithful character in $\mathrm{Irr}(G)$ of degree 9 contains $\mathbf{reg}_P$ on restriction to P and so must occur in γ^G, a contradiction.

Case 4: $S = \mathrm{PSL}(3,4)$. Taking $g_5 \in G$ of order 5, we see that g_5 fulfills the conditions of 3.2(ii) with α specified in (3.3). Taking $g_7 \in G$ of order 7, we see that g_7 fulfills the conditions of 3.2(ii) with α specified in (3.4). Thus if $p \neq 5$ then by Lemma 3.2 applied to g_5 we have that $\gamma^G(g_5) = 0$ and $\alpha + \alpha^* = 0$. As $b5 + b5^* = -1$, we conclude that $5|D$. Likewise, if $p \neq 7$ then Lemma 3.2 applied to g_7 yields that $7|D$.

Now if $p \neq 5, 7$, then we have that $35|D$; in particular, $|\mathbf{Z}(G)| \leq 2$. If $|\mathbf{Z}(G)| = 1$, then $D = 35$, and $\chi_i(g_2) = 3 \neq 0$ and $\chi_i(g_3) = -1 \neq 0$ for an element $g_2 \in G$ of order 2 and an element $g_3 \in G$ of order 3. This contradicts Lemma 3.2 applied to g_3 when $p \neq 3$ and to g_2 when $p \neq 2$. Likewise, if $|\mathbf{Z}(G)| = 2$, then $D = 70$, and $\chi_i(g_2) = -2 = \chi_i(g_3)$ for an element $g_2 \in G$ of order 2 and an element $g_3 \in G$ of order 3, again a contradiction.

Assume now that $p = 5$ or 7, whence $H = P \times \mathbf{Z}(G)$. In each of these cases, one can find two faithful characters in $\mathrm{Irr}(G)$ of distinct degrees that both contain $\mathbf{reg}_P$ on restriction to P, contradicting Lemma 3.2(iii).

Case 5: $S = \mathrm{PSU}(4,3)$. Taking $g_5 \in G$ of order 5, we see that g_5 fulfills the conditions of 3.2(ii) with α specified in (3.3). Taking $g_7 \in G$ of order 7, we see that g_7 fulfills the conditions of 3.2(ii) with α specified in (3.4). Thus if $p \neq 5$ then by Lemma 3.2 applied to g_5 we have that $\gamma^G(g_5) = 0$ and $\alpha + \alpha^* = 0$. As $b5 + b5^* = -1$, we conclude that $5|D$. Likewise, if $p \neq 7$ then Lemma 3.2 applied to g_7 yields that $7|D$.

Now if $p \neq 5, 7$, then we have that $35|D$. In all of these cases, one of the following holds:

(3.5.a) One can find a p'-element $h \in G \smallsetminus \mathbf{Z}(G)$ and $\beta \neq 0$ such that $\chi_i(h) = \beta$ for all χ_i occurring in (3.2).

(3.5.b) One can find two p'-elements $h, h' \in G \smallsetminus \mathbf{Z}(G)$ and pairs $(\beta_1, \beta_1') \in \mathbb{C}^2$ and $(\beta_2, \beta_2') \in \mathbb{C}^2$ such that $(\chi_i(h), \chi_i(h')) = (\beta_1, \beta_1')$ or (β_2, β_2') for all χ_i occurring in (3.2). Furthermore, the system of equations $x_1\beta_1 + x_2\beta_2 = x_1\beta_1' + x_2\beta_2' = 0$ have only one solution $x_1 = x_2 = 0$.

Certainly, (3.5.a) contradicts Lemma 3.2(iii). In the case of (3.5.b), if we let x_1 be the number of χ_i in (3.2) with $(\chi_i(h), \chi_i(h')) = (\beta_1, \beta_1')$ and x_2 the number of the remaining χ_i, then by Lemma 3.2(i) we must have

$$x_1\beta_1 + x_2\beta_2 = \gamma^G(h) = 0 = \gamma^G(h') = x_1\beta_1' + x_2\beta_2',$$

whence $x_1 + x_2 = k = 0$, again a contradiction.

Assume now that $p = 5$ or 7, whence $H = P \times \mathbf{Z}(G)$. In each of these cases, one can find two faithful characters in $\mathrm{Irr}(G)$ of distinct degrees that both contain $\mathbf{reg}_P$ on restriction to P (in fact, they can be chosen to have p-defect 0, unless $\mathbf{Z}(G) = 12_2$ in the notation of [At]). This contradicts Lemma 3.2(iii). $\square$

Remark 3.3. Note that Theorem 3.1 does not hold when $(S, p) = (\mathrm{Alt}(5), 5)$ and $(\mathrm{Alt}(6), 3)$, even with $\gamma \in \mathrm{Irr}(H)$ assumed to be linear. Indeed, taking $H = P \times \mathbf{Z}(G)$ with $P \in \mathrm{Syl}_p(G)$, $\gamma|_P = 1_P$, and $\gamma|_{\mathbf{Z}(G)}$ to be faithful, we have $\gamma^G = 2\chi$ for some irreducible character χ of degree 6 of $G = 2\mathrm{Alt}(5)$ in the former case, and $\gamma^G = 2(\chi + \chi')$ for some irreducible characters χ and χ' of degree 10 of $G = 2\mathrm{Alt}(6)$ in the latter case.

4. Induction and Central Products

Suppose that the finite group E is the central product of subgroups $X_1, \ldots, X_n$. By this we mean that $X_i \leq E$, $[X_i, X_j] = 1$ for $i \neq j$, $Z = \bigcap_{j=1}^n X_i$, and $E/Z = (X_1/Z) \times \cdots \times (X_n/Z)$, that is, $(\prod_{j \neq i} X_j) \cap X_i = Z$ for all i. We fix $\lambda \in \mathrm{Irr}(Z)$.

Suppose that χ_i is a character of X_i all of whose irreducible constituents lie over λ. We claim that there is a unique character $\chi_1 \cdot \ldots \cdot \chi_n$ of E, all of whose irreducible constituents lie over λ, such that

$$(\chi_1 \cdot \ldots \cdot \chi_n)(x_1 \cdots x_n) = \chi_1(x_1) \cdots \chi_n(x_n)$$

for $x_i \in X_i$.

Let $E^* = X_1 \times \cdots \times X_n > Z \times \cdots \times Z$ and let

$$A = \{(x_1, \ldots, x_n) \in Z \times \cdots \times Z |\ x_1 \cdots x_n = 1\}.$$

Then E^*/A is naturally isomorphic to E, via the homomorphism f given by $(x_1, \ldots, x_n) \mapsto x_1 \cdots x_n$ with kernel A. Notice that A is contained in the kernel of $\chi = \chi_1 \times \cdots \times \chi_n$, and therefore χ naturally corresponds to a unique character ψ of E such that $\psi(f(g)) = \chi(g)$ for $g \in E^*$. The character ψ is what we have called $\chi_1 \cdot \ldots \cdot \chi_n$.

Furthermore, by [N_2, Theorem 10.7], the map

$$\mathrm{Irr}(X_1|\lambda) \times \cdots \times \mathrm{Irr}(X_n|\lambda) \to \mathrm{Irr}(E|\lambda) \tag{4.1}$$

given by $(\theta_1, \ldots, \theta_n) \mapsto \theta_1 \cdot \ldots \cdot \theta_n$ is a bijection.

Lemma 4.1. *Suppose now that we have a subgroup K of E of the form $K = K_1 \cdots K_n$, where $Z \leq K_i \leq X_i$. Suppose that $\gamma_i \in \mathrm{Irr}(K_i|\lambda)$. Then*

$$(\gamma_1 \cdot \ldots \cdot \gamma_n)^E = (\gamma_1)^{X_1} \cdot \ldots \cdot (\gamma_n)^{X_n}.$$

Proof. Again, let E^*, A and $f : E^* \mapsto E$ as before. Let $K^* = K_1 \times \cdots \times K_n$, so $A < K^*$ and $K^*/A \simeq f(K^*) = K$. Since the map defined in (4.1) commutes with the induction of characters, it is enough to check that

$$(\gamma_1 \times \cdots \times \gamma_n)^{X_1 \times \cdots \times X_n} = \gamma_1^{X_1} \times \cdots \times \gamma_n^{X_n},$$

but this is an easy exercise. $\square$

5. Proof of Theorem A

In order to prove Theorem A, we shall use the following.

Theorem 5.1. *Suppose that H is a nilpotent subgroup of G, and $N \lhd G$ is nilpotent. If $\gamma \in \mathrm{Irr}(H)$ is such that $\gamma^{HN} \in \mathrm{Irr}(HN)$, then HN is nilpotent.*

Proof. This is Corollary 2.3 of [N_1]. □

Next we prove our main result.

Theorem 5.2. *Let G be a finite group and let H be a nilpotent subgroup of G. Suppose that $\gamma \in \mathrm{Irr}(H)$ is such that $\gamma^G \in \mathrm{Irr}(G)$. Then $\mathbf{F}^*(G) = \mathbf{F}(G)$.*

Proof. Let G be a counterexample to Theorem 5.2 such that $|G|$ is as small as possible. Of course, if $H = G$, then G is nilpotent and there is nothing to prove. We may assume therefore that H is a maximal subgroup of G with respect to being nilpotent.

Since $\mathbf{F}^*(G)$ strictly contains $\mathbf{F}(G)$, the group G is not only non-solvable but it contains a subnormal quasi-simple group, say X. Let E be the product of all H-conjugates of X in G. Since G is a minimal counterexample, we have $G = HE$.

The group E is the layer of G and is a perfect central extension of a direct product of simple groups which are transitively permuted by H. Thus $E/\mathbf{Z}(E)$ is a perfect minimal normal subgroup of $G/\mathbf{Z}(E)$. Write $E/\mathbf{Z}(E)$ as $E/\mathbf{Z}(E) = S_1 \times \cdots \times S_n$ for some pairwise isomorphic simple groups S_i with i in $\{1, \ldots, n\}$ and integer n. Put $\chi = \gamma^G$.

Step 1. The irreducible character χ must be faithful.

The kernel of χ is equal to $U = \cap_{x \in G}(\ker(\gamma))^x$ by [I, Lemma 5.11] which is a normal subgroup of G contained in H. Thus U is nilpotent. Now γ may be viewed as a character of H/U which induces an irreducible character of G/U. Since QU/U is a component of G/U, it easily follows that G/U is a counterexample to the conjecture with order at most $|G|$. This can only happen if $U = 1$.

Step 2. If $A \lhd G$ is nilpotent, then $A \leq H$ and χ_A is a multiple of some $\tau \in \mathrm{Irr}(A)$. In particular, if $\tau(1) = 1$, then $A \subseteq \mathbf{Z}(G)$ is cyclic. Hence, every normal abelian subgroup of G is cyclic and central.

By Theorem 5.1, we have that $A \leq H$ by using that H is a maximal subgroup of G subject to being nilpotent. Let $\tau \in \mathrm{Irr}(A)$ be an irreducible constituent of γ_A. Hence τ also lies under χ. Let T be the stabilizer of τ in G. Since $[E, \mathbf{F}(G)] = 1$ and $A \leq \mathbf{F}(G)$, we have that $E \subseteq T$. Now, if $\epsilon \in \mathrm{Irr}(T \cap H|\tau)$ is the Clifford correspondent of γ over τ, it follows that $\epsilon^G = \gamma^G = \chi$ is irreducible. Hence, ϵ^T is also irreducible. Since X is a component of T, we would have a contradiction in case $T < G$. Thus $T = G$, which means that χ_A is a multiple of τ by Clifford theory. Since χ is faithful by Step 1, so is τ. If τ is linear, then $A \leq \mathbf{Z}(G)$ since τ and A are G-invariant. It also follows that A is cyclic.

Step 3. We have $\mathbf{F}(G) = C_G(E/\mathbf{Z}(E))$ and $\mathbf{F}(G) \cap E = \mathbf{Z}(E)$.

Let M be the largest normal solvable subgroup of G. Since $\mathbf{Z}(E) \leq M$ and $E/\mathbf{Z}(E)$ is a direct product of non-abelian simple groups, it follows that $M \cap E = \mathbf{Z}(E)$. The group $M/\mathbf{Z}(E)$ is isomorphic to ME/E which is nilpotent (because it is isomorphic to a subgroup of $H/(H \cap E)$). Since $\mathbf{Z}(E)$ is contained in $\mathbf{Z}(G)$ by Step 2, it follows that M is nilpotent. Now, it is clear that $M \leq \mathbf{F}(G) \leq C_G(E)$. Since $C_G(E)/\mathbf{Z}(E)$ is isomorphic to a subgroup of the nilpotent group G/E, so $C_G(E)$ is a normal solvable subgroup of G, which proves that $C_G(E) \leq M$. Thus, $M = \mathbf{F}(G) = C_G(E)$ follows. Finally, $C_G(E) \leq C_G(E/\mathbf{Z}(E))$ is clear, while if $g \in C_G(E/\mathbf{Z}(E))$, then $[x,y]^g = [x^g, y^g] = [x,y]$ holds for every $x, y \in E$, so $g \in C_G(E)$ by using that E is perfect.

Step 4. $|H/\mathbf{F}(G)|^2 \geq |G/\mathbf{F}(G)|$.

Write $F = \mathbf{F}(G)$. By Step 2, we know that χ_F is a multiple of $\tau \in \mathrm{Irr}(F)$. Now we use the theory of character triples, as developed in Section 5.4 of [N$_2$], and with the same notation. Let (G^*, F^*, τ^*) be a character triple isomorphic to (G, F, τ), where F^* is central, and τ^* is faithful. Let $(H/F)^* = H^*/F^*$. If $\gamma^* \in \mathrm{Irr}(H^*|\tau^*)$ corresponds to γ, then we have that $(\gamma^*)^{G^*} = \chi^*$ is irreducible. (See the discussion before Lemma 5.8 of [N$_2$].) Using that $|G^* : F^*| = |G : F|$ and $|H^* : F^*| = |H : F|$ we get that

$$|G:F| = |G^*:F^*| \geq \chi^*(1)^2 = \gamma^*(1)^2 \cdot |G^*:H^*|^2 = \gamma^*(1)^2 \cdot \frac{|G^*:F^*|^2}{|H^*:F^*|^2} \geq \frac{|G:F|^2}{|H:F|^2},$$

where the first inequality follows from [I, Lemma 2.27(f) and Corollary 2.30] and from the fact that F^* is central in G^*.

Step 5. We have that $n > 1$ and $E \cap H \neq \mathbf{Z}(E)$.

If $n = 1$, then $G/\mathbf{F}(G)$ is almost-simple and $|H/\mathbf{F}(G)|^2 < |G/\mathbf{F}(G)|$ by Theorem C contradicting Step 4.

Suppose now that $E \cap H = \mathbf{Z}(E)$, so $H \cap (\mathbf{F}(G)E) = \mathbf{F}(G)$ and $E/Z(E) \cong \mathbf{F}(G)E/\mathbf{F}(G)$. By Step 3, the kernel of the action of H on $E/Z(E) \simeq S_1 \times \ldots \times S_n$ equals $\mathbf{F}(G)$, so this induces an inclusion of the nilpotent group $H/\mathbf{F}(G)$ into $W = \mathrm{Out}(S_1) \wr \mathrm{Sym}(n)$. If ψ denotes the natural map from W to $W/(\mathrm{Out}(S_1))^n$, then $\psi(H/\mathbf{F}(G))$ may be viewed as a nilpotent subgroup of $\mathrm{Sym}(n)$. We have $|\psi(H/\mathbf{F}(G))| \leq 2^{n-1}$ by [D, Theorem 3]. Thus $|H/\mathbf{F}(G)| \leq |\mathrm{Out}(S_1)|^n \cdot 2^{n-1}$. By a remark after [GMP, Lemma 7.7], it follows that $|\mathrm{Out}(S_1)| < |S_1|^{1/2}/2$. We conclude that

$$|H/\mathbf{F}(G)|^2 < |\mathrm{Out}(S_1)|^{2n} \cdot 2^{2n} \leq |S_1|^n = |\mathbf{F}(G)E/\mathbf{F}(G)| \leq |G/\mathbf{F}(G)|.$$

Again, this contradicts the bound $|H/\mathbf{F}(G)|^2 \geq |G/\mathbf{F}(G)|$ obtained in Step 4, thus $n > 1$ and $E \cap H \neq \mathbf{Z}(E)$ as claimed.

Step 6. We have $(E \cap H)/\mathbf{Z}(E) = L_1 \times \cdots \times L_n$ for some pairwise isomorphic nilpotent subgroups $L_i < S_i$ with i in $\{1, \ldots, n\}$, which are transitively permuted by H.

Let L_i be the projection of $(E \cap H)/\mathbf{Z}(E)$ to S_i for $1 \leq i \leq n$. Since $H/\mathbf{Z}(E)$ acts transitively on the set $\{S_1, \ldots, S_n\}$ by conjugation, it also acts transitively on the set $\{L_1, \ldots, L_n\}$ by conjugation. Thus the groups L_i are pairwise isomorphic and nilpotent for i in $\{1, \ldots, n\}$.

Let K be the preimage of $L_1 \times \cdots \times L_n$ in E. Clearly, K is a nilpotent subgroup of E. We claim that $K = E \cap H$. By Theorem 5.1 and the maximality of H, it is sufficient to show that K is normalized by H. For this it is sufficient to see that the preimage K_1 of L_1 in E has the property that $K_1^h \leq K$ for every $h \in H$. But this is clear.

The group $H/\mathbf{Z}(E)$ acts by conjugation on $E/\mathbf{Z}(E)$. As noted in the proof of Step 6, this action induces a transitive action on the set $\Omega = \{S_1, \dots, S_n\}$ of simple direct factors of $E/\mathbf{Z}(E)$. Let $B/\mathbf{Z}(E)$ be the kernel of this action. Thus B is a normal subgroup of H containing $\mathbf{Z}(E)$ with the property that H/B can be considered as a transitive subgroup of $\mathrm{Sym}(\Omega)$.

Step 7. Both $(E \cap H)/\mathbf{Z}(E)$ and H/B are p-groups for the same prime p.

By Step 5 we know that $n > 1$ and $L_1 \neq 1$. Consider H/B as a transitive subgroup of $\mathrm{Sym}(\Omega)$. By our assumptions, H/B is a non-trivial nilpotent group. Let p be a prime divisor of $|H/B|$. Since no non-trivial normal subgroup of H/B can stabilize $S_1 \in \Omega$, the Sylow subgroup $\mathbf{O}_p(H/B)$ of H/B cannot stabilize S_1. It follows that neither the Sylow subgroup $\mathbf{O}_p(H/\mathbf{Z}(E))$ of $H/\mathbf{Z}(E)$ can stabilize S_1. Let x be a p-element in $H/\mathbf{Z}(E)$ which does not stabilize S_1. Then x cannot stabilize $L_1 \leq (E \cap H)/\mathbf{Z}(E)$ either. By Step 6 and its proof, we know that $L_1 \cap (L_1)^x = 1$. Since $H/\mathbf{Z}(E)$ is nilpotent, this can only occur if L_1 is a p-group. Since L_1 is a non-trivial p-group where p is an arbitrary prime divisor of $|H/B|$, we conclude that both L_1 and H/B are (non-trivial) p-groups for the same prime p. The result now follows by Step 6.

Let T denote the preimage in G of the kernel of the action of $G/\mathbf{Z}(E)$ on Ω. So $H \cap T = B$ and $T = BE$. Recall that $\mathrm{LIST} = \{\mathrm{Alt}(5), \mathrm{Alt}(6), \mathrm{PSL}(2,7), \mathrm{PSL}(3,4), \mathrm{PSU}(4,3)\}$.

Define $c = 1$ if $S_1 \in \mathrm{LIST}$, L_1 is a Sylow p-subgroup of S_1 and $p = 2$ in case $S_1 = \mathrm{Alt}(5)$ or $\mathrm{Alt}(6)$. Otherwise, define $c = 2$.

Step 8. $|B/\mathbf{F}(G)|^2 \leq c^{-n} \cdot |T/\mathbf{F}(G)|$.

By Step 3, the group $T/\mathbf{F}(G)$ is isomorphic to a subgroup $\overline{T}$ of $\mathrm{Aut}(S_1) \times \cdots \times \mathrm{Aut}(S_n)$ containing the normal subgroup $S_1 \times \cdots \times S_n$ and the group $B/\mathbf{F}(G)$ may be viewed as a nilpotent subgroup $\overline{B}$ of $\overline{T}$. We show the claim by induction on n. First, for $n = 1$ the claim follows from Theorem C if $c = 1$, while it follows from Theorem 2.1 if $c = 2$. Now, let us assume that $n > 1$ and that the claim is true for $n - 1$. Let π be the natural projection of $\overline{T}$ to $\mathrm{Aut}(S_1)$. Then $c \cdot |\pi(\overline{B})|^2 \leq |\pi(\overline{T})|$. Moreover, $|\ker(\pi) \cap \overline{B}|^2 \leq c^{-n+1} \cdot |\ker(\pi) \cap \overline{T}|$ by the fact that the claim is true for $n - 1$. Thus

$$\begin{aligned} |B/\mathbf{F}(G)|^2 = |\overline{B}|^2 = |\pi(\overline{B})|^2 \cdot |\ker(\pi) \cap \overline{B}|^2 \leq \\ \leq c^{-n} \cdot |\pi(\overline{T})| \cdot |\ker(\pi) \cap \overline{T}| = c^{-n} \cdot |\overline{T}| = c^{-n} \cdot |T/\mathbf{F}(G)|. \end{aligned}$$

Step 9. $S_1 \in \mathrm{LIST}$ and L_1 is a Sylow p-subgroup of S_1. Moreover, $p = 2$ in case S_1 is $\mathrm{Alt}(5)$ or $\mathrm{Alt}(6)$.

By Steps 5 and 7, we know that $n > 1$ and $L_1 \neq 1$ is a p-group for some prime p. By the definition of c, we need to prove that $c = 1$. We have $|B/\mathbf{F}(G)|^2 \leq c^{-n} \cdot |T/\mathbf{F}(G)|$ by Step 8. Thus

$$|H/\mathbf{F}(G)|^2 = |H/B|^2 \cdot |B/\mathbf{F}(G)|^2 \leq c^{-n} \cdot |H/B|^2 \cdot |T/\mathbf{F}(G)|. \tag{5.1}$$

Since $G = HT$ and $H \cap T = B$, we have

$$|H/B| \cdot |T/\mathbf{F}(G)| = \frac{|H||T|}{|B||\mathbf{F}(G)|} = \frac{|HT|}{|\mathbf{F}(G)|} = |G/\mathbf{F}(G)|. \tag{5.2}$$

Inequalities (5.1) and (5.2) give

$$|H/\mathbf{F}(G)|^2 \leq c^{-n} \cdot |H/B| \cdot |G/\mathbf{F}(G)| \leq c^{-n} \cdot 2^{n-1} \cdot |G/\mathbf{F}(G)| \tag{5.3}$$

where the second bound follows from Step 7, noting that H/B is a p-subgroup of the symmetric group on n letters and thus it has size at most 2^{n-1}. (It is a well-known fact that the p-part of $n!$ is at most this number.) Now Step 4 and (5.3) give

$$|G/\mathbf{F}(G)| \leq |H/\mathbf{F}(G)|^2 \leq c^{-n} \cdot 2^{n-1} \cdot |G/\mathbf{F}(G)|$$

from which $1 \leq c^{-n} \cdot 2^{n-1}$ follows, forcing $c = 1$.

Step 10. The character $\gamma_{H\cap E}$ is irreducible.

Let U be any proper subgroup of H containing $H \cap E$. We claim that $\mu^H \neq \gamma$ for every irreducible character μ of U. For a contradiction assume that $\mu^H = \gamma$ for some $\mu \in \mathrm{Irr}(U)$. Since γ^G is irreducible, μ^{EU} is also irreducible. Since $|EU : U| = |G : H|$ and $|EU| < |G|$, by induction we will have that $\mathbf{F}^*(EU)$ is nilpotent, but this cannot happen.

Let $H = U_0 > \ldots > U_t = H \cap E$ be a chain of normal subgroup of H with $t \geq 1$ maximal. By repeated application of [I, Theorem 6.18] and the claim in the previous paragraph, the character γ_{U_i} is homogeneous for every index i with $0 \leq i \leq t$. Moreover, since H is nilpotent and t is maximal, $|U_i/U_{i+1}|$ is prime for every index i with $0 \leq i \leq t-1$, and so γ_{U_i} is irreducible for every index i with $0 \leq i \leq t$. In particular $\gamma_{H\cap E}$ is irreducible.

As in the proof of Step 6, for every i with $1 \leq i \leq n$, let $K_i \leq H \cap E$ be the preimage of $L_i < S_i \leq E/\mathbf{Z}(E)$ in E. In particular $L_i \cong K_i/\mathbf{Z}(E)$.

Step 11. $H \cap E$ is a central product of the subgroups $K_1, \ldots, K_n$ amalgamating $\mathbf{Z}(E)$.

For each i with $1 \leq i \leq n$, the group K_i contains $\mathbf{Z}(E)$. By Step 6, $H \cap E = \langle K_1, \ldots, K_n \rangle$. Since every distinct pair of components of G commute, we have $[K_i, K_j] = 1$ for every i and j with $1 \leq i < j \leq n$. Finally, for every index i with $1 \leq i \leq n$, the intersection of K_i with $\langle K_1, \ldots, K_{i-1}, K_{i+1}, \ldots K_n \rangle$ is $\mathbf{Z}(E)$.

Final Step.

By Step 10, $\gamma_{H\cap E} \in \mathrm{Irr}(H \cap E)$, so $\gamma_{H\cap E} = \gamma_1 \cdot \ldots \cdot \gamma_n$ for $\gamma_i \in \mathrm{Irr}(K_i|\lambda)$, where $\chi_{\mathbf{Z}(E)} = \chi(1)\lambda$ by Step 11 and by Section 4.

For every i with $1 \leq i \leq n$, let X_i be the preimage of $S_i \leq E/\mathbf{Z}(E)$ in E, so E is the central product of $X_1, \ldots, X_n$. By Mackey and Clifford's theorem, we have that all irreducible constituents of

$$\chi_E = (\gamma_{H\cap E})^E = (\gamma_1 \cdot \ldots \cdot \gamma_n)^E$$

have equal degrees. By Lemma 4.1, this character equals

$$(\gamma_1)^{X_1} \cdot \ldots \cdot (\gamma_n)^{X_n}.$$

For $k > 1$, fix an irreducible constituent ρ_k of $(\gamma_k)^{X_k}$. By Theorem 3.1, let ξ_1 and ξ_2 be irreducible constituents of $(\gamma_1)^{X_1}$ with different degrees. Then $\xi_i \cdot \rho_2 \cdot \ldots \cdot \rho_n$ with $i \in \{1, 2\}$ are two irreducible constituents of χ_E with different degrees. This contradiction proves the theorem. □

Notice that Theorem A easily follows from Theorem 5.1 and Theorem 5.2.

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