AVERAGE DIMENSION OF FIXED POINT SPACES WITH APPLICATIONS

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Abstract. Let G be a finite group, F a field, and V a finite dimensional FG-module such that G has no trivial composition factor on V. Then the arithmetic average dimension of the fixed point spaces of elements of G on V is at most $(1/p)\dim V$ where p is the smallest prime divisor of the order of G. This answers and generalizes a 1966 conjecture of Neumann which also appeared in a paper of Neumann and Vaughan-Lee and also as a problem in The Kourovka Notebook posted by Vaughan-Lee. Our result also generalizes a recent theorem of Isaacs, Keller, Meierfrankenfeld, and Moretó. We also classify precisely when equality can occur. Various applications are given. For example, another conjecture of Neumann and Vaughan-Lee is proven and some results of Segal and Shalev are improved and/or generalized concerning BFC groups.

Dedicated to Peter M. Neumann on the occasion of his 70th birthday.

1. Introduction

Let G be a finite group, F a field, and V a finite dimensional FG-module. For a non-empty subset S of G we define

$$\operatorname{avgdim}(S,V) = \frac{1}{|S|}\sum_{s\in S}\dim C_V(s)$$

to be the arithmetic average dimension of the fixed point spaces of all elements of S on V. Here $C_V(s)$ is the set of fixed points of s on V. In his 1966 DPhil thesis Neumann [12] conjectured that if V is an irreducible non-trivial FG-module then $\operatorname{avgdim}(G,V) \le (1/2)\dim V$. This problem was restated in 1977 by Neumann and Vaughan-Lee [13] and was posted in 1982 by Vaughan-Lee in The Kourovka Notebook [9] as Problem 8.5. The conjecture was proved by Neumann and Vaughan-Lee [13] for solvable groups G and also in the case when $|G|$ is invertible in F. Later Segal and Shalev [17] showed that $\operatorname{avgdim}(G,V) \le (3/4)\dim V$ for an arbitrary finite group G. This was improved by Isaacs, Keller, Meierfrankenfeld, and Moretó

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[8] to $\text{avgdim}(G, V) \leq ((p+1)/2p) \dim V$ where p is the smallest prime factor of $|G|$. Our first main theorem is

Theorem 1.1. *Let G be a finite group, F a field, and V a finite dimensional FG-module. Let N be a normal subgroup of G that has no trivial composition factor on V. Then* $\text{avgdim}(Ng, V) \leq (1/p) \dim V$ *for every $g \in G$ where p is the smallest prime factor of the order of G.*

Theorem 1.1 not only solves the above-mentioned conjecture of Neumann and Vaughan-Lee but it also generalizes and improves the resul in many ways. First of all, G need not be irreducible on V; the only restriction we impose is that G has no trivial composition factor on V. Secondly, we prove the bound $(1/2) \dim V$ not just for $\text{avgdim}(G, V)$ but for $\text{avgdim}(S, V)$ where S is an arbitrary coset of a normal subgroup of G with a certain property. Thirdly, Theorem 1.1 involves a better general bound, namely $(1/p) \dim V$ where p is the smallest prime divisor of $|G|$.

We next turn to the question of when we can have equality in Theorem 1.1. Note that the example [8, Page 3129] of a completely reducible FG-module V for an elementary abelian p-group G shows that $\text{avgdim}(G, V) = (1/p) \dim V$ can occur in Theorem 1.1. There are examples for equality in Theorem 1.1 even when V is an irreducible module. Let p be an arbitrary odd prime, let G be the extraspecial p-group of order p^{1+2a} (for a positive integer a) of exponent p, let $N = Z(G)$, let F be an algebraically closed field of characteristic different from p, and let V be an irreducible FG-module of dimension p^a. Then for every element $x \in G \setminus N$ we have $\dim C_V(x) = (1/p) \dim V$ and so $\text{avgdim}(Ng, V) = (1/p) \dim V$ for every $g \in G$. In particular we have $\text{avgdim}(H, V) = (1/p) \dim V$ for every subgroup H of G containing N.

We give a different proof of Theorem 1.1 in characteristic 0 and combine the ideas of that proof with Theorem 1.1 to show:

Theorem 1.2. *Let G be a finite group, F a field, and V a finite dimensional FG-module with no trivial composition factors. Let p be the smallest prime factor of $|G|$. Then* $\text{avgdim}(G, V) = (1/p) \dim V$ *if and only if $G/C_G(V)$ is a group of exponent p.*

In his DPhil thesis [12] Neumann showed that if V is a non-trivial irreducible FG-module for a field F and a finite solvable group G then there exists an element of G with small fixed point space. Specifically, he showed that there exists $g \in G$ with $\dim C_V(g) \leq (7/18) \dim V$. Neumann conjectured that in fact, there should exists $g \in G$ such that $\dim C_V(g) \leq (1/3) \dim V$. Segal and Shalev [17] proved, for an arbitrary finite group G, that there exists an element $g \in G$ with $\dim C_V(g) \leq (1/2) \dim V$. Later, under milder conditions (V is a completely reducible FG-module with $C_V(G) = 0$), Isaacs, Keller, Meierfrankenfeld, and Moretó [8] showed that there exists an element $g \in G$ with $\dim C_V(g) \leq (1/p) \dim(V)$ where p is the smallest prime divisor of $|G|$. Under even weaker conditions we improve this latter result.

Corollary 1.3. *Let G be a finite group, F a field, and V a finite dimensional FG-module. Let N be a normal subgroup of G that has no trivial composition factor on*

V. Let x be an element of G and let p be the smallest prime factor of the order of G. Then there exists an element $g \in Nx$ with $\dim C_V(g) \leq (1/p) \dim V$ and there exists an element $g \in N$ with $\dim C_V(g) < (1/p) \dim V$.

Note that Corollary 1.3 follows directly from Theorem 1.1 just by noticing that $\dim C_V(1) = \dim V$. Note also that if V is irreducible and faithful in Corollary 1.3 then no non-trivial normal subgroup of G has a non-zero fixed point on V and so the N above can be any non-trivial normal subgroup of G. During the last stage of the writing of this paper Neumann's above-mentioned conjecture was proved in [6]; if V is a non-trivial irreducible FG-module for a finite group G then there exists an element $g \in G$ such that $\dim C_V(g) \leq (1/3) \dim V$.

Let $\mathrm{cl}_G(g)$ denote the conjugacy class of an element g in a finite group G, and for a positive integer n and a prime p let n_p denote the p-part of n. In [8] Isaacs, Keller, Meierfrankenfeld, and Moretó conjecture that for any primitive complex irreducible character χ of a finite group G the degree of χ divides $|\mathrm{cl}_G(g)|$ for some element g of G. Using their result mentioned before the statement of Corollary 1.3 they showed that if χ is an arbitrary primitive complex irreducible character of a finite solvable group G and p is a prime divisor of $|G|$ then $\chi(1)_p$ divides $(|\mathrm{cl}_G(g)|)^3$ for some $g \in G$. Using Theorem 1.1 we may prove more than this.

Corollary 1.4. *Let χ be an arbitrary primitive complex irreducible character of a finite solvable group G and let p be a prime divisor of $|G|$. Then the number of $g \in G$ for which $\chi(1)_p$ divides $(|\mathrm{cl}_G(g)|)^3$ is at least $(2|G|)/(1+k)$ where $k = \log_p |G|_p$. Furthermore if $\chi(1)_p > 1$ then there exists a p'-element $g \in G$ for which $p^3 \cdot \chi(1)_p$ divides $(|\mathrm{cl}_G(g)|)^3$.*

Recall that a chief factor of a finite group is a section X/Y of G with $Y < X$ both normal in G such that there is no normal subgroup of G strictly between X and Y. Note that X/Y is a direct product of isomorphic simple groups. If X/Y is abelian, then it is an irreducible G-module. If X/Y is non-abelian, then G permutes the direct factors transitively. A chief factor is called central if G acts trivially on X/Y and non-central otherwise. Let G be a finite group acting on another finite group Z by conjugation. For a non-empty subset S of G define

$$\mathrm{geom}(S,Z) = \left(\prod_{s \in S} |C_Z(s)| \right)^{1/|S|}$$

to be the geometric mean of the sizes of the centralizers of elements of S acting on Z. Similarly, for a non-empty subset S of G define

$$\mathrm{avg}(S,Z) = \frac{1}{|S|} \sum_{s \in S} |C_Z(s)|$$

to be the arithmetic mean of the sizes of the centralizers of elements of S acting on Z. Our next result is a non-abelian version of Theorem 1.1 proved using some recent work of Fulman and the first author [5].

Theorem 1.5. *Let G be a finite group with $X/Y = M$ a non-abelian chief factor of G with X and Y normal subgroups in G. Then, for any $g \in G$, we find that $\mathrm{geom}(Xg, M) \leq \mathrm{avg}(Xg, M) \leq |M|^{.41}$.*

Let $\mathrm{ccf}(G)$ and $\mathrm{ncf}(G)$ be the product of the orders of all central and non-central chief factors (respectively) of a finite group G. (In case these are not defined put them equal to 1.) These invariants are independent of the choice of the chief series of G. Let $F(G)$ denote the Fitting subgroup of G. Note that $F(G)$ acts trivially on every chief factor of G. Using Theorems 1.1 and 1.5 we prove

Theorem 1.6. *Let G be a finite group. Then $\mathrm{geom}(G,G) \leq \mathrm{ccf}(G) \cdot (\mathrm{ncf}(G))^{1/p}$ where p is the smallest prime factor of the order of $G/F(G)$.*

By taking the reciprocals of both sides of the inequality of Theorem 1.6 and multiplying by $|G|$, we obtain the following result.

Corollary 1.7. *Let G be a finite group. Then $\mathrm{ncf}(G) \leq \left(\prod_{g \in G} |\mathrm{cl}_G(g)|\right)^{p/((p-1)|G|)}$ where p is the smallest prime factor of the order of $G/F(G)$.*

A group is said to be a BFC group if its conjugacy classes are finite and of bounded size. A group G is called an n-BFC group if it is a BFC group and the least upper bound for the sizes of the conjugacy classes of G is n. One of B. H. Neumann's discoveries was that in a BFC group the commutator subgroup G' is finite [11]. One of the purposes of this paper is to give an upper bound for $|G'|$ in terms of n for an n-BFC group G. Note that $C_G(G')$ is a finite index nilpotent subgroup. Thus, $F(G)$ is well defined for BFC groups.

If G is a BFC group, then there is a finitely generated subgroup H with $H' = G'$ and $G = HC_G(G') = HF(G)$. Then H has a finite index central torsionfree subgroup N. Set $J = H/N$. So J' and G' are G-isomorphic. In particular, $\mathrm{ncf}(J) = \mathrm{ncf}(G)$. Clearly, $G/F(G) \cong J/F(J)$. Thus, for the next result, it suffices to consider finite groups. Our first main theorem on BFC groups follows from Corollary 1.7 (by noticing that $|\mathrm{cl}_G(1)| = 1$ and that in that result, we may always assume the action is faithful).

Theorem 1.8. *Let G be an n-BFC group with $n > 1$. Then $\mathrm{ncf}(G) < n^{p/(p-1)} \leq n^2$, where p is the smallest prime factor of the order of $G/F(G)$.*

Theorem 1.8 solves [13, Conjecture A].

Not long after B. H. Neumann's proof that the commutator subgroup G' of a BFC group is finite, Wiegold [20] produced a bound for $|G'|$ for an n-BFC group G in terms of n and conjectured that $|G'| \leq n^{(1/2)(1+\log n)}$ where the logarithm is to base 2. Later Macdonald [10] showed that $|G'| \leq n^{6n(\log n)^3}$ and Vaughan-Lee [19] proved Wiegold's conjecture for nilpotent groups. For solvable groups the best bound to date is $|G'| \leq n^{(1/2)(5+\log n)}$ obtained by Neumann and Vaughan-Lee [13]. In the same paper they showed that for an arbitrary n-BFC group G we have $|G'| \leq n^{(1/2)(3+5\log n)}$. Using the Classification of Finite Simple Groups (CFSG) Cartwright [2] improved this bound to $|G'| \leq n^{(1/2)(41+\log n)}$ which was later further sharpened by Segal and Shalev [17] who obtained $|G'| \leq n^{(1/2)(13+\log n)}$. Applying Theorem 1.8 at the bottom of [17, Page 511] we arrive at a further improvement of the general bound on the order of the derived subgroup of an n-BFC group.

Theorem 1.9. *Let G be an n-BFC group with $n > 1$. Then $|G'| < n^{(1/2)(7+\log n)}$.*

A word ω is an element of a free group of finite rank. If the expression for ω involves k different indeterminates, then for every group G, we obtain a function from G^k to G by substituting group elements for the indeterminates. Thus we can consider the set G_ω of all values taken by this function. The subgroup generated by G_ω is called the verbal subgroup of ω in G and is denoted by $\omega(G)$. An outer commutator word is a word obtained by nesting commutators but using always different indeterminates. In [4] Fernández-Alcober and Morigi proved that if ω is an outer commutator word and G is any group with $|G_\omega| = m$ for some positive integer m then $|\omega(G)| \leq (m-1)^{m-1}$. They suspect that this bound can be improved to a bound close to one obtainable for the commutator word $\omega = [x_1, x_2]$. By noticing that every conjugacy class of a group G has size at most the number of commutators of G we see that Theorem 1.9 yields

Corollary 1.10. *Let G be a group with m commutators for some positive integer m at least 2. Then $|G'| < m^{(1/2)(7+\log m)}$.*

Segal and Shalev [17] showed that if G is an n-BFC group with no non-trivial abelian normal subgroup then $|G| < n^4$. We improve and generalize this result in Theorem 1.11. For a finite group X, let $k(X)$ denote the number of conjugacy classes of X.

Theorem 1.11. *Let G be an n-BFC group with $n > 1$. If the Fitting subgroup $F(G)$ of G is finite, then $|G| < n^2 k(F(G))$. In particular, if G has no non-trivial abelian normal subgroup then $|G| < n^2$.*

Since $F(G)$ has finite index in G, the hypotheses of Theorem 1.11 imply that G is finite. Note that even more is true than Theorem 1.11; if G is a finite group then $|G| \leq a^2 k(F(G))$ where $a = |G|/k(G)$ is the (arithmetic) average size of a conjugacy class in G (this is [7, Theorem 10 (i)]). If b denotes the maximal size of a set of pairwise non-commuting elements in G then, by Turán's theorem [18] applied to the complement of the commuting graph of G, we have $a < b+1$. Thus if G is a finite group with no non-trivial abelian normal subgroup then $|G| < (b+1)^2$. This should be compared with the bound $|G| < c^{(\log b)^3}$ holding for some universal constant c with $c \geq 2^{20}$ which implicitly follows from [15, Lemma 3.3 (ii)] and should also be compared with the remark in [15, Page 294] that for a non-abelian finite simple group G we have $|G| \leq 27 \cdot b^3$.

The final main result concerns n-BFC groups with a given number of generators. Segal and Shalev [17] proved that in such groups the order of the commutator subgroup is bounded by a polynomial function of n. In particular they obtained the bound $|G'| \leq n^{5d+4}$ for an arbitrary n-BFC group G that can be generated by d elements. By applying Theorem 1.8 to [17, Page 515] we may improve this result.

Corollary 1.12. *Let G be an n-BFC group that can be generated by d elements. Then $|G'| \leq n^{3d+2}$.*

Finally, the following immediate consequence of Corollary 1.12 sharpens [17, Corollary 1.5].

Corollary 1.13. *Let G be a d-generator group. Then*

$$|\{[x,y] : x, y \in G\}| \geq |G'|^{1/(3d+2)}.$$

The example $T_m(p)$ [13, Page 213] shows that Theorem 1.9, Corollary 1.10, Corollary 1.12, and Corollary 1.13 are close to best possible.

We point out that Theorem 1.1 for p odd requires only the Feit-Thompson Odd Order Theorem [3]. However, most of the results in this paper depend on CFSG as do the results in [17] and [8] (for groups of even order).

2. Proof of Theorem 1.1

Our first lemma sharpens and generalizes [13, Theorem 6.1].

Lemma 2.1. *Let G be a finite group, F a field, and V a finite dimensional FG-module. Let N be an elementary abelian normal subgroup of G such that $C_V(N) = 0$. Then $\mathrm{avgdim}(Ng, V) \leq (1/p) \dim V$ for every $g \in G$ where p is the smallest prime factor of the order of G.*

Proof. We may assume that F is algebraically closed. Let us consider a counterexample with $|G|$ and $\dim V$ minimal. It clearly suffices to assume that $G = \langle g, N\rangle$. We may assume that V is irreducible (since if we have the inequality on each composition factor of V we have it on V). Finally, we may assume that N acts faithfully on V. If N does not act homogeneously on V, then g transitively permutes the components in an orbit of size $t \geq p$ and so every element in Ng has a fixed point space of dimension at most $(1/t) \dim V \leq (1/p) \dim V$. So we may assume that the elementary abelian group N acts homogeneously on V. This means that it acts as scalars on V. Thus $N \leq Z(G)$ and $G/Z(G)$ is cyclic. It follows that G is abelian and so $\dim V = 1$. At most 1 element in the coset Ng is the identity and so $\mathrm{avgdim}(Ng, V) \leq (1/|N|) \dim V \leq (1/p) \dim V$. The result follows. □

We first need a result about generation of finite groups. This is an easy consequence of the proof of the main results of [1].

Theorem 2.2. *Let G be a finite group with a minimal normal subgroup $N = L_1 \times \ldots \times L_t$ for some positive integer t with $L_i \cong L$ for all i with $1 \leq i \leq t$ for a non-abelian simple group L. Assume that $G/N = \langle xN\rangle$ for some $x \in G$. Then there exists an element $s \in L_1 \leq N$ such that $|\{g \in Nx : G = \langle g, s\rangle\}| > (1/2)|N|$.*

Proof. First suppose that $t = 1$. This is an immediate consequence of [1, Theorem 1.4] unless G is one of $Sp(2n, 2), n > 2$, S_{2m+1} or $L = \Omega^+(8, 2)$ or A_6.

If $G = Sp(2n, 2), n > 2$, then the result follows by [1, Proposition 5.8]. If $G = S_{2m+1}$, then apply [1, Proposition 6.8].

Suppose that $L = A_6$. Note that the proper overgroups of s of order 5 in A_6 are two subgroups isomorphic to A_5 (of different conjugacy classes) and the normalizer of the subgroup generated by s. The result follows trivially from this observation.

Finally consider $L = \Omega^+(8, 2)$. We take s to be an element of order 15. It follows by the discussion in [1, Section 4.1] that given G, there is an element of order 15 satisfying the result (although it is possible that the choice of s depends on which G occurs).

Now assume that $t > 1$. Write $x = (u_1, \dots, u_t)\sigma$ where σ just cyclically permutes the coordinates of N (sending L_i to L_{i+1} for $i < t$) and $u_i \in \mathrm{Aut}(L_i)$. By conjugating by an element of the group $\mathrm{Aut}(L_1) \times \dots \times \mathrm{Aut}(L_t)$ we may assume that $u_2 = \dots = u_t = 1$ (we do not need to do this but it just makes the computations easier).

Let $f : Nx \to \mathrm{Aut}(L_1)$ be the map sending wx to the projection of $(wx)^t$ in $\mathrm{Aut}(L_1)$. Write $w = (w_1, \dots, w_t)$ with $w_i \in L_i$. Then $f(wx) = w_t w_{t-1} \dots w_1 u_1$ is in $L_1 u_1$. Moreover, we see that every fiber of f has the same size. By the case $t = 1$, we know that the probability that $\langle f(wx), s\rangle = \langle L_1, u_1\rangle$ is greater than $1/2$.

We claim that if $L_1 \le \langle f(wx), s\rangle$, then $G = \langle wx, s\rangle$. The claim then implies the result. So assume that $L_1 \le \langle (f(wx), s\rangle$ and set $H = \langle wx, s\rangle$. Let $M \le N$ be the normal closure of s in $J := \langle (wx)^t, s\rangle$. This projects onto L_1 by assumption, but is also contained in L_1, whence $M = L_1$. So $L_1 \le H$. Since any element of Nx acts transitively on the L_i, it follows that $N \le H$ and so $G = H$. □

The next result we need is Scott's Lemma [16].

Lemma 2.3 (Scott's Lemma)**.** *Let G be a subgroup of $\mathrm{GL}(V)$ with V a finite dimensional vector space. Suppose that $G = \langle g_1, \dots, g_r\rangle$ with $g_1 \cdots g_r = 1$. Then*

$$\sum_{i=1}^{r} \dim[g_i, V] \ge \dim V + \dim[G, V] - \dim C_V(G).$$

We will apply this in the case $r = 3$. Noting that $\dim V = \dim[x, V] + \dim C_V(x)$ for any x, we can restate this as:

$$\sum_{i=1}^{3} \dim C_V(g_i) \le \dim V + \dim C_V(G) + \dim V/[G, V].$$

Theorem 2.4. *Let G be a finite group. Assume that G has a normal subgroup E that is a central product of quasisimple groups. Let V be a finite dimensional FG-module for some field F such that E has no trivial composition factor on V. If $g \in G$, then* $\mathrm{avgdim}(gE, V) \le (1/2)\dim V$.

Proof. Let us consider a counterexample with $|G|$ and $\dim V$ minimal. There is no loss of generality in assuming that F is algebraically closed, $G = \langle E, g\rangle$, and then assuming that V is an irreducible (hence absolutely irreducible) and faithful FG-module. If $Z(E) \ne 1$, the result follows by Lemma 2.1 (by taking $N = Z(E)$ and noting that $Z(E)$ is completely reducible on V with $C_V(Z(E)) = 0$ (since V is a faithful FG-module)). So we may assume that E is a direct product of non-abelian simple groups. If V is not a homogeneous FE-module, then g transitively permutes the homogeneous components and so any element in gE has fixed point space of dimension at most $(1/2)\dim V$. So we may assume that V is a homogeneous FE-module. Thus $E = L_1 \times \dots \times L_m$ with the L_i's non-abelian simple groups. So V is a direct sum of say t copies of $V_1 \otimes \dots \otimes V_m$ where V_i is an irreducible nontrivial FL_i-module. (Since G/E is cyclic and V is irreducible, it follows that $t = 1$ (by Clifford theory) but we will not use this fact.) We may replace E by a minimal normal subgroup of G contained in E (the hypothesis on the minimal normal subgroup will hold by Clifford's theorem) and so assume that g transitively permutes the isomorphic subgroups $L_1, \dots, L_m$.

Let $s \in L_1 \leq E$ be chosen so that $Y := \{y \in gE : \langle y, s\rangle = G\}$ has size larger than $(1/2)|E|$. Such an element exists by Theorem 2.2. Set $c = \dim C_V(s)$. If $y \in Y$ then, by Lemma 2.3 (applied to the triple $(y, s, (ys)^{-1})$), we have

$$c + \dim C_V(y) + \dim C_V(ys) \leq \dim V.$$

For any $y \in Y' := gE \setminus Y$, we at least have

$$\dim C_V(y) + \dim C_V(ys) \leq \dim V + c.$$

Thus,

$$2 \sum_{y \in gE} \dim C_V(y) = \sum_{y \in gE} \Big(\dim C_V(y) + \dim C_V(ys) \Big)$$

is at most

$$|Y|(\dim V - c) + |Y'|(\dim V + c) < |E| \dim V.$$

This gives $\mathrm{avgdim}(gE) \leq (1/2) \dim V$ as required. □

We now prove Theorem 1.1. As usual, we may assume that F is algebraically closed, V is an irreducible FG-module, and N acts faithfully on V. Let A be a minimal normal subgroup of G contained in N. Since V is a faithful completely reducible FN-module, A has no trivial composition factor on V. Now apply Lemma 2.1 and Theorem 2.4 to conclude that $\mathrm{avgdim}(Ag, V) \leq (1/p) \dim V$ where p is the smallest prime divisor of $|G|$. Since Ng is the union of cosets of A, the result follows.

3. Proof of Theorem 1.2

We first consider fields of characteristic 0.

Lemma 3.1. *Let G be a finite group, $\mathbb{C}$ the field of complex numbers, and V a finite dimensional $\mathbb{C}G$-module. For an element $g \in G$ and a root of unity $a \in \mathbb{C}$ let a_g denote the multiplicity of a as an eigenvalue of g. Then $\sum_{g \in G} a_g = \sum_{g \in G} b_g$ as long as a and b have the same order in $\mathbb{C}^*$.*

Proof. Let a and b be roots of unity of the same order. Let m be the exponent of G with μ a primitive m-th root of unity. Let σ be an element of the automorphism group of the field $\mathbb{Q}(\mu)$ with $\sigma(a) = b$. Let e be a positive integer such that $\sigma(\mu) = \mu^e$. Then e is relatively prime to m and hence also to $|G|$. Thus, the map $G \to G$ with $g \mapsto g^e$ is a bijection on G and so $\sum_{g \in G} b_g = \sum_{g \in G} b_{g^e} = \sum_{g \in G} a_g$, whence the result. □

The Möbius function $\mu(n)$ of a positive integer n is 0 if n is not square free and is $(-1)^m$ if n is square free and the number of (distinct) prime divisors of n is m. For a positive integer n let $s(n)$ be the sum of primitive nth roots of unity (in $\mathbb{C}$). We recall the following well known result.

Lemma 3.2. *For a positive integer n we have $s(n) = \mu(n)$.*

Proposition 3.3. *Let G be a finite group, let F be a field such that $|G|$ is invertible in F, let V be a finite dimensional FG-module with no trivial FG-composition factor, and let p be the smallest prime divisor of the order of $G/C_G(V)$. Then* $\operatorname{avgdim}(G,V) \leq (1/p)\dim V$ *with equality if and only if the exponent of $G/C_G(V)$ is p.*

Proof. By $\operatorname{avgdim}(G,V) = \operatorname{avgdim}(G/C_G(V),V)$ we see that it is sufficient to assume that $C_G(V) = 1$. Since $|G|$ is invertible, there is no loss in assuming that $\operatorname{char}(F) = 0$.

Let χ be the character of the FG-module V. Then, by hypothesis, $\langle 1_G, \chi\rangle = 0$, that is, $\sum_{g\in G}\chi(g) = 0$. Let $n_1, n_2, \ldots, n_m$ be the possible distinct orders of elements of G with $n_1 = 1$ and $n_2 = p$. Since $\chi(g)$ is the sum of the eigenvalues of the matrix of g acting on V, Lemma 3.1 shows that there exist positive integers $k_1, k_2, \ldots, k_m$ with

$$0 = \sum_{g\in G}\chi(g) = \sum_{i=1}^{m} k_i s(n_i).$$

Letting $\varphi(n)$ denote the Euler function of n, we may write the previous equation as

$$0 = \sum_{i=1}^{m}(k_i\varphi(n_i))(s(n_i)/\varphi(n_i)) \geq k_1 - (|G|\dim V - k_1)(1/(p-1))$$

since $s(n_i)/\varphi(n_i) > (-1)/(p-1)$ for all i with $2 < i \leq m$. This gives $k_1 \leq (1/p)|G|\dim V$ with equality if and only if the exponent of G is p. $\square$

Now we prove Theorem 1.2. By Proposition 3.3, we know that equality always occurs when $G/C_G(V)$ is a group of exponent p. Hence, it remains to show that whenever $\operatorname{avgdim}(G,V) = (1/p)\dim V$, then $G/C_G(V)$ is a group of exponent p.

Choose a minimal counterexample to this latter statement with respect to $|G|$ and $\dim V$. As before, we may assume that $C_G(V) = 1$. By Proposition 3.3, we may also assume that $r := \operatorname{char}(F)$ divides the order of G.

We claim that V is an irreducible FG-module. For suppose not and W is a non-trivial proper submodule of V. By the minimality of $\dim V$ and by the fact that

$$\operatorname{avgdim}(G,V) \leq \operatorname{avgdim}(G,W) + \operatorname{avgdim}(G,V/W) \leq (1/p)\dim W + (1/p)\dim V/W,$$

we have that $G/C_G(W)$ and $G/C_G(V/W)$ are groups of exponent p. Let N be the normal subgroup of G which acts trivially on both W and V/W. Note that N is an r-group. So $G = PN$ where P is a Sylow p-subgroup of G of exponent p. Since G is a counterexample to the above statement, $N \neq 1$. For any element $g \in P$ we have $\operatorname{avgdim}(gN,V) \leq \dim C_V(g)$. (This can be seen by observing that some power of an arbitrary element gn is conjugate to g. Moreover, $\operatorname{avgdim}(N,V) \leq (1/r)\dim V < (1/p)\dim V$. Thus,

$$\operatorname{avgdim}(G,V) = |P|^{-1}\sum_{g\in P}\operatorname{avgdim}(gN,V) < \operatorname{avgdim}(P,N) = (1/p)\dim V,$$

a contradiction.

So we may assume that V is an irreducible FG-module. Let M be a minimal normal subgroup of G. By Theorem 1.1, we have $\mathrm{avgdim}(Mg, V) \leq (1/p)\dim V$ for each coset Mg of M in G, so $\mathrm{avgdim}(Mg, V) = (1/p)\dim V$ must hold for each coset Mg of M in G. In particular, by the minimality of G, the group M is an elementary abelian p-group. Since G is not a p-group, we can choose $g \in G$ of prime order $s > p$ such that $G = \langle g, M\rangle$ (by the minimality of G). (The module V remains an irreducible FG-module (by the minimality of $\dim V$) and $C_G(V) = 1$ continues to hold since both M and g acts faithfully on V.) If M is central in G, then G is abelian and $\dim V = 1$. In this case $\mathrm{avgdim}(G, V) = (1/|G|)\dim V < (1/p)\dim V$, a contradiction. If M is not central, then g permutes the eigenspaces of M in an orbit of size $s > p$ (for some divisor t of s) and so $\mathrm{avgdim}(Mg, V) \leq (1/t)\dim V < (1/p)\dim V$, which is again a contradiction. This proves Theorem 1.2.

4. Proof of Corollary 1.4

Let us first prove the first statement of Corollary 1.4. By making the assumptions of the proof of [8, Corollary D], it is sufficient to show that the number of $g \in G$ such that $\dim C_V(g) \leq (1/2)\dim V$ is at least

$$\frac{2|G|}{1 + \log_p |G|_p} \leq \frac{2|G|}{2 + \dim V}.$$

But this is clear by Theorem 1.1 noting that $\dim V$ is even.

Let us prove the second statement of Corollary 1.4. Use the notations and assumptions of the last part of the proof of [8, Corollary D]. Let H be a Hall p'-subgroup of G. Since V is a completely reducible G-module with $C_V(G) = 0$, the vector space V is also a completely reducible H-module with $C_V(H) = 0$. Hence applying Corollary 1.3 to the H-module V we get that there exists $g \in H$ with $\dim C_V(g) < (1/2)\dim V$. So the last displayed inequality of the proof of [8, Corollary D] becomes

$$\frac{|\mathrm{cl}_G(g)|_p}{p} \geq \chi(1)^{1/3}$$

since $\dim V$ is even. From this we get that $p^3\chi(1) \leq {|\mathrm{cl}_G(g)|_p}^3$.

5. Proof of Theorem 1.5

Note that Y centralizes M and so there is no loss in working in G/Y and assuming that $X = M$ is a minimal normal subgroup of G. Set $H = \langle M, g\rangle$ and so assume that g acts transitively on the direct factors of M.

We compute the arithmetic mean of the positive integers $|C_M(x)|$ for $x \in gM$. All elements in a given M-conjugacy class in gM have the same centralizer size. If $h \in gM$, then the M-conjugacy class of h has $|M : C_M(h)|$ elements. Thus, we see that the arithmetic mean is precisely the number of M-conjugacy classes in gM. By [5, Lemma 2.1], this is at most $k(M)$, the number of conjugacy classes in M. By [5, Proposition 5.3], this is at most $|M|^{.41}$. Since the geometric mean is bounded above by the arithmetic mean, the result follows.

6. Proof of Theorem 1.6

Let us fix a chief series for a finite group G. Let $\mathcal{N}$ be the set of non-central chief factors of this series. Let p be the smallest prime factor of the order of $G/F(G)$. If $N \in \mathcal{N}$ is abelian then, by Theorem 1.1 (noting that $F(G)$ acts trivially on N), we have $\mathrm{geom}(G,N) \leq |N|^{1/p}$. If $N \in \mathcal{N}$ is non-abelian then, by Theorem 1.5 and the Feit-Thompson Odd Order Theorem [3], we again have $\mathrm{geom}(G,N) \leq |N|^{1/p}$. Notice also that for any $g \in G$ we have the inequality $|C_G(g)| \leq \mathrm{ccf}(G) \prod_{N \in \mathcal{N}} |C_N(g)|$. From these observations Theorem 1.6 already follows since

$$\mathrm{geom}(G,G) = \Big(\prod_{g \in G} |C_G(g)| \Big)^{1/|G|} \leq \mathrm{ccf}(G) \Big(\prod_{g \in G} \prod_{N \in \mathcal{N}} |C_N(g)| \Big)^{1/|G|} =$$

$$= \mathrm{ccf}(G) \Big(\prod_{N \in \mathcal{N}} \prod_{g \in G} |C_N(g)| \Big)^{1/|G|} = \mathrm{ccf}(G) \Big(\prod_{N \in \mathcal{N}} \mathrm{geom}(G,N) \Big) \leq$$

$$\leq \mathrm{ccf}(G) \Big(\prod_{N \in \mathcal{N}} |N|^{1/p} \Big) = \mathrm{ccf}(G) \cdot (\mathrm{ncf}(G))^{1/p}.$$

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