On generalized blocks for alternating groups *

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Abstract

In a recent paper Külshammer, Olsson, Robinson gave a d- analogue for the Nakayama conjecture for symmetric groups where $d \geq 2$ is an arbitrary integer. We prove that there is a natural d-analogue of the Nakayama conjecture for alternating groups whenever d is 2 or an arbitrary odd integer greater than 1. This generalizes an old result of Kerber.

1 Introduction

Let d be a prime. A d-block of a finite group G is usually defined to be a minimal two-sided ideal of the group algebra FG where F is an algebraically closed field of characteristic d. It is a basic fact of modular representation theory that each complex irreducible character of G is assigned to a unique d-block. In this case we say that the character is in the d-block.

Recently, motivated by earlier work [2] on Hecke algebras, Külshammer, Olsson, Robinson [6] extended the definition of a d-block of a finite group for all integers $d \geq 2$. Let $\mathcal{C}$ be the union of a set of conjugacy classes of a finite group G, and let $\mathrm{Irr}(G)$ be the set of complex irreducible characters of G. They defined a $\mathcal{C}$-block to be a non-empty subset B of $\mathrm{Irr}(G)$ which is minimal subject to the following condition. If $\chi \in B$, $\psi \in \mathrm{Irr}(G)$, and if there exists a natural number k and a sequence $\chi = \chi_0, \ldots, \chi_k = \psi$ so that for all $0 \leq i < k$ the truncated inner product of χ_i and χ_{i+1} across $\mathcal{C}$, that is, the inner product

$$\langle \chi_i, \chi_{i+1} \rangle_{\mathcal{C}} := \frac{1}{|G|} \sum_{g \in \mathcal{C}} \chi_i(g) \overline{\chi_{i+1}(g)}, \tag{1}$$

is not 0, then $\psi \in B$. If the expression (1) is not 0 for the irreducible characters χ_i and χ_{i+1}, then it is said that the characters are directly $\mathcal{C}$-linked. We indeed got a more general definition of a block, since if d is prime and if $\mathcal{C}$ is the set of all elements of G with orders relatively prime to d, then the $\mathcal{C}$-blocks of G are precisely the subsets of $\mathrm{Irr}(G)$ corresponding to the usual d-blocks of G.

To state a main result of [6] we need some more definitions from that paper. The complex irreducible characters χ_λ of the symmetric group S_n are labelled naturally by partitions λ of n. Let $d \geq 2$ be an arbitrary integer. The d-core γ_λ of the partition λ is the partition that we get after removing all d-hooks from λ. We say that $B \subseteq \mathrm{Irr}(S_n)$ is a combinatorial d-block of S_n if B consists of

*2000 Mathematics Subject Classification 20C30, (20C15, 20C20) all irreducible characters of S_n that are labelled by partitions with the same d-core. Külshammer, Olsson, Robinson [6] proved that if $d \geq 2$, then $\mathcal{C}$-blocks and combinatorial d-blocks for S_n are the same if $\mathcal{C}$ is the set of all elements of S_n which have no cycle (of their disjoint cycle decompositions) of length divisible by d. If d is a prime, then this gives the so-called Nakayama conjecture proved by Brauer and Robinson [1]. The Nakayama conjecture has a natural analogue for alternating groups A_n proved by Kerber [5] (see also Theorem 6.1.46 of [4], and for a partial result see the doctoral thesis of Puttaswamaiah [10]). Let λ' denote the associate partition of the partition λ. (The diagram of λ' is obtained from the diagram of λ by interchanging its rows with its columns.) Let $Res^{S_n}_{A_n}(\chi_\lambda)$ be the restriction of the character χ_λ to the group A_n, and let d be a prime. The theorem of Kerber states that if $\lambda = \gamma_\lambda$, then each irreducible constituent of $Res^{S_n}_{A_n}(\chi_\lambda)$ is the only element in its d-block, and if $\lambda \neq \gamma_\lambda$, then to the d-block of an irreducible constituent of $Res^{S_n}_{A_n}(\chi_\lambda)$ there belong just the constituents of such restrictions $Res^{S_n}_{A_n}(\chi_\mu)$, where $\gamma_\mu = \gamma_\lambda$ or $\gamma_\mu = \gamma_{\lambda'}$. Now let $d \geq 2$ be an arbitrary integer, and define a combinatorial d-block of A_n by the combinatorial description of a d-block we just mentioned.

In this paper we prove the following generalization of the above-mentioned result of Kerber.

Theorem 1.1. *Let d be 2 or an arbitrary odd integer greater than 1. Let $\mathcal{C}$ be the set of all permutations of the alternating group A_n so that no cycle of their disjoint cycle decompositions has length divisible by d. Then $\mathcal{C}$-blocks and combinatorial d-blocks coincide for A_n.*

When d is a prime, then this theorem reduces to Kerber's result, so there is nothing to prove in case $d = 2$. In fact, most of our argument for the proof of Theorem 1.1 only works for $d > 1$ odd in which case we are able to prove an even stronger result (see Theorem 2.1) using ideas from [8]. If d is an even integer greater than 2, then there are integers n for which the conclusion of Theorem 1.1 is false.

Finally, we note that $\mathcal{C}$-blocks are investigated for other classes of groups also. In connection with Broué's conjectures, Jean-Baptiste Gramain [3] studied generalized blocks of some groups of Lie rank one.

2 Preliminaries

Let H be an arbitrary (finite or infinite) set of positive integers. Following [8] we say that a permutation (of finite order) is H-regular if for all $h \in H$ no cycle (of its disjoint cycle decomposition) has length equal to h. Let G be the symmetric group S_n or the alternating group A_n, and let $H(G)$ denote the set of H-regular elements of G. For complex irreducible characters α, β of G we define $\langle \alpha, \beta \rangle_{H(G)}$ as in (1). We will prove the following generalization of Theorem 1.1 for $d > 1$ odd.

Theorem 2.1. *Let $d > 1$ be odd, and let H be a set of positive integers so that $d \in H$ and all elements of H are divisible by d. Then $H(A_n)$-blocks and combinatorial d-blocks are the same for A_n.*

To perform our calculations for the proof of Theorem 2.1, we will first need to move from A_n to S_n and then to a generalized symmetric group $Z_d \wr S_w$ for some integer w.

How can we label the characters of A_n? As in the Introduction, for any partition λ, we denote the associate partition by λ'. If $\lambda = \lambda'$, then it is said that the partition λ is self-associate. Otherwise it is non-associate. The irreducible characters of A_n are canonically labelled by symbols λ^0, μ^+, μ^- where $\{\lambda, \lambda'\}$ and μ run through the set of all associate pairs of non-associate and the set of all self-associate partitions of n, respectively. Denote the irreducible character of A_n associated to the non-associate partition λ by χ_{λ^0}. This is exactly the restriction to A_n of both irreducible characters χ_λ and $\chi_{\lambda'}$ of S_n. For convenience, let us call such a character a stalk. Denote the irreducible characters of A_n associated to the self-associate partition μ by χ_{μ^+} and χ_{μ^-}. These are characters of A_n such that the restriction to A_n of the irreducible character $\chi_\mu = \chi_{\mu'}$ of S_n is $\chi_{\mu^+} + \chi_{\mu^-}$. Let us call such characters cherries. For the time being, this is all we will need to know about characters of A_n. Later we will introduce formulas (4) and (5). For more information the reader is referred to [4].

To prepare our step of moving from S_n to the relevant generalized symmetric group, we need some more definitions. Let $d > 1$ be an arbitrary integer. For a partition λ we denote the d-core of λ by γ_λ, and the d-quotient by β_λ. (The d-core, γ_λ (as mentioned earlier) is the partition obtained by removing all d-hooks from λ, and the d-quotient is a d-tuple of partitions recording the d-hook removals obtained by moving from λ to its d-core.) Note that the empty partition, the partition of 0, could occur as a d-core. There are $1-1$ correspondences between partitions of n with a fixed d-core and their d-quotients. Let us fix one. If there are w hooks of length d to be removed from λ to go to its d-core, then w is called the d-weight of β_λ. The d-quotients serve as a natural index set for the conjugacy classes and the irreducible characters of the generalized symmetric group $Z_d \wr S_w$. If λ is a partition of n so that β_λ has d-weight w, then we denote the associated character of $Z_d \wr S_w$ by χ_{β_λ}. Again, the reader is referred to [4] for detailed information. Finally, by Pages 61-63 of [9] and by Theorem 4.53 of [11], for each partition λ, there is a sign, σ_λ, called the d-sign of λ which is equal to $(-1)^{\sum_i l_i}$ where l_i is the leg length of the i-th d-hook to be removed from λ when 'moving down' to its d-core γ_λ along an arbitrary path. We will make use of the simple observation that if $d > 1$ is odd, then $\sigma_\lambda = \sigma_{\lambda'}$.

We will next describe a certain set of elements of the group $Z_d \wr S_w$. Let $d > 1$ be a positive integer, and let H be any set of positive integers so that all elements of H are divisible by d. Put $H_d = \{h/d : h \in H\}$. Following [8], we define an H_d-regular element of $Z_d \wr S_w$ to be an element $(a_1, \ldots, a_w)\sigma$ where $(a_1, \ldots, a_w)$ is in the base group ${Z_d}^w$ (which we consider to be the w-th power of the group of complex d-th roots of unity) and σ is a permutation of S_w, such that for all $h \in H$, the product of the a_j's corresponding to each h-cycle of σ is different from 1. The set of all H_d-regular elements is a union of conjugacy classes of $Z_d \wr S_w$. Let us denote this set by $H_d(Z_d \wr S_w)$. For complex irreducible characters α, β of $Z_d \wr S_w$ we define $\langle \alpha, \beta \rangle_{H_d(Z_d \wr S_w)}$ as in (1).

With the notations above, Theorem 5.1 of [8] states that if $d > 1$ is an arbitrary integer, and λ, μ are arbitrary partitions of n, then

$$\langle \chi_\lambda, \chi_\mu \rangle_{H(S_n)} = \langle \sigma_\lambda \chi_{\beta_\lambda}, \sigma_\mu \chi_{\beta_\mu} \rangle_{H_d(Z_d \wr S_w)} \tag{2}$$

holds. Formula (2) is a generalization of the 'perfect isometry' of [6].

In this paper, as in [6] and [8], the 'perfect isometry' will only be applied in the special case when χ_{β_λ} is the trivial character $\mathbf{1}$ of the generalized symmetric group $Z_d \wr S_w$. A special case of Theorem 5.2 of [8], and a slight extension of Theorem 5.12 of [6], is the following. If $d \in H$, then for all irreducible characters χ_{β_μ} of $Z_d \wr S_w$, the algebraic integer

$$\frac{d^w w! \cdot \langle \mathbf{1}, \chi_{\beta_\mu} \rangle_{H_d(Z_d \wr S_w)}}{\chi_{\beta_\mu}(1)}$$

is an integer so that

$$\frac{d^w w! \cdot \langle \mathbf{1}, \chi_{\beta_\mu} \rangle_{H_d(Z_d \wr S_w)}}{\chi_{\beta_\mu}(1)} \equiv (-1)^w \pmod{d}. \tag{3}$$

We are now in the position to prove our theorems.

3 The proof of Theorem 2.1

Let $n \geq 2$ and $d > 1$ be arbitrary integers. Let H be a set of positive integers so that $d \in H$ and all elements of H are divisible by d.

We prove Theorem 2.1 in steps.

Lemma 3.1. *Let $d > 1$ be an arbitrary odd integer. For all $\alpha \in Irr(A_n)$, the truncated inner product $\langle 1_{A_n}, \alpha \rangle_{H(A_n)}$ is non-zero if and only if α is inside the combinatorial d-block of the trivial character 1_{A_n}.*

Proof. If α is a cherry, then by Frobenius reciprocity, we have $\langle 1_{A_n}, \alpha \rangle_{H(A_n)} = \langle 1_{S_n}, \chi_\lambda \rangle_{H(S_n)}$ for some self-associate partition λ of n where 1_{S_n} denotes the trivial character of S_n. Now by the 'perfect isometry' (2) and by (3) we see that $\langle 1_{S_n}, \chi_\lambda \rangle_{H(S_n)} \neq 0$ if and only if 1_{S_n} and χ_λ are in the same combinatorial d-block of A_n.

If α is a stalk, then by Frobenius reciprocity again, we have $\langle 1_{A_n}, \alpha \rangle_{H(A_n)} = \langle 1_{S_n}, \chi_\mu + \chi_{\mu'} \rangle_{H(S_n)}$ for some non-associate partition μ of n. We have four cases to consider. If $\gamma_\mu = \gamma_{(n)} \neq \gamma_{\mu'}$ or if $\gamma_\mu \neq \gamma_{(n)} = \gamma_{\mu'}$, then again by the 'perfect isometry' (2) and by (3) we see that $\langle 1_{S_n}, \chi_\mu + \chi_{\mu'} \rangle_{H(S_n)} \neq 0$. If $\gamma_\mu \neq \gamma_{(n)} \neq \gamma_{\mu'}$, then $\langle 1_{S_n}, \chi_\mu + \chi_{\mu'} \rangle_{H(S_n)} = 0$. Finally, in case $\gamma_\mu = \gamma_{(n)} = \gamma_{\mu'}$, we claim that $\langle 1_{S_n}, \chi_\mu + \chi_{\mu'} \rangle_{H(S_n)} \neq 0$.

Let $n = dw + r$ where w and r are integers such that $0 \leq r \leq d-1$. Let the characters of $Z_d \wr S_w$ corresponding to χ_μ and $\chi_{\mu'}$ be χ_{β_μ} and $\chi_{\beta_{\mu'}}$, respectively. By (2) we have

$$d^w w! \langle 1_{S_n}, \chi_\mu + \chi_{\mu'} \rangle_{H(S_n)} = d^w w! \langle 1_{Z_d \wr S_w}, \sigma_\mu \chi_{\beta_\mu} + \sigma_{\mu'} \chi_{\beta_{\mu'}} \rangle_{H_d(Z_d \wr S_w)},$$

where $\sigma_\mu = \sigma_{\mu'}$ is the d-sign of μ and μ'. These are equal since d is odd. Notice that the quotients β_μ and $\beta_{\mu'}$ are associate to each other as necklaces, which implies that $\chi_{\beta_\mu}(1) = \chi_{\beta_{\mu'}}(1)$. Hence, by (3), we see that

$$\frac{\sigma_\mu d^w w! \langle 1_{Z_d \wr S_w}, \chi_{\beta_\mu} + \chi_{\beta_{\mu'}} \rangle_{H_d(Z_d \wr S_w)}}{\chi_{\beta_\mu}(1)}$$

is an integer congruent to $2\sigma_\mu (-1)^w$ modulo d. This is never 0. The proof of the lemma is complete.

□

By reading the proof of Lemma 3.1 more carefully, one can see that for any stalk α and any irreducible character χ of A_n, the truncated inner product $\langle \alpha, \chi \rangle_{H(A_n)}$ is 0 if α and χ lie in different combinatorial d-blocks of A_n.

Next we investigate the truncated inner products of cherries. But before we do so, we recall a few results from Page 67 of [4].

Let μ be a self-associate partition of n, and let $h(\mu)$ be the partition (of n) with parts consisting of the main hooks of μ. Let $d(\mu)$ denote the product of all parts of this partition $h(\mu)$. Now $Res^{S_n}_{A_n}(\chi_\mu) = \chi_{\mu^+} + \chi_{\mu^-}$. For all permutations π of A_n of cycle-shape different from $h(\mu)$, it is known that $\chi_{\mu^+}(\pi) = \chi_{\mu^-}(\pi) = \chi_\mu(\pi)/2$. Otherwise, if π has cycle-shape equal to $h(\mu)$, then it is a member of one of two conjugacy classes of A_n. Let π^+ and π^- denote two representatives of these conjugacy classes with respect to the following identities (see Theorem 2.5.13 of [4]):

$$\chi_{\mu^+}(\pi^\pm) = \frac{1}{2}\Big(\chi_\mu(\pi) \pm \sqrt{\chi_\mu(\pi) \cdot d(\mu)}\Big) \tag{4}$$

$$\chi_{\mu^-}(\pi^\pm) = \frac{1}{2}\Big(\chi_\mu(\pi) \mp \sqrt{\chi_\mu(\pi) \cdot d(\mu)}\Big). \tag{5}$$

We are now in the position to state

Lemma 3.2. *Let $d > 1$ be an integer, and let α, β be irreducible characters of A_n lying in different combinatorial d-blocks of A_n. Then $\langle \alpha, \beta \rangle_{H(A_n)} = 0$.*

Proof. By the remark after Lemma 3.1 we may (and do) suppose that α and β are cherries. There are two possibilities to consider: α and β are associate with the same stalk or they are not.

Let us start with the case when α and β are associate with different stalks. There are four cases to be dealt with. These are $\alpha = \chi_{\mu^\pm}$ and $\beta = \chi_{\lambda^\pm}$ where λ, μ are different self-associate partitions of n. However, we only need to consider one of these cases. Indeed, by Frobenius reciprocity we have

$$\langle \chi_{\mu^+}, \chi_{\lambda^-} \rangle_{H(A_n)} + \langle \chi_{\mu^+}, \chi_{\lambda^+} \rangle_{H(A_n)} = \langle \chi_\mu, \chi_\lambda \rangle_{H(S_n)} =$$

$$= \langle \chi_{\mu^-}, \chi_{\lambda^+} \rangle_{H(A_n)} + \langle \chi_{\mu^-}, \chi_{\lambda^-} \rangle_{H(A_n)},$$

where $\langle \chi_\mu, \chi_\lambda \rangle_{H(S_n)} = 0$. So suppose that $\alpha = \chi_{\mu^+}$ and $\beta = \chi_{\lambda^+}$. Define $\epsilon(\mu)$ to be 1 if the partition $h(\mu)$ is H-regular and to be 0 if it is not. Define $\epsilon(\lambda)$ similarly. Now evaluating $\langle \chi_{\mu^+}, \chi_{\lambda^+} \rangle_{H(A_n)}$ using formula (4), we get

$$\langle \chi_{\mu^+}, \chi_{\lambda^+} \rangle_{H(A_n)} = \frac{1}{2} \cdot \langle \chi_\mu, \chi_\lambda \rangle_{H(A_n)} +$$

$$+ \frac{1}{4} \cdot \epsilon(\mu) n! (2z_\mu)^{-1} \Big(\sqrt{\chi_\mu(g) \cdot d(\mu)} \chi_\lambda(g^{-1}) - \sqrt{\chi_\mu(g) \cdot d(\mu)} \chi_\lambda(g^{-1}) \Big) +$$

$$+ \frac{1}{4} \cdot \epsilon(\lambda) n! (2z_\lambda)^{-1} \Big(\sqrt{\chi_\lambda(h) \cdot d(\lambda)} \chi_\mu(h^{-1}) - \sqrt{\chi_\lambda(h) \cdot d(\lambda)} \chi_\mu(h^{-1}) \Big) =$$

$$= \frac{1}{2} \cdot \langle \chi_\mu, \chi_\lambda \rangle_{H(A_n)},$$

where $H(A_n)$ is considered as a union of conjugacy classes in S_n, where g and h are permutations of cycle-shape μ and λ, respectively, and z_μ, z_λ are the orders of the centralizers of g and h, respectively. Since μ and λ are self-associate partitions, both χ_μ and χ_λ vanish outside A_n, so we conclude that

$$\frac{1}{2}\cdot\langle\chi_\mu,\chi_\lambda\rangle_{H(A_n)}=\frac{1}{2}\cdot\langle\chi_\mu,\chi_\lambda\rangle_{H(S_n)}=0.$$

Suppose now that α and β are associate with the same stalk. Without loss of generality, put $\alpha=\chi_{\mu^+}$ and $\beta=\chi_{\mu^-}$ for some self-associate d-core partition, μ. Let us calculate $\langle\chi_{\mu^+},\chi_{\mu^-}\rangle_{H(A_n)}$. We use formulas (4), (5), and the facts that $z_{h(\mu)}=d(\mu)$ (since $h(\mu)$ is a partition with distinct parts) and $\chi_\mu(\pi)=\pm 1$ where π is an element of A_n of cycle-shape $h(\mu)$ (this follows from the Murnaghan-Nakayama formula). Also note that π is an H-regular permutation since μ is a d-core partition.

We have

$$\langle\chi_{\mu^+},\chi_{\mu^-}\rangle_{H(A_n)}=\frac{1}{2n!}\Big(\sum\chi_\mu(g)\overline{\chi_\mu(g)}+$$

$$+\frac{n!}{2z_{h(\mu)}}\Big(\chi_\mu(\pi)+\sqrt{\chi_\mu(\pi)d(\mu)}\Big)\cdot\Big(\chi_\mu(\pi)-\chi_\mu(\pi)\sqrt{\chi_\mu(\pi)d(\mu)}\Big)+$$

$$+\frac{n!}{2z_{h(\mu)}}\Big(\chi_\mu(\pi)-\sqrt{\chi_\mu(\pi)d(\mu)}\Big)\cdot\Big(\chi_\mu(\pi)+\chi_\mu(\pi)\sqrt{\chi_\mu(\pi)d(\mu)}\Big)\Big)=$$

$$=\frac{1}{2}\cdot\langle\chi_\mu,\chi_\mu\rangle_{H(A_n)}-\frac{1}{2},$$

where the sum is over all elements g of $H(A_n)$ of cycle-shape different from $h(\mu)$ and where the last truncated inner product means that we are only summing over the subset $H(A_n)$ of S_n.

Since μ is a self-associate partition of n, we have

$$\langle\chi_\mu,\chi_\mu\rangle_{H(A_n)}=\langle\chi_\mu,\chi_\mu\rangle_{H(S_n)}.$$

Also, since μ is a d-core partition, by part (d) of Example 1.8 of [7] and by the Murnaghan-Nakayama formula (see Example 7.5 of [7]), we see that χ_μ vanishes off the set of H-regular permutations of S_n. From this we conclude that $\langle\chi_\mu,\chi_\mu\rangle_{H(A_n)}=1$, and hence that $\langle\chi_{\mu^+},\chi_{\mu^-}\rangle_{H(A_n)}=0$.

□

So far we know that for all integers $d>1$, combinatorial d-blocks for A_n are unions of $H(A_n)$-linked blocks.

Let B be a combinatorial d-block of S_n consisting of characters labelled by partitions with d-quotients of weight w. Let $\chi_\tau\in B$ be the character of S_n for which χ_{β_τ} is the trivial character of $Z_d\wr S_w$. There are two cases to consider: τ is a non-associate partition and τ is a self-associate partition.

Suppose that τ is a non-associate partition of n. In this case, notice that in the proof of Lemma 3.1 we may replace the character 1_{A_n} by χ_{τ^0} (and the partition (n) by τ), and we may conclude that χ_{τ^0} is directly $H(A_n)$-linked to (this definition is found after formula (1)) all characters in its combinatorial d-block.

Let τ be a self-associate partition of n. First of all, we may (and do) suppose that τ is *not* a d-core partition.

We claim that χ_{τ^+} is directly $H(A_n)$-linked to every irreducible character α different from χ_{τ^-} of its combinatorial d-block. If $\alpha:=Res^{S_n}_{A_n}(\chi_\lambda)$ is a stalk where λ is some non-associate partition of n, then $\langle \chi_{\tau^+}, \alpha \rangle_{H(A_n)} = \langle \chi_\tau, \chi_\lambda \rangle_{H(S_n)} \neq 0$. If $\alpha := \chi_{\mu^+}$ is a cherry where μ is some self-associate partition of n, then the calculations in Lemma 3.2 yield

$$\langle \alpha, \chi_{\tau^+} \rangle_{H(A_n)} = \frac{1}{2} \langle \chi_\mu, \chi_\tau \rangle_{H(A_n)} = \frac{1}{2} \langle \chi_\mu, \chi_\tau \rangle_{H(S_n)} \neq 0.$$

The same is true if $\alpha = \chi_{\mu^-}$. This proves the claim.

Similarly, it is also true that χ_{τ^-} is directly $H(A_n)$-linked to every irreducible character α different from χ_{τ^+} of its combinatorial d-block.

By the fact that τ is not a d-core partition and by the two claims above, we conclude that there exists a third character α in the combinatorial d-block of A_n containing χ_{τ^+} and χ_{τ^-} which is directly $H(A_n)$-linked to both χ_{τ^+} and χ_{τ^-}.

The proof of Theorem 2.1 is now complete.

What if $d > 1$ is arbitrary, not necessarily odd? From the above, we know that combinatorial d-blocks are unions of $H(A_n)$-linked blocks for A_n. Can we say more?

We remark, that by part (e) of Example 1.8 of [7], it follows - in the last case of the proof above - that χ_τ can only correspond to the trivial character of $Z_d \wr S_w$ if the d-quotient of τ is equal (as a necklace) to the d-quotient $\beta_{(n)}$ where one entry is (w) and all other entries are the empty partitions. This happens only if $w = 0$ or if $w = 1$. If $w = 1$ is the case, then γ_τ is self-associate and hence d has to be odd. So in the case when τ is a self-associate partition of n, there is no need to assume in the beginning that d is odd.

By Theorem 6.1.46 of [4], we see that the two notions coincide when $d = 2$ and when H is the set of all positive even integers. However, if $d > 2$ is even, then combinatorial d-blocks and $H(A_n)$-linked blocks do *not* coincide for the groups A_d, A_{d+1}, or A_{d+3} if H is the set of all positive integers divisible by d. Indeed, in the first two cases, there is no $H(A_n)$-linked block containing at least two irreducible characters, however the combinatorial d-block containing the trivial character contains at least two irreducible characters. The group A_{d+3} contains precisely one conjugacy class of *non* H-regular elements. If the value of an irreducible character of A_{d+3} is 0 on this conjugacy class, then that character forms a separate $H(A_n)$-linked block of its own. All irreducible characters not vanishing on that conjugacy class form one $H(A_n)$-linked block. So there is at most one $H(A_n)$-linked block of A_{d+3} containing more than one irreducible character. On the other hand, there are precisely two combinatorial d-blocks having more than one irreducible character. (One associated to the d-core (3) and the other to the d-core $(2, 1)$.)

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