MINIMAL LENGTH FACTORIZATIONS OF FINITE SIMPLE GROUPS OF LIE TYPE BY UNIPOTENT SYLOW SUBGROUPS

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Abstract. We prove that every finite simple group G of Lie type satisfies $G = UU^{-}UU^{-}$ where U is a unipotent Sylow subgroup of G and U^{-} is its opposite. We also characterize the cases for which $G = UU^{-}U$. These results are best possible in terms of the number of conjugates of U in the above factorizations.

1. Introduction

Let G be a finite simple group of Lie type with defining characteristic p. We address the problem of finding the minimal number m such that G is equal to the product of m Sylow p-subgroups (unipotent Sylows) of G. This question has already been considered by several authors before us. Liebeck and Pyber had proved [8, Theorem D] that G is a product of no more than 25 Sylow p-subgroups. In [1] it was claimed that the 25 can be replaced by 5, however no complete proof has been published. A sketch of a proof of this claim for exceptional Lie type groups can be found in a survey by Pyber and Szabó [10, Theorem 15]. Smolensky, Sury and Vavilov [16, Theorem 1] considered the problem of unitriangular factorizations of Chevalley groups over commutative rings of stable rank 1. When specializing their results to elementary Chevalley groups over finite fields, they get that any non-twisted finite simple group of Lie type is a product of four unipotent Sylows. Later on, these results were extended by Smolensky in [11] to cover some twisted Chevalley groups over finite fields or the field of complex numbers.

Here we give a unified self-contained treatment of the problem of finding minimal length products of unipotent Sylows for all finite simple groups of Lie type, by exploiting their split BN-pair structure. Our main result is the following theorem.

Theorem 1 *Let G be a simple group of Lie type with defining characteristic p. Let U be a Sylow p-subgroup of G* and U^{-} its opposite. *Then $G = (UU^{-})^2$, and moreover, $G = UU^{-}U$ if and only if U is self-normalizing. In both cases these factorizations are of minimal length.*

After the completion of our work, and in parallel to its publication in preprint form [6], Smolensky made available a preprint in which he shows that every Suzuki

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and Ree group is a product of four unipotent Sylow subgroups [12]. Thus, the results in [14], [15], [16], [13], [11] and [12], combine to give a different proof of the four Sylow claim of Theorem 1.

2. Preliminaries

The proof of Theorem 1 consists of two main steps: 1. A reduction to the case where the Weyl group is Z_2 (i.e., to groups G of rank 1), which is carried within the framework of groups with a split BN-pair, and some extra assumptions to be detailed in the sequel. 2. A derivation of a general necessary and sufficient criterion for rank 1 groups satisfying an extended set of split BN-pair assumptions, that is then verified to hold for the special case of groups with a σ-setup, using a result from [5].

We would like to point out that although the proof of [16, Theorem 1] also uses a "reduction to rank 1 argument" which is due to Tavgen′ [14], we do not know if there is a more direct relation between this approach and ours.

We treat simple groups of Lie type in the setting of groups with a σ-setup as in [7, Definition 2.2.1]. For this fix a prime p, a simple algebraic group $\overline{K}$ defined over $\overline{\mathbb{F}}_p$ and a Steinberg endomorphism σ of $\overline{K}$, and consider K - the subgroup of $C_{\overline{K}}(\sigma)$ generated by all p-elements. All groups K obtained in this way are said to have a σ-setup given by the pair $(\overline{K}, \sigma)$. The set of all groups possessing a σ-setup for the prime p is denoted by $\mathcal{L}ie(p)$. Set $\mathcal{L}ie := \cup \mathcal{L}ie(p)$ where the union is over all primes p. We have surjective homomorphisms $K_u \to K \to K_a$ with central kernels [7, Theorem 2.2.6], where the groups K_u, $K_a \in \mathcal{L}ie(p)$ are called the universal and the adjoint version of K, respectively. Any finite simple group of Lie type can be realized as the adjoint version K_a of some $K \in \mathcal{L}ie$ [7, Definition 2.2.8] (note that the Tits group ${}^2F_4(2)'$ is not in $\mathcal{L}ie$). For $G \in \mathcal{L}ie$ we have [3, Chapter 2]:

(i) G is a group with a split BN-pair (B, N) and a finite Weyl group W, where $B = H \ltimes U$,
(ii) U is a Sylow p-subgroup of G,
(iii) G is generated by its p-elements.

3. Reduction to the case $|W| = 2$ for groups with a split BN-pair

For our purposes we will call a triple (H, U, N) a split BN-pair for a group G if $(H \ltimes U, N)$ satisfies the axioms of split BN-pairs in [3, §2.5] with respect to H and U. We assume that the Weyl group $W = N/H$ of the BN-pair is finite (this certainly holds for finite groups), and so the longest element $w_0 = n_0 H$ of W exists and defines subgroups $U^- := U^{n_o}$ and $B^- := B^{n_o}$. For $w \in W$ we sometimes use $\dot{w}$ to denote an arbitrary choice of an element of N such that $w = \dot{w}H$. We use the notation U^-, X_i, U_i, X_{-i}, U_w from [3, §2.5] for the BN-pair (B, N) and we label with the upper-script '$-$' the corresponding subgroups for (B^-, N), i.e. when U and B are replaced by U^- and B^- everywhere: $(U^-)^-$, X_i^-, U_i^-, X_{-i}^-, U_w^-. Note that $(U^-)^- = U$, $X_i^- = X_{-i}$, $X_{-i}^- = X_i$ and that $X_i = U_s$ and $X_{-i} = U_s^-$ for some simple reflection s. In addition, define $L_s := \langle U_s, U_s^-, H\rangle$ and $G_s := \langle U_s, U_s^-\rangle$ for any simple reflection $s \in W$. Furthermore, we assume that the root subgroups X_α, $\alpha \in \Phi$ (Φ is the set of roots associated with W) satisfy the commutator relations [3, p.61].

Lemma 3.1. *Let G be a group with a split BN-pair (H, U, N). Suppose that G is a product of $k = 2m + \varepsilon \geq 3$ conjugates of U where $m \geq 1$ is an integer and $\varepsilon \in \{0, 1\}$. Then $G = (UU^-)^m U^\varepsilon$.*

Proof. By assumption, $G = U^{x_1} U^{x_2} \cdots U^{x_k}$ for some elements $x_1, \ldots, x_k \in G$. This is equivalent to $G = Ug_1U \cdots Ug_{k-1}U$ for some $g_1, \ldots, g_{k-1} \in G$ [2, §2 Lemma 1]. By the Bruhat expression of elements (w.r.t. H and U) we may assume that $g_i \in N$ for all i. Indeed, by [3, Theorem 2.5.14], for each $1 \leq i \leq k-1$, we have $g_i = u_i h_i \dot{w}_i u'_i$ where $u_i \in U$, $h_i \in H$, $w_i \in W$ and $u'_i \in U_w \leq U$. Since the h_i lie in $N_G(U)$ we have $Ug_1U \cdots Ug_{k-1}U = Uh_1\dot{w}_1U \cdots Uh_{k-1}\dot{w}_{k-1}U = U\dot{w}_1U \cdots U\dot{w}_{k-1}U(h_1^{\dot{w}_1 \cdots \dot{w}_{k-1}} \cdots h_{k-1})$. Thus we have proved that G is a product of k conjugates of U if and only if $G = Ug_1U \cdots Ug_{k-1}U$ for some $g_1, \ldots, g_{k-1} \in N$, which is equivalent to $G = UU^{g_1^{-1}} U^{g_2^{-1}} \cdots U^{g_{k-1}^{-1}}$ (same g_i - see proof of [2, §2 Lemma 1]).

Let $n \in N$ be arbitrary, and let $w = nH$. By [3, Proposition 2.5.12] we have

$$U = U_{w_0w}U_w = \left(U \cap U^{n_0(n_0n)}\right)\left(U \cap U^{n_0n}\right),$$

which gives

$$U^{n^{-1}} = \left(U \cap U^{n_0(n_0n)}\right)^{n^{-1}} \left(U \cap U^{n_0n}\right)^{n^{-1}} = \left(U^{n^{-1}} \cap U\right)\left(U^{n^{-1}} \cap U^{n_0}\right) \leq UU^-.$$

However, since U, U_{w_0w}, U_w are all subgroups, $U = U_{w_0w}U_w$ implies $U = U_wU_{w_0w}$, and hence we also get $U^{n^{-1}} \leq U^-U$. Therefore

$$G = UU^{g_1^{-1}} U^{g_2^{-1}} \cdots U^{g_{k-1}^{-1}} \subseteq U\left(UU^-\right)\left(U^-U\right)\left(UU^-\right)\left(U^-U\right) \cdots = UU^-UU^-U \cdots,$$

where we have used $U^2 = U$ and $(U^-)^2 = U^-$, and the claim follows. □

In the following lemma we collect known results about minimal (non-abelian) Levi subgroups which will be used in the sequel. First note that for a fixed split BN-pair (H, U, N) we have *the (split) BN-pair opposite to* (B, N) given by (H, U^-, N). Clearly, for any $g \in G$, (B^g, N^g) is a split BN-pair, and if $g \in N$ then $B^g \cap N = H$ so $B^g = H \ltimes U^g$. In particular this applies to $g = n_0$.

Lemma 3.2. *Let G have a split BN-pair (H, U, N). Let w_0 be the longest element of the Weyl group N/H and $s = n_sH$ a simple reflection with respect to (H, U, N). Then*

(a) $U = U_sU_{w_0s} = U_{w_0s}U_s$ *and* $U^- = U_s^-U_{w_0s}^- = U_{w_0s}^-U_s^-$,
(b) $L_s \subseteq N_G(U_{w_0s}) \cap N_G(U_{w_0s}^-)$.

Proof. Any $s = n_sH$ in W is simple with respect to (H, U, N) if and only if it is simple with respect to (H, U^-, N). This follows from [3, Propositions 2.2.6 and 2.2.7] and the fact that the positive roots with respect to (H, U^-, N) are the negative roots with respect to (H, U, N). So $I := \{s_1, ..., s_l\}$, the set of simple reflections for (H, U, N), is a set of simple reflections for both of these BN pairs and $s = s_i$ for some $i \in \{1, \ldots, l\}$. Since L_s is the subgroup $X_iH \cup X_iHn_iX_i = \langle X_i, X_{-i}, H\rangle$ [3, Corollary 2.6.2], $X_i^- = X_{-i}$ and $X_{-i}^- = X_i$, it follows that $L_s = \langle X_i^-, X_{-i}^-, H\rangle$ is a (minimal) standard Levi subgroup with respect to both (B, N) and (B^-, N).

Now (a) follows from [3, Proposition 2.5.11] and (b) is a particular case of [3, Proposition 2.6.4]. □

The following lemma is [16, Lemma 4].

Lemma 3.3. *Let G be a group and let $X \subseteq G$ satisfy $X = X^{-1}$ and $G = \langle X \rangle$. If $\emptyset \neq Y \subseteq G$ is such that $XY \subseteq Y$ then $Y = G$.*

Proposition 3.4. *Let G be a group with a split BN-pair such that the conjugates of U in G generate G. Let $k \geq 2$ be an integer and assume further that $G_s = \left(U_s U_s^-\right)^k$ for every simple reflection s then $G = \left(UU^-\right)^k$.*

Proof. Set $X := \{u^g | u \in U, g \in G\}$. Then $X = X^{-1}$ since U is a subgroup of G, and $G = \langle X \rangle$ since G is the normal closure of U. Set $Y := \left(UU^-\right)^k$. By Lemma 3.3 our claim will follow if we show that $XY \subseteq Y$. Thus it suffices to show that $u^g Y \subseteq Y$ for any $u \in U$ and $g \in G$. By [3, Theorem 2.5.14], for any $g \in G$ there exist $u' \in U$, $h \in H$, $w \in W$ and $u'' \in U_w \leq U$ such that $g = u'hn_w u''$. Hence $u^g = u^{u'hn_w u''} = \left(u^{u'h}\right)^{n_w u''}$. But $u^{u'h} \in U$, so it is sufficient to prove that $u^{nv} Y \subseteq Y$ for all $u, v \in U$ and $n \in N$. Now we claim that the last statement follows if we prove that N normalizes Y. For suppose that N normalizes $Y = \left(UU^-\right)^k$. We have:

$$\begin{aligned} u^{nv}\left(UU^-\right)^k &= v^{-1}n^{-1}unv\left(UU^-\right)^k = v^{-1}n^{-1}un\left(UU^-\right)^k \\ &= v^{-1}n^{-1}u\left(UU^-\right)^k n = v^{-1}n^{-1}\left(UU^-\right)^k n \\ &= v^{-1}\left(UU^-\right)^k n^{-1}n = \left(UU^-\right)^k. \end{aligned}$$

Thus we prove that N normalizes $\left(UU^-\right)^k$. Since H clearly normalizes $\left(UU^-\right)^k$, and N is generated by a set I of representatives for simple reflections together with H, it is sufficient to prove that $\left(UU^-\right)^k$ is normalized by all n in I. Fix a simple reflection $s = nH$. By Lemma 3.2.(a), $UU^- = U_s U_{w_0 s} U_{w_0 s}^- U_s^-$. By Lemma 3.2.(b), each of U_s and U_s^- commutes with both $U_{w_0 s}$ and $U_{w_0 s}^-$. This, and the assumption $G_s = \left(U_s U_s^-\right)^k$, give:

$$\left(UU^-\right)^k = \left(U_s U_s^-\right)^k \left(U_{w_0 s} U_{w_0 s}^-\right)^k = G_s \left(U_{w_0 s} U_{w_0 s}^-\right)^k.$$

Since $L_s = G_s H$ we can assume $n \in G_s$ and hence $nG_s = G_s n$. Since $n \in G_s \leq L_s$, Lemma 3.2.(b) gives $n\left(U_{w_0 s} U_{w_0 s}^-\right)^k = \left(U_{w_0 s} U_{w_0 s}^-\right)^k n$. Combining everything together yields:

$$\begin{aligned} n\left(UU^-\right)^k &= nG_s\left(U_{w_0 s} U_{w_0 s}^-\right)^k = G_s n\left(U_{w_0 s} U_{w_0 s}^-\right)^k \\ &= G_s\left(U_{w_0 s} U_{w_0 s}^-\right)^k n = \left(UU^-\right)^k n, \end{aligned}$$

and the proof that N normalizes $\left(UU^-\right)^k$ is concluded. □

Remark 3.5. *If G is generated by U and U^- then the use of Lemma 3.3 in the above proof can be avoided as follows. If N normalizes $\left(UU^-\right)^k$ then $\left(UU^-\right)^k$ is stable under conjugation by $n_0 H$ and so it is equal to $\left(U^-U\right)^k$. It is easy to see that if this equality holds then $G = (UU^-)^k$.*

4. The case $|W| = 2$

In this section we prove (Lemma 4.2) a criterion for a group G of rank 1 to satisfy $G = \left(UU^-\right)^2$.

Lemma 4.1. *Let G be a group with a split BN-pair (H,U,N) and a Weyl group $W=\{1,s_1\}$. Set $(U^-)^* := U^- - \{1\}$. Fix an arbitrary $n_1\in N$ such that $s_1=n_1H$, and set*

$$\widetilde{H} := \left\{h\in H | \exists u^- \in \left(U^-\right)^*,\ Uu^-U = Un_1hU\right\}.$$

Then:

(a) $U\left(U^-\right)^* U = Un_1\widetilde{H}U$.
(b) $UU^-UU^- = UU^-\left(\{1\}\cup n_1\widetilde{H}n_1\widetilde{H}\right)\cup Un_1\widetilde{H}$.

Proof. (a) Since $W=\{1,s_1\}$ we have

$$G = B\cup Bn_1B = UH\cup UHn_1HU = UH\cup Un_1HU,$$

where the union on the right is disjoint. By [3, Proposition 2.5.5(i)], $B\cap U^-=1$. Hence $(U^-)^*\subseteq Un_1HU$. Thus, for every $u^-\in(U^-)^*$ there exists $h\in H$ such that $Uu^-U=Un_1hU$. But, by definition, $h\in\widetilde{H}$, so this proves $U\left(U^-\right)^*U\subseteq Un_1\widetilde{H}U$. The reverse inclusion is also clear and hence $U\left(U^-\right)^*U=Un_1\widetilde{H}U$.

(b) Note that since each element of H normalizes both U and U^-, the set $\widetilde{H}$ commutes with U. Also, $w_0=s_1$ and hence $n_1Un_1^{-1}=U^-$ and $n_1U^-n_1^{-1}=U$. Given this and the relation in (a) we get:

$$\begin{aligned}
UU^-UU^- &= U\left(U^-\right)^* UU^-\cup UU^- = Un_1\widetilde{H}UU^-\cup UU^-\\
&= UU^-n_1\widetilde{H}U^-\cup UU^- = UU^-Un_1\widetilde{H}\cup UU^-\\
&= U\left(U^-\right)^* Un_1\widetilde{H}\cup Un_1\widetilde{H}\cup UU^-\\
&= Un_1\widetilde{H}Un_1\widetilde{H}\cup Un_1\widetilde{H}\cup UU^-\\
&= UU^-n_1\widetilde{H}n_1\widetilde{H}\cup Un_1\widetilde{H}\cup UU^-\\
&= UU^-\left(\{1\}\cup n_1\widetilde{H}n_1\widetilde{H}\right)\cup Un_1\widetilde{H}.
\end{aligned}$$

□

Lemma 4.2. *Let G be a group with a split BN-pair (H,U,N) and Weyl group $W=\{1,s_1\}$. Using the notation of Lemma 4.1, the following conditions are equivalent:*

(a) $\left(U^-\right)^*\cap Un_1hU\neq\emptyset$ *for all* $h\in H$. *Equivalently* $H=\widetilde{H}$.
(b) $U\left(U^-\right)^*U=Un_1HU$.
(c) $G=\left(UU^-\right)^2$.

Proof. By definition $\widetilde{H}\subseteq H$ and by Lemma 4.1 (a), $U\left(U^-\right)^*U=Un_1\widetilde{H}U$. Hence (a) and (b) are equivalent. To finish the proof observe that

$$\begin{aligned}
G = B\cup Bn_1B = (B\cup Bn_1B)\,n_1 = Bn_1\cup Bn_1Bn_1 = Bn_1\cup BB^-\\
= Un_1H\cup UU^-H,
\end{aligned}$$

where the union on the r.h.s. is disjoint. Since the sets $\widetilde{H}$ and $n_1\widetilde{H}n_1$ are both contained in H, we have $Un_1\widetilde{H}\subseteq Un_1H$, and $UU^-\left(\{1\}\cup n_1\widetilde{H}n_1\widetilde{H}\right)\subseteq UU^-H$. Since $G=Un_1H\cup UU^-H$ is a disjoint union, Lemma 4.1 (b) implies that $G=\left(UU^-\right)^2$ if and only if $Un_1\widetilde{H}=Un_1H$ and $UU^-\left(\{1\}\cup n_1\widetilde{H}n_1\widetilde{H}\right)=UU^-H$. Thus, by Lemma 4.1 (a), we get that (c) implies (b), and it is also clear that (a) implies (c).

□

5. Groups with a σ-setup

Any $K \in \mathcal{L}ie$ has a split BN-pair (H, U, N), where U is a Sylow p-subgroup for the defining characteristic p, descending from the algebraic group $\overline{K}$ [7, Theorem 2.3.4]. More precisely, if $\overline{T} \subseteq \overline{B}$ is a pair of σ-stable maximal torus and Borel subgroup of $\overline{K}$ then $B = \overline{B} \cap K$ and $N = N_{\overline{K}}(\overline{T}) \cap K$ form a BN-pair for K and if $\overline{U}$ is the unipotent radical of $\overline{K}$, i.e., $\overline{B} = \overline{T} \ltimes \overline{U}$, then $B = H \ltimes U$ where $H = \overline{T} \cap K$ and $U = \overline{U} \cap K$ is a Sylow p-subgroup of K (as in [7, Section 3.4]).

Remark 5.1. *1.) Some groups in $\mathcal{L}ie$ have split BN-pairs for different primes p, e.g. $A_1(4) = A_1(5)$* [7, Theorem 2.2.10].

2.) For any simple reflection s, if $K \in \mathcal{L}ie(p)$ then $K_s \in \mathcal{L}ie(p)$, and if K is universal then so is K_s by [7, Theorem 2.6.5.(f)].

3.) Note also that if $\overline{K}$ is universal [7, Theorem 1.10.4] *then, by a result of Steinberg* [9, Theorem 24.15], *$K_u = C_{\overline{K}}(\sigma)$ so B, N, H and U are the centralizers of σ in $\overline{B}$, $\overline{N}$, $\overline{T}$ and $\overline{U}$ respectively.*

Lemma 5.2. *Let $K_u \in \mathcal{L}ie(p)$ be universal of rank 1 and let U be a Sylow p-subgroup of K_u. Then $K_u = (UU^-)^2$.*

Proof. First note that since K_u is universal, the corresponding algebraic group $\overline{K}_u$ is universal (or simply connected in a different terminology [7, Definition 1.10.5]). By Remark 5.1.3, K_u is a finite group of Lie type. The possible types for rank 1 are A_1, ${}^2A_2$, ${}^2B_2$ and ${}^2G_2$ (see for example [9, Table 23.1]) and the possibilities for $\overline{K}_u$ can be read off from [7, Theorem 1.10.7].

Let p be the defining characteristic of K_u. By [3, §1.19], we need to consider, for all powers q of p, the groups $\mathrm{SL}_2(q)$, $\mathrm{SU}_3(q^2)$, ${}^2B_2(q^2)$ if $p = 2$ and $q^2 = 2^{2n-1}$ for some $n \geq 0$ and ${}^2G_2(q^2)$ if $p = 3$ and $q^2 = 3^{2n-1}$ for some $n \geq 0$.

Now K_u satisfies the assumptions of Lemma 4.2, so, in particular we use the notation of Lemma 4.2. For $K_u = \mathrm{SL}_2(q)$ condition (a) of the lemma is easily verified - for the calculation see [4, §6.1]. For the remaining cases we use [5, Proposition 4.1]. By this result, for every $h \in H$ there exists $y \in U$ such that $yn_1 \in U^-hU = n_1^{-1}Un_1hU$. Multiplying by n_1^{-1} on the left, and using $\left(n_1^{-1}\right)^2 \in H$, we obtain $n_1^{-1}yn_1 \in Un_1\left(\left(n_1^{-1}\right)^2 h\right)U$. Observe that $1 \notin Un_1\left(\left(n_1^{-1}\right)^2 h\right)U$, and hence $n_1^{-1}yn_1 \in (U^-)^*$. Moreover, as h varies over H, so does $\left(n_1^{-1}\right)^2 h$. Hence, condition (a) of Lemma 4.2 holds for this case, and the claim follows. $\square$

Remark 5.3. *Note that the groups denoted by $\mathrm{SU}_n(q^2)$ in* [3, §1.19] *are denoted by $\mathrm{SU}_n(q)$ in* [9, Example 21.2]. *Note also that for the groups ${}^2B_2(2^{2n-1})$ the universal and the adjoint versions are isomorphic* [3, §1.19]. *Moreover since the center $Z(K_u)$ lies in $C_{Z(\overline{K}_u)}(\sigma)$* [9, Corollary 24.13] *it follows that $Z(K_u) = 1$ except if $K_u = \mathrm{SL}_2(q)$ and q is odd (here $Z(K_u) = Z_2$) or $K_u = \mathrm{SU}_3(q^2)$ and 3 divides $q+1$ (here $Z(K_u) = Z_3$). Excluding these exceptions, K_u is isomorphic to its adjoint version, i.e. $K_u \cong K_a$ and condition (a) of Lemma 4.2 can be checked with the calculation in* [4, §13.7].

The next lemma is an analogue for the split BN-pair setting, of an observation of [16].

Lemma 5.4. *If G is a group with a split BN-pair (H, U), then $H \cap UU^-U = \{1\}$.*

Proof. Let $h \in H \cap UU^{-}U$. Then $h \in u_1 U^{-} u_2$, with $u_1, u_2 \in U$. Equivalently, $u_1^{-1} h u_2^{-1} = h \left(u_1^{-1}\right)^h u_2^{-1} \in U^{-}$. But $h \left(u_1^{-1}\right)^h u_2^{-1} \in B = HU$, and hence $h \left(u_1^{-1}\right)^h u_2^{-1} \in B \cap U^{-} = \{1\}$ [3, Proposition 2.5.5(i)]. Using $H \cap U = \{1\}$ this gives $h = 1$. □

Proof of Theorem 1. We will show that $G = (UU^{-})^2$ for each $G \in \mathcal{L}ie$. As explained in Section 2, this set of groups includes all finite simple groups of Lie type (and some more). Since G satisfies the assumptions of Proposition 3.4, we can assume that G is in $\mathcal{L}ie$ of rank 1. Moreover, since there is a surjective homomorphism $K_u \to K$ which maps unipotent Sylows onto unipotent Sylows, we can assume that G is universal. A universal G in $\mathcal{L}ie$ of rank 1 satisfies $G = (UU^{-})^2$ by Lemma 5.2.

Suppose $H \neq 1$. By Lemma 5.4, $H \cap UU^{-}U = \{1\}$ and so, employing Lemma 3.1, G is a product of at least four unipotent Sylow subgroups, and hence $G = (UU^{-})^2$ is of minimal length. Suppose now that $H = 1$. By [2, Theorem 5], $G = BB^{-}B$, where $B = UH$ and $B = U^{-}H$. Substituting $H = 1$ gives $G = UU^{-}U$. Moreover, $H = 1$ if and only if U is self-normalizing. □

References

[1] L. Babai, N. Nikolov, and L. Pyber. Product growth and mixing in finite groups. In *Proceedings of the Nineteenth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 248–257. ACM, New York, 2008.

[2] J. Cannon, M. Garonzi, D. Levy, A. Maróti, and I. Simion. Groups equal to a product of three conjugates subgroups. 2014. To appear in Israel journal of Mathematics.

[3] R. W. Carter. *Finite groups of Lie type*. Pure and Applied Mathematics (New York). John Wiley & Sons, Inc., New York, 1985. Conjugacy classes and complex characters, A Wiley-Interscience Publication.

[4] R. W. Carter. *Simple groups of Lie type*. Wiley Classics Library. John Wiley & Sons Inc., New York, 1989. Reprint of the 1972 original, A Wiley-Interscience Publication.

[5] V. Chernousov, E. W. Ellers, and Nikolai G., Gauss decomposition with prescribed semisimple part: short proof. *J. Algebra*, 229(1):314–332, 2000.

[6] M. Garonzi, D. Levy, A. Maróti, and I. Simion. Minimal length products of unipotent Sylow subgroups in finite simple groups of Lie type. arXiv:1501.05678v1, 2015.

[7] D. Gorenstein, R. Lyons, and R. Solomon. *The classification of the finite simple groups. Number 3. Part I. Chapter A*, volume 40 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1998. Almost simple K-groups.

[8] M. W. Liebeck and L. Pyber. Finite linear groups and bounded generation. *Duke Math. J.*, 107(1):159–171, 2001.

[9] G. Malle and D. M. Testerman. *Linear algebraic groups and finite groups of Lie type*, volume 133 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2011.

[10] L. Pyber and E. Szabó. Growth in linear groups. In *Thin groups and superstrong approximation*, volume 61 of *Math. Sci. Res. Inst. Publ.*, pages 253–268. Cambridge Univ. Press, Cambridge, 2014.

[11] A. Smolensky. Unitriangular factorization of twisted Chevalley groups. *Internat. J. Algebra Comput.*, 23(6):1497–1502, 2013.

[12] A. Smolensky. Products of Sylow subgroups in Suzuki and Ree groups. arXiv: 1501.05234v1,arXiv:1501.05234v2, 2015.

[13] A. Smolensky, B. Sury, and N. Vavilov. Gauss decomposition for Chevalley groups, revisited. *Int. J. Group Theory*, 1(1):3–16, 2012.

[14] O. I. Tavgen′. Bounded generability of Chevalley groups over rings of S-integer algebraic numbers. *Izv. Akad. Nauk SSSR Ser. Mat.*, 54(1):97–122, 221–222, 1990.

[15] O. I. Tavgen′. Bounded generability of Chevalley groups over rings of S-integer algebraic numbers.Proceedings of the International Conference on Algebra Dedicated to the Memory of A. I. Mal'cev, Part 1, vol. CONM/131.1, Contemp. Math, 1992, pp. 409–421.

[16] N. A. Vavilov, A. V. Smolenskiĭ, and B. Sury. Unitriangular factorizations of Chevalley groups. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 388(Voprosy Teorii Predstavlenii Algebr i Grupp. 21):17–47, 309–310, 2011.

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