NORMAL COVERINGS OF LINEAR GROUPS

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Abstract. For a non-cyclic finite group G, let $\gamma(G)$ denote the smallest number of conjugacy classes of proper subgroups of G needed to cover G. In this paper we show that if G is in the range $\mathrm{SL}_n(q) \leq G \leq \mathrm{GL}_n(q)$ for $n > 2$, then $n/\pi^2 < \gamma(G) \leq (n+1)/2$. This result complements recent work of Bubboloni, Praeger and Spiga on symmetric and alternating groups. We give various alternative bounds, and derive explicit formulas for $\gamma(G)$ in some cases.

1. Introduction

1.1. **Normal coverings.** Let G be a non-cyclic finite group. We write $\gamma(G)$ for the smallest number of conjugacy classes of proper subgroups of G needed to cover it. In other words, $\gamma(G)$ is the least k for which there exist subgroups $H_1, \ldots, H_k < G$ such that

$$G = \bigcup_{i=1}^{k} \bigcup_{g \in G} {H_i}^g.$$

We say that the set of conjugacy classes $\{{H_i}^G \mid i = 1, \ldots, k\}$ is a *normal covering* for G.

Bubboloni and Praeger [7] have recently investigated $\gamma(G)$ in the case that G is a finite symmetric or alternating group. They show, for example, that if n is an odd composite number then

$$\frac{\phi(n)}{2} + 1 \leq \gamma(S_n) \leq \frac{n-1}{2},$$

where ϕ is Euler's totient function. Similar results are established for all values of n, and for both S_n and A_n. Part of the motivation for their work comes from an application in number theory.

It is a well-known theorem of Jordan that no finite group is covered by the conjugates of any proper subgroup. A paraphrase of this statement is that $\gamma(G) > 1$ for any finite group G. It is known that there exists a finite solvable group G with $\gamma(G) = k$ for every $k > 1$ [9]. It has been shown in [4] that if G is one of the groups $\mathrm{GL}_n(q), \mathrm{SL}_n(q), \mathrm{PGL}_n(q)$ or $\mathrm{PSL}_n(q)$, then $\gamma(G) = 2$ if and only if $n \in \{2, 3, 4\}$. (Notice that γ is undefined for $n = 1$, since the groups are cyclic in this case.) Other groups of Lie type possessing a normal covering of size 2 have been studied in [5] and [6].

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In this paper we give bounds on $\gamma(G)$, where $\mathrm{SL}_n(q) \leq G \leq \mathrm{GL}_n(q)$, for all values of n. In some cases we are able to give an exact value. Our bounds extend without change to $G/Z(G)$.

We introduce some notation. We write $\lfloor x \rfloor$ for the integer part of a real number x. As already noted above, ϕ denotes Euler's function. We shall also use Lehmer's *partial totient function*, which we define here.

Definition. Let k and t be such that $0 \leq t < k < n$. We define the partial totient $\phi(k,t,n)$ to be the number of integers x, coprime with n, such that

$$\frac{nt}{k} < x < \frac{n(t+1)}{k}.$$

We give two separate upper bounds on $\gamma(G)$.

Theorem 1.1. *Let $n \in \mathbf{N}$, and let $\nu = \nu(n)$ be the number of prime factors of n. Let $p_1, \ldots, p_\nu$ be the distinct prime factors of n, with $p_1 < p_2 < \cdots < p_\nu$. Let G be a group such that $\mathrm{SL}_n(q) \leq G \leq \mathrm{GL}_n(q)$. Then*

(1) *If $\nu \geq 2$ then*

$$\gamma(G) \leq \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\frac{n}{2} + 2.$$

(2) *If $n > 6$ then*

$$\gamma(G) \leq \left\lfloor \frac{n}{3} \right\rfloor + \phi(6,2,n) + \nu.$$

A great deal of information is given in [14, §6] about the function $\phi(6,t,n)$, from which the following statement can be derived.

$$\frac{\phi(n)}{6} - \phi(6,2,n) = \begin{cases} 0 & \text{if } n \text{ is divisible either by 9, or by a prime} \\ & \quad \text{of the form } 3k+1 \text{ for } k \in \mathbf{N}, \\ \frac{1}{12}\lambda(n)2^\nu & \text{otherwise, if } n \text{ is divisible by 3,} \\ \frac{1}{6}\lambda(n)2^\nu & \text{otherwise, if } n \text{ is not divisible by 3,} \end{cases}$$

in which $\lambda(n) = (-1)^\ell$, where ℓ is the number of prime divisors of n counted with multiplicity.

1.2. **Independent sets of conjugacy classses.** Let $\kappa(G)$ be the size of the largest set of conjugacy classes of G such that any pair of elements from distinct classes generates G. We call such a set an *independent set of classes*. Guralnick and Malle [12] have shown that $\kappa(G) \geq 2$ for any finite simple group G. It is clear that whenever $\gamma(G)$ is defined, we have the inequality

$$\kappa(G) \leq \gamma(G),$$

since if $\mathcal{C}$ is a normal covering of G, and if $\mathcal{I}$ is a independent set of classes, then each element of $\mathcal{C}$ covers at most one element of $\mathcal{I}$.

We establish two lower bounds for $\kappa(G)$. By the observation of the previous paragraph, these also operate as lower bounds for $\gamma(G)$.

Theorem 1.2. *Let $n \in \mathbf{N}$, and let $\nu = \nu(n)$ be the number of prime factors of n. Let $p_1, \ldots, p_\nu$ be the distinct prime factors of n, with $p_1 < p_2 < \cdots < p_\nu$. Let G be a group such that $\mathrm{SL}_n(q) \leq G \leq \mathrm{GL}_n(q)$.*

(1) *If* $\nu \geq 2$ *then*

$$\frac{\phi(n)}{2} + \nu(n) \leq \kappa(G).$$

(2) *If* $\nu \geq 3$*, and if* n *is not equal to* $6p$ *or* $10p$ *for any prime* p*, then*

$$\left\lfloor \frac{n+6}{12} \right\rfloor + \phi(12, 1, 3n) + \nu \leq \kappa(G).$$

Furthermore, if $\mathrm{hcf}(n, 6) = 1$ *then*

$$\left\lfloor \frac{n+6}{12} \right\rfloor + \phi(12, 1, 3n) + \phi(12, 0, n) + \nu \leq \kappa(G).$$

The values $t = 0, 1$ are not amongst those for which the function $\phi(12, t, n)$ is evaluated explicitly in [14]. However, Theorem 10 of [14] gives the following general estimate,

$$|\phi(n) - k\phi(k, t, n)| \leq (k-1)2^{\nu},$$

where ν is the number of prime divisors of n. This yields the lower bound

$$\phi(12, t, n) \geq \frac{\phi(n)}{12} - \frac{11}{12}2^{\nu}.$$

There are certain cases in which an upper bound for $\gamma(G)$ coincides with a lower bound for $\kappa(G)$. In these cases we must have $\gamma(G) = \kappa(G)$, and we obtain a precise formula.

Theorem 1.3. *Let* G *be a group such that* $\mathrm{SL}_n(q) \leq G \leq \mathrm{GL}_n(q)$*.*

(1) *If* $n = p^a$*, where* p *is a prime and* $a \in \mathbf{N}$*, and if* $n > 2$*, then*

$$\gamma(G) = \kappa(G) = \left(1 - \frac{1}{p}\right)\frac{n}{2} + 1.$$

(2) *If* $n = p^a q^b$ *where* p *and* q *are distinct primes and* $a, b \in \mathbf{N}$*, then*

$$\gamma(G) = \kappa(G) = \left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{q}\right)\frac{n}{2} + 2.$$

(3) *If* $n = 6p$ *where* p *is a prime, then* $\gamma(G) = \kappa(G) = p + 2$*.*

(4) *If* $n = 10p$ *where* p *is a prime, then* $\gamma(G) = \kappa(G) = 2p + 2$*.*

Certain cases of Theorem 1.3 will require independent treatment, as they arise as exceptional cases in the proof of Theorem 1.2.

1.3. **Linear bounds.** Theorem 1.1 (1), Theorem 1.2 (2), and Theorem 1.3, taken together, imply that

$$\frac{n}{12} < \kappa(G) \leq \gamma(G) \leq \frac{n+1}{2}, \tag{1}$$

for all $n > 2$. The upper bound is exact when n is an odd prime. (When $n = 2$ it is known that $\gamma(G) = 2$; see [4], or the remark after Proposition 4.1 below. It is also easy to show that $\kappa(G) = 2$ in this case.) It follows immediately that

$$\limsup \frac{\gamma(G)}{n} \;=\; \frac{1}{2}. \tag{2}$$

The lower bound for γ can be improved, as the following theorem indicates.

Theorem 1.4. *Let* G *be a group such that* $\mathrm{SL}_n(q) \leq G \leq \mathrm{GL}_n(q)$*. Then* $\frac{n}{\pi^2} < \gamma(G)$*.*

From the first part of Theorem 1.1 and from Theorem 1.4, it is easy to show that

$$\frac{1}{\pi^2} \leq \liminf \frac{\gamma(G)}{n} \leq \frac{1}{6}. \tag{3}$$

It follows from the theorems which we have stated, that $\gamma(G)$ and $\kappa(G)$ are bounded above and below by monotonic functions which grow linearly with n. It appears that the situation for symmetric groups is similar. It has been announced in [3, §1.1], and will be demonstrated in a forthcoming paper [8] now in preparation, that $\gamma(S_n)$ and $\gamma(A_n)$ are bounded above and below by linear functions of n. In fact the numbers $\gamma(S_n)$ and $\gamma(\mathrm{GL}_n(q))$ seem to be closely related; in all cases where both are known exactly, they differ by at most 1. It is not hard to show, and it is worth remarking in this connection, that the upper bounds stated for $\gamma(G)$ in Theorem 1.1 are also upper bounds for $\gamma(S_n)$, improving marginally on those of [7, Theorem A]. It should also be noted that all of our bounds are independent of the field size q.

We establish the upper bounds of Theorem 1.1 in Section 2, by exhibiting explicit normal coverings of the necessary sizes. This builds on work described in [2], in which coverings of $\mathrm{GL}_n(q)$ by proper subgroups are constructed. The two lower bounds of Theorem 1.2 are proved in Section 3. Both are proved by exhibiting an independent set of classes. This requires an account of overgroups of certain special elements in $\mathrm{GL}_n(q)$. For such an account we rely on [11], which provides a classification of subgroups whose orders are divisible by primitive prime divisors of q^d-1, for all $d > n/2$. The remaining cases of Theorem 1.3 are brought together in Section 4. Finally, Theorem 1.4 is established in Section 5. Its proof relies on work from the doctoral thesis of Joseph DiMuro [10], which extends the classification of [11] to cover all $d \geq n/3$.

The classes of subgroups in our normal covering remain distinct, proper and non-trivial in the quotient of G by $Z(G)$. This is true also of the classes of maximal overgroups which cover the conjugacy classes in our independent sets. It follows that the bounds which we have stated for $\gamma(G)$ and for $\kappa(G)$ hold equally for $\gamma(G/Z(G))$ and for $\kappa(G/Z(G))$.

2. Normal coverings of G

We shall write V for the space ${\mathbf{F}_q}^n$. Throughout the paper, we assume that $\mathrm{SL}(V) \leq G \leq \mathrm{GL}(V)$.

We begin by introducing the classes of subgroups which we shall need for our coverings. Proposition 2.1 below contains standard information about certain subgroups of $\mathrm{GL}_n(q)$, and we shall not prove it here.

Proposition 2.1. (1) *Let d be a divisor of n. There exist embeddings of $\mathrm{GL}_{n/d}(q^d)$ into $\mathrm{GL}_n(q)$. All such embeddings are conjugate by elements of $\mathrm{SL}_n(q)$, and each has index d in its normalizer in $\mathrm{GL}_d(q)$. If d is prime then the normalizer is a maximal subgroup of $\mathrm{GL}_n(q)$.*

(2) *Suppose that $1 \leq k < n$, and let U be a k-dimensional subspace of V. Then the set stabilizer G_U of U in G is a maximal subgroup of G. If W is another k-dimensional subspace, then G_U and G_W are conjugate in G.*

It will be convenient to have concise notation for these subgroups.

Definition. (1) We refer to the maximal subgroups of Proposition 2.1 (1) as *extension field subgroups* of degree d, and we write $\mathrm{efs}(d)$ for the conjugacy class consisting of the intersections of all such subgroups with the group G.
(2) We refer to the subgroups of Proposition 2.1 (2) as *subspace stabilizers* of dimension k, and we write $\mathrm{ss}(k)$ for the conjugacy class consisting of all such subgroups.

The following technical lemma will be useful.

Lemma 2.2. (1) *Suppose that $X \in \mathrm{GL}(V)$, and that X stabilizes a k-dimensional subspace of V. Then X stabilizes a subspace whose dimension is $n-k$.*
(2) *Let $X \in \mathrm{GL}(V)$, and let p be a prime dividing n. If X lies in no extension field subgroup of degree p, then it stabilizes a subspace of V whose dimension is coprime with p.*

Proof. (1) Suppose X stabilizes a space U of dimension k. Then the transpose X^t acts on the dual space V^*, and stabilizes the annihilator of U, which has dimension $n-k$.
(2) If X stabilizes no subspace whose dimension is coprime with p, then every irreducible divisor of its characteristic polynomial has degree divisible by p, and must therefore split into p factors over $\mathbf{F}_{q^p}$. Suppose that the elementary divisors of X are $f_1^{a_1}, \ldots, f_t^{a_t}$. For each i, let g_i be an irreducible factor of f_i over $\mathbf{F}_{q^p}$, and let $Y \in \mathrm{GL}_{n/p}(q^p)$ have elementary divisors $g_1^{a_1}, \ldots, g_t^{a_t}$. Then it is not hard to see that any embedding of $\mathrm{GL}_{n/p}(q^p)$ into $\mathrm{GL}_n(q)$ must map Y to a conjugate of X.

□

We are now in a position to exhibit some normal coverings of G.

Lemma 2.3. (1) *Let p be a prime dividing n. Then there is a normal covering $\mathcal{C}_p$ for G given by*

$$\mathcal{C}_p = \{\mathrm{efs}(p)\} \cup \{\mathrm{ss}(k) \mid 1 \le k \le n/2,\ p \nmid k\}.$$

The size of $\mathcal{C}_p$ is

$$|\mathcal{C}_p| = \left\lfloor \left(1 - \frac{1}{p}\right) \frac{n}{2} \right\rfloor + 1 + \epsilon,$$

where

$$\epsilon = \begin{cases} 1 & \text{if } p = 2 \text{ and } n/2 \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

This is minimized when p is the smallest prime divisor of n.
(2) *Let p_1 and p_2 be distinct prime divisors of n. Then there is a normal covering $\mathcal{C}_{p_1,p_2}$ for G given by*

$$\mathcal{C}_{p_1,p_2} = \{\mathrm{efs}(p_1), \mathrm{efs}(p_2)\} \cup \{\mathrm{ss}(k) \mid 1 \le k < n/2,\ p_1, p_2 \nmid k\}.$$

The size of $\mathcal{C}_{p_1,p_2}$ is

$$|\mathcal{C}_{p_1,p_2}| = \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\frac{n}{2} + 2.$$

This is minimized when p_1 and p_2 are the two smallest prime divisors of n.

Proof. The sizes of the sets $\mathcal{C}_p$ and $\mathcal{C}_{p_1,p_2}$ are easily seen to be as stated. That $\mathcal{C}_p$ is a normal covering follows immediately from Lemma 2.2. So it remains only to prove that C_{p_1,p_2} is a normal covering.

Let $X \in G$, let f_X be the characteristic polynomial of X, and let $g_1, \ldots, g_s$ be the irreducible factors of f_X over $\mathbf{F}_q$, with degrees $d_1, \ldots, d_s$ respectively. Then clearly there exist X-invariant subspaces $U_1, \ldots, U_s$ such that $\dim U_i = d_i$ for all i, and such that $U_i \cap U_j = \{0\}$ whenever $i \neq j$. If any d_i is divisible by neither of the primes p_1 and p_2, then X is contained in a subspace stabilizer from one of the classes in $\mathcal{C}_{p_1,p_2}$. So we assume that each d_i is divisible by at least one of p_1 or p_2. Suppose that d_a is divisible by p_1 but not by p_2, and that d_b is divisible by p_2 but not by p_1. Then $U_a \oplus U_b$ is an X-invariant subspace, and its dimension is coprime with p_1 and p_2; so again, X is in a subspace stabilizer from $\mathcal{C}_{p_1,p_2}$. But if no such d_a and d_b can be found, then either all of the d_i are divisible by p_1, or they are all divisible by p_2. In this case, X lies in an extension field subgroup either of degree p_1 or of degree p_2. □

We note that the argument of the last paragraph of this proof does not extend to the case of three primes, p_1, p_2, p_3. It is possible to find matrices whose invariant subspaces all have dimensions divisible by one of those primes, but which lie in no extension field subgroup. In the case that the primes are $2, 3$ and 5, for instance, there are 30-dimensional matrices whose irreducible invariant spaces have dimensions 2, 3 and 25. (Another example is used in the proof of Proposition 4.4 below.) This is the explanation for the appearance of the two smallest prime divisors of n in the first upper bound of Theorem 1.1, which may at first seem a little curious.

The second upper bound of Theorem 1.1 is proved in a somewhat similar fashion.

Lemma 2.4. *Let $p_1, \ldots, p_\nu$ be the distinct primes dividing n. Then there is a normal covering $\mathcal{D}$ of G given by*

$$\begin{aligned}\mathcal{D} = &\{\mathrm{ss}(k) \mid 1 \leq k \leq n/3\}\\ &\cup\ \{\mathrm{ss}(k) \mid n/3 < k \leq n/2,\ \mathrm{hcf}(k,n) = 1\}\\ &\cup\ \{\mathrm{efs}(p_i) \mid 1 \leq i \leq \nu\}.\end{aligned}$$

For $n > 6$, the size of $\mathcal{D}$ is

$$\left\lfloor \frac{n}{3} \right\rfloor + \phi(6,2,n) + \nu.$$

Proof. Let $X \in G$. Suppose that X is reducible, and that its smallest non-trivial invariant subspace has dimension k. If $k > n/3$ then it is not hard to see (for instance, by considering the irreducible factors of the characteristic polynomial) that X stabilizes at most one other proper non-trivial subspace, of dimension $n-k$. It follows that if p is a prime dividing both n and k, then X is contained in an element of $\mathrm{efs}(p)$. It is now a straightforward matter to show that $\mathcal{D}$ is a normal covering, and we omit further details. The size of $\mathcal{D}$ follows immediately from its definition. □

3. Lower bounds for $\kappa(G)$

Recall that $\mathrm{GL}_n(q)$ contains elements of order $q^n - 1$, often known as Singer elements. Such elements stabilize no non-trivial proper subspace of V. The determinant of a Singer element generates the multiplicative group of $\mathbf{F}_q$.

In order to handle all groups G in the range $\mathrm{SL}_n(q) \leq G \leq \mathrm{GL}_n(q)$ together, we define a parameter $\alpha \in \mathbf{N}$ as follows.

$$\alpha = \begin{cases} 0 & \text{if } G = \mathrm{SL}_n(q), \\ -|\mathrm{GL}_n(q) : G| & \text{otherwise.} \end{cases}$$

Let ζ be a generator of the multiplicative group of $\mathbf{F}_q$. Then we have

$$\frac{G}{\mathrm{SL}_d(q)} \cong \langle \zeta^\alpha \rangle.$$

Definition. (1) For $d = 1, \ldots, n$, let Γ_d be a Singer element with determinant ζ in $\mathrm{GL}_d(q)$.

(2) For $k < n/2$, define

$$\Sigma_k = \mathrm{diag}({\Gamma_k}^{\alpha-1}, \Gamma_{n-k}).$$

(3) For $j < (n-2)/4$, define

$$T_j = \mathrm{diag}({\Gamma_j}^{\alpha-2}, \Gamma_{j+1}, \Gamma_{n-2j-1}).$$

The reasons for defining α as above will be clear from the following remark.

Remark. (1) Since $\det \Sigma_k = \det T_j = \zeta^\alpha$, we have $\Sigma_k, T_j \in G$.

(2) It is clear from the definition of α that $(1-q) < \alpha \leq 0$, and hence that $|\alpha - 2| < q + 1$. It follows easily that the actions of the matrices ${\Gamma_k}^{\alpha-1}$ and ${\Gamma_j}^{\alpha-2}$ are irreducible for all k and j. Therefore the module $\mathbf{F}_q\langle\Sigma_k\rangle$ decomposes into precisely two irreducible summands, and $\mathbf{F}_q\langle T_j\rangle$ decomposes into precisely three irreducible summands.

Lemma 3.1. *Suppose that $n > 4$. Let $k < n/2$, and if $q = 2$ then suppose that $n - k \neq 6$. Let $j < (n-2)/4$, and if $q = 2$ then suppose that $n - 2j - 1 \neq 6$.*

(1) *If M is a maximal subgroup of G containing Γ_n then M is an extension field subgroup of prime degree.*

(2) *If M is a maximal subgroup of G containing Σ_k then M is either an extension field subgroup whose degree is a prime divisor of $\gcd(k, n)$, or else the stabilizer of a subspace of dimension k or $n - k$.*

(3) *Let n have at least 3 distinct prime divisors. If M is a maximal subgroup of G containing T_j, then M is the stabilizer of a subspace whose dimension is one of j, $j+1$, $2j+1$, $n-2j-1$, $n-j-1$ or $n-j$.*

Proof. Part (1) of the lemma is a result of Kantor [13].

For $(n, q) \neq (11, 2)$, part (2) of the lemma follows from part (2) of Theorem 4.1 of [2]. However a few comments are to be made about this assertion. The matrix that we have called Σ_k is referred to as GLk in [2]. The result in [2] is stated only for the groups $\mathrm{GL}_n(q)$ and $\mathrm{SL}_n(q)$, but the proof given there applies equally to intermediate subgroups. Finally, the proof in [2] relies on the existence of primitive prime divisors of $q^{n-k} - 1$ (where $n - k > 2$), which is given by Zsigmondy's Theorem [17] for all pairs $(q, n-k)$ except $(2, 4)$ and $(2, 6)$; the second of these exceptions accounts for the excluded case in the statement of the present lemma. The argument uses the classification in [11], of subgroups of $\mathrm{GL}_n(q)$ whose order is divisible by a prime divisor of $q^e - 1$, where $e > n/2$.

To finish the proof of part (2) of the present lemma, we must consider the exceptional case of the group $\mathrm{GL}_{11}(2)$. In this case we require a reference directly to the lists of [11]. We find that there are several irreducible subgroups whose

order is divisible by a primitive prime divisor 11 of $2^{10}-1$; we must show that none of these contains Σ_1. All of these subgroups are almost simple, and have a socle which is isomorphic either to one of the Mathieu groups M_{23} or M_{24}, or to the unitary group $\mathrm{PSU}_5(2)$, or to a linear group $\mathrm{SL}_2(11)$ or $\mathrm{SL}_2(23)$. (These subgroups may be found in Table 5 (lines 12 and 14) and Table 8 (lines 2, 7 and 9) of [11].) Information about these groups may be found in [1]. None of these groups themselves, nor any of their outer automorphism groups, have order divisible by 31. Therefore an almost simple group of one of these types can contain no element of order $2^{10}-1 = 3\cdot 11\cdot 31$, which is the order of the element Σ_1.

For the proof of part (3) of the lemma, we refer once again to the classification of [11], this time for matrix groups whose order is divisible by a primitive prime divisor of $q^{n-2j-1}-1$. It is not hard to see that T_j has no overgroups of classical type. The condition that n has 3 distinct prime divisors rules out the small dimensional sporadic examples contained in Tables 1–7. Other examples are ruled out because their order is less than $q^{n-2j-1}-1$, which is the order of the summand Γ_{n-2j-1} of T_j. □

We define a set of classes which will help us to establish the first of our lower bounds for $\kappa(G)$.

Definition. Define a set Φ of classes of G by

$$\Phi = \{[\Sigma_p] \mid p|n,\ p \text{ prime}, p < n/2\} \cup \{[\Sigma_k] \mid k < n/2,\ \mathrm{hcf}(n,k) = 1\},$$

where $[g]$ denotes the conjugacy class of g.

Lemma 3.2. *Let $n > 2$, and let $\nu(n)$ be the number of prime factors of n. Then*

$$|\Phi| = \phi(n)/2 + \nu(n) - \epsilon,$$

where

$$\epsilon = \begin{cases} 1 & \text{if } n = 2p \text{ for some odd prime } p, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. This is immediate from the definition of Φ. □

Lemma 3.2, together with the following two lemmas, will imply the first part of Theorem 1.2.

Lemma 3.3. *Φ is an independent set of classes.*

Proof. Suppose that $q \neq 2$, or that $[\Sigma_{n-6}] \notin \Phi$. Then Lemma 3.1 provides full information about the maximal subgroups of G which contain elements of Φ, and it is easy to check that the result holds in this case.

Next suppose that $q = 2$ and $[\Sigma_{n-6}] \in \Phi$. (This implies that $n \in \{7, 8, 9, 11\}$.) Lemma 3.1 gives full information about the maximal subgroups of G covering elements of the classes in Φ other than $[\Sigma_{n-6}]$. No class of subgroups contains elements of more than one such class, and it is easy to check that none covers the element Σ_{n-6} itself. □

Lemma 3.4. *Let $n = 2p$ where $p > 2$ is a prime. Then $\kappa(G) \geq |\Phi| + 1$.*

Proof. The proof of Lemma 3.3 shows that in any normal covering of G, the distinct classes in Φ are covered by distinct classes of subgroups. We add an extra conjugacy class to Φ, namely the class represented by $\Sigma_p = \mathrm{diag}({\Gamma_p}^{\alpha-1}, \Gamma_p)$, where Γ_p is a Singer element in $\mathrm{GL}_p(q)$. This element stabilizes no subspace of dimension k for

any k coprime with n; nor does it stabilize a subspace of dimension 2 or $n-2$. Therefore, by part (2) of Lemma 3.1, if $\Phi \cup \{[\Sigma_p]\}$ is not an independent set of classes, then Σ_p must lie in a subgroup in efs(2).

Note that since 2 and p are coprime, ${\Sigma_p}^2$ has two irreducible summands of dimension p. It is not hard to show that these submatrices are not conjugate, and neither of them is reducible over $\mathbf{F}_{q^2}$; it follows that ${\Sigma_p}^2$ is not contained in any embedding of $\mathrm{GL}_p(q^2)$ into G. Hence Σ_p itself is not contained in an embedding of $\mathrm{GL}_p(q^2) \cdot 2$. □

Lemmas 3.2, 3.3 and 3.4 complete the proof of part (1) of Theorem 1.2.

We define a second independent set of classes which yields the second lower bound of Theorem 1.2. We shall require the following lemma.

Lemma 3.5. *Let p be a prime divisor of n. Suppose that n has at least 3 distinct prime divisors, and that n is not equal to $6q$ or $10q$ for any prime q. Then there exists an integer w_p such that $(n-2)/4 \leq w_p < n/2$, and such that w_p is divisible by p, and by no other prime divisor of n. If $p \neq 3$ then w_p may be chosen so that it is not divisible by 3.*

Proof. Bertrand's Postulate states that for every $k > 3$ there is a prime r such that $k < r < 2k-2$. The conditions on n imply that $n \geq 12p$. So there is a prime $r > 3$ such that

$$\frac{n}{4p} < r < \frac{n}{2p}.$$

If r is not itself a prime divisor of n, or if it is equal to p, then we may take $w_p = pr$. On the other hand, if r is a prime divisor of n other than p then clearly $n = 3pr$, and since we have assumed that $n \geq 12p$, we have $r \geq 5$. Now we see that there exists m equal either to $r+1$ or to $r+2$, such that m is not divisible by 3, and we may take $w_p = pm$. □

Definition. Let n be a number with at least 3 distinct prime divisors, and not equal to $6p$ or $10p$ for any prime p. We define a set Ψ of classes of G by

$$\begin{aligned}\Psi = &\{[T_j] \mid j < (n-2)/4, j \equiv 1 \bmod 3\}\\ &\cup\ \{[\Sigma_k] \mid n/4 < k < n/2,\ \mathrm{hcf}(3n,k) = 1\}\\ &\cup \{[\Sigma_{6b}] \mid b < n/12, \mathrm{hcf}(n,6b) = 1\}\\ &\cup\ \{[\Sigma_{w_p}] \mid p|n,\ p \text{ prime}\},\end{aligned}$$

where w_p is as constructed in Lemma 3.5, and where $[g]$ denotes the conjugacy class of g.

To describe the size of the set Ψ we use Lehmer's partial totient function $\phi(k,t,n)$, which was defined before the statement of Theorem 1.1 above.

Lemma 3.6. *Let n have ν distinct prime divisors, where $\nu \geq 3$, and suppose that n is not equal to $6p$ or $10p$ for any prime p.*

(1) *If 2 or 3 divides n, then*

$$|\Psi| = \left\lfloor \frac{n+6}{12} \right\rfloor + \phi(12,1,3n) + \nu.$$

(2) *If* $\mathrm{hcf}(n,6)=1$, *then*

$$|\Psi| = \left\lfloor \frac{n+6}{12} \right\rfloor + \phi(12,1,3n) + \phi(12,0,n) + \nu.$$

Proof. We write $\lceil x \rceil$ for the least integer not less than x. The size X of the set $\{[T_j] \mid j < (n-2)/4, j \equiv 1 \bmod 3\}$ is $\lceil N/3 \rceil$, where $N = \lfloor (n-2)/4 \rfloor$. By examining residues modulo 12, it is not hard to show that $X = \lfloor (n+6)/12 \rfloor$, the first term in our sum.

It is immediate from the definition of the function $\phi(k,t,n)$ that the size of the set $\{[\Sigma_k] \mid n/4 < k < n/2,\ \mathrm{hcf}(3n,k)=1\}$ is $\phi(12,1,3n)$. We observe that the set $\{[\Sigma_{6b}] \mid b < n/12, \mathrm{hcf}(n,6b)=1\}$ is empty if $\mathrm{hcf}(n,6) \neq 1$; otherwise it has size $\phi(12,0,n)$. And clearly the set $\{[\Sigma_{w_p}] \mid p|n,\ p \text{ prime}\}$ has size ν as required. □

To establish the second lower bound in Theorem 1.2, it will suffice to show that any normal covering for G has size at least $|\Psi|$. This is done in the following lemma.

Lemma 3.7. *Let n have at least 3 distinct prime divisors, and not equal to $6p$ or $10p$ for any prime p. Then Ψ is an independent set of classes.*

Proof. Lemma 3.1 describes the maximal subgroups of G which contain elements of the classes in Ψ. The elements T_j lie only in members of $\mathrm{ss}(\ell)$ or $\mathrm{ss}(n-\ell)$, where $\ell \in \{j, j+1, 2j+1\}$. Notice that if $\ell > n/4$ then $\ell = 2j+1$, and hence $\ell \equiv 3 \bmod 6$. The elements Σ_k, where k is coprime with n, lie only in members of $\mathrm{ss}(k)$ or $\mathrm{ss}(n-k)$. And the elements Σ_{w_p} lie in subspace stabilizers and also in elements of $\mathrm{efs}(p)$. It is easy to check that the values permitted for j, k, b and w_p ensure that no two elements of distinct classes in Ψ stabilize subspaces of the same dimension. Therefore no two classes in Ψ can be covered by a single class of subgroups. □

4. Several equalities

In this section we establish the various claims of Theorem 1.3. We do this simply by comparing upper and lower bounds from earlier parts of the paper.

Proposition 4.1. *If $n = p^a$, where p is a prime and $a \in \mathbf{N}$, and if $n > 2$, then*

$$\gamma(G) = \kappa(G) = \left(1 - \frac{1}{p}\right)\frac{n}{2} + 1.$$

Proof. Lemma 2.3 and Lemma 3.3 together tell us that

$$|\Phi| \leq \kappa(G) \leq \gamma(G) \leq |\mathcal{C}_p|.$$

But it is easy to check, using Lemma 3.2, that $|\Phi| = |\mathcal{C}_p|$, and that this number is as claimed in the proposition. □

Remark. If $n = 2$, then the covering $\mathcal{C}_2$ has size 2. Since no finite group is covered by a single class of proper subgroups, it follow that $\gamma(G) = 2$ in this case.

Proposition 4.2. *If $n = p^a q^b$ where p and q are distinct primes and $a, b \in \mathbf{N}$, then*

$$\gamma(G) = \kappa(G) = \left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{q}\right)\frac{n}{2} + 2.$$

Proof. As in the proof above, Lemma 2.3 with Lemmas 3.3 and 3.4 yield that

$$|\Phi| + \epsilon \leq \kappa(G) \leq \gamma(G) \leq |\mathcal{C}_{p,q}|,$$

where $\epsilon = 1$ if $n = 2p$ (or $n = 2q$), and $\epsilon = 0$ otherwise. But we see that $|\Phi| + \epsilon = |\mathcal{C}_{p,q}|$, with this number being as claimed in the proposition. □

Proposition 4.3. *If $n = 6p$ where p is a prime, then*

$$\gamma(G) = \kappa(G) = p + 2.$$

Proof. In this case we have

$$|\Phi| \leq \kappa(G) \leq \gamma(G) \leq |\mathcal{C}_{2,3}|,$$

and it is easy to calculate that $|\Phi| = |\mathcal{C}_{2,3}| = p + 2$. □

Proposition 4.4. *If $n = 10p$ where p is a prime, then*

$$\gamma(G) = \kappa(G) = 2p + 2.$$

Proof. If p is 2 or 5 then the result follows from Proposition 4.2; if $p = 3$ then it follows from Proposition 4.3. So we may assume that $p > 5$. Then we have

$$|\Phi| \leq \kappa(G) \leq \gamma(G) \leq |\mathcal{C}_{2,5}|,$$

but in this case we see that $|\Phi| = 2p + 1$ whereas $|\mathcal{C}_{2,5}| = 2p + 2$. To prove that the upper bound is sharp for $\kappa(G)$, it will be sufficient to exhibit an element Y of G which cannot be covered by any class of subgroups containing an element of any conjugacy class in Φ. We define

$$Y = \mathrm{diag}({\Gamma_p}^{\alpha-2}, \Gamma_5, \Gamma_{n-p-5}).$$

Notice that $n - p - 5$ is even, and coprime with 5 and with p. It follows that Y does not stabilize a subspace of dimension coprime with n. But certainly Y lies in no extension field subgroup, and so it satisfies the required condition. □

5. Proof of Theorem 1.4

For a positive integer n, let $f(n)$ be the number of partitions of n with exactly three parts. By an elementary counting argument the following formula can be found for $f(n)$.

Lemma 5.1.

$$f(n) = \begin{cases} \frac{1}{12}(n-1)(n-2) + \frac{1}{2}\lfloor (n-1)/2 \rfloor & \text{if } 3 \nmid n, \\ \frac{1}{12}(n-1)(n-2) + \frac{1}{2}\lfloor (n-1)/2 \rfloor + \frac{1}{3} & \text{if } 3 \mid n. \end{cases}$$

It follows from Lemma 5.1 that

$$\left| f(n) - \frac{n^2}{12} \right| \leq \frac{1}{3}.$$

We define $\epsilon_n = f(n) - n^2/12$.

Let $P(n)$ be the set of partitions of n into three parts having no common divisor greater than 1. Let $g(n) = |P(n)|$. Then we have $f(n) = \sum_{d|n} g(d)$. By the Möbius Inversion Formula, we obtain

$$\begin{aligned} g(n) &= \sum_{d|n} \mu(d) f(n/d) = \left(\sum_{d|n} \mu(d) \frac{1}{12} (n/d)^2 \right) + \left(\sum_{d|n} \mu(d) \epsilon_{n/d} \right) \\ &> \frac{n^2}{12} \left(\sum_{d|n} \frac{\mu(d)}{d^2} \right) + \left(\sum_{d|n} \mu(d) \epsilon_{n/d} \right) \\ &> \frac{n^2}{12} \left(\prod_{p \text{ prime}} \left(1 - \frac{1}{p^2} \right) \right) + \left(\sum_{d|n} \mu(d) \epsilon_{n/d} \right). \end{aligned}$$

Since

$$\prod_{p \text{ prime}} \left(1 - \frac{1}{p^2} \right) = \frac{6}{\pi^2},$$

we have

$$g(n) > \left(\frac{n^2}{2\pi^2} \right) + \left(\sum_{d|n} \mu(d) \epsilon_{n/d} \right).$$

Now since the number of divisors of n is less than $2\sqrt{n}$, we obtain the following lemma.

Lemma 5.2.

$$\frac{n^2}{2\pi^2} - \frac{2}{3}\sqrt{n} < g(n).$$

The next lemma is the principal step in our proof. It gives information about the maximal overgroups in G, of an element of the form $\mathrm{diag}({\Gamma_a}^{\alpha-2}, \Gamma_b, \Gamma_c)$, where the degrees a, b and c are coprime. The proof relies on knowledge of the subgroups of $\mathrm{GL}_n(q)$ whose order is divisible by a primitive prime divisor of $q^d - 1$, where $d > n/3$. An account of such subgroups has been given in the doctoral dissertation of Joseph DiMuro [10]; this work extends the classification of [11], which deals with the case $d > n/2$.[1]

Lemma 5.3. *Let $\nu(n) \geq 3$ and let $n \geq 98$. For $\lambda = (a, b, c) \in P(n)$, with $a \leq b \leq c$, and with a, b, c coprime, let $g = g_\lambda = \mathrm{diag}({\Gamma_a}^{\alpha-2}, \Gamma_b, \Gamma_c)$. Then every maximal overgroup M of g in G is a subspace stabilizer, except possibly in the following cases.*

(i) *$2|n$, $c = n/2$, and $M \cong G \cap (\mathrm{GL}_{n/2}(q) \wr C_2)$;*
(ii) *$4|n$, $(a, b, c) = (2, (n-2)/2, (n-2)/2)$, and either $M \cong G \cap (\mathrm{GL}_{n/2}(q) \wr C_2)$, or $M \cong G \cap (\mathrm{GL}_{n/2}(q) \circ \mathrm{GL}_2(q))$. (Here $\circ$ is used to denote a central product.)*

[1] DiMuro's dissertation aims to classify elements of $\mathrm{GL}_n(q)$ of prime power order which act faithfully and irreducibly on a subspace of dimension $n/3$ or greater. However we have been informed by its author that there is at present a gap in the argument concerning those elements whose orders are prime powers but not prime. For our purposes, only the results concerning elements of prime order are required.

Proof. We observe that V may be decomposed as $V_a \oplus V_b \oplus V_c$, where V_a, V_b and V_c are g-invariant subspaces of dimensions a, b and c respectively. The action of g on each of these summands is irreducible. It follows that g lies in the stabilizers of proper subspaces of at least 4 different dimensions; and so g is covered by the class $\mathrm{ss}(k)$ for at least 4 values of k.

Note that $c > n/3$, and that $q^c - 1$ divides the order of g. Hence a maximal overgroup M of g must belong to one of the classes of groups mentioned in Section 1.2 of [10]. We observe firstly that owing to our assumption that $\nu \geq 3$ and $n \geq 98$, the subgroup M cannot be any of those in Tables 1.1–1.9 of [10]; this immediately rules out several of the Examples listed there. We shall go through the remaining Examples.

Example 1. *Classical examples.* The determinant of g is a generator of the quotient $G/\mathrm{SL}_n(q)$, and so M cannot contain $\mathrm{SL}_n(q)$.

Any element of a symplectic or orthogonal group is similar to its own inverse; an element g of a unitary group is similar to its conjugate-inverse $g^{-\tau}$, where τ is induced by an involutory field automorphism. (See [16], Section 2.6, or (3.7.2) for groups in characteristic 2.)

If M normalizes a symplectic or orthogonal group H, then g^{q-1} lies in H itself, and so g^{q-1} is similar to its own inverse. Then it is clear that ${\Gamma_c}^{q-1}$ is similar to its own inverse (it does not matter here whether or not $b = c$). But this cannot be the case since $c > 2$.

Similarly, if M normalizes a unitary group U then g^{q+1} lies in U, and it follows that g^{q+1} is similar to its conjugate-inverse. But then it follows that ${\Gamma_c}^{q+1}$ is similar to its conjugate-inverse, and it is easy to show that this is not the case.

Example 3. *Imprimitive examples.* Here M preserves a decomposition $V = U_1 \oplus \cdots \oplus U_t$ for $t \geq 2$. Let $\dim U_i = m$, so that $n = mt$. Recall that the $\langle g \rangle$-module V is the direct sum of 3 irreducible submodules V_a, V_b, V_c of dimensions a, b, c respectively. So $\langle g \rangle$ has at most 3 orbits on the set of spaces U_i.

Let r be the smallest integer such that V_c is contained in the direct sum of r of the spaces U_i. We observe that $n/3 < c \leq rm$, and so $m > n/3r$. Without loss of generality, we may assume that $V_c \leq W = U_1 \oplus \cdots \oplus U_r$. It is clear that W is g-invariant. Let $\overline{g}$ be the restriction of g to W. Then $\langle \overline{g} \rangle$ acts transitively on $\{U_1, \ldots, U_r\}$. Since $\overline{g}^r$ acts in the same way on each U_i for $i \leq r$, an upper bound for the order of $\overline{g}$ is $(q^m - 1)r$. But since $m \leq n/r$, and since $n \geq 98$ by assumption, we see that $(q^m - 1)r < q^{n/3} - 1$ if $r \geq 4$. Therefore we must have $r \leq 3$.

It follows that V_c is a simple $\mathbf{F}_q\langle \overline{g}^r \rangle$-module. Now since $\overline{g}^r$ commutes with the projections of W onto its summands U_i, we see that at least one of the spaces U_i contains an $\overline{g}^r$-invariant subspace of dimension c. So $m > n/3$, and hence $r \neq 3$.

Suppose that $r = 2$. Since $\overline{g}^r$ has two fixed spaces of dimension m, we see that $b = c = m$, and that $V_b \oplus V_c \leq W$. If $W < V$, then $W = V_b \oplus V_c$. Now we see that m divides each of a and $b + c = 2c$. Since a, b, c are coprime, it follows that $m = 2$. But this implies that $n < 6$, which contradicts the assumption that $n \geq 98$. So we may suppose that $W = V$. Then it is not hard to show that V_a has two irreducible summands as an $\langle \overline{g}^2 \rangle$-module. But this can occur only when $a = 2$, and this accounts for the first of the exceptional cases of the lemma.

Finally, if $r = 1$ then $m \geq c > n/3$, and so $t = 2$. It is easy to see, in this case, that $c = m = n/2$, and this accounts for the second exceptional case of the lemma.

Example 4. *Extension field examples.* If g stabilizes an $\mathbf{F}_{q^r}$-structure on V, then g^r lies in the image of an embedding of $\mathrm{GL}_{n/r}(q^r)$ into $\mathrm{GL}_n(q)$. Now if this is the case then it is not hard, by considering the degrees of the eigenvalues of g over the fields $\mathbf{F}_q$ and $\mathbf{F}_{q^r}$, to show that r must divide each of a, b, c. But this implies that $r = 1$, since a, b, c are coprime.

Example 5. *Tensor product decomposition examples.* Here M stabilizes a non-trivial tensor product decomposition $V = V_1 \otimes V_2$. There is an embedding of the central product $\mathrm{GL}(V_1) \circ \mathrm{GL}(V_2)$ into $\mathrm{GL}_n(q)$, and M is the intersection of this group with G. For $x_1 \in \mathrm{GL}(V_1)$ and $x_2 \in \mathrm{GL}(V_2)$, we write (x_1, x_2) for the corresponding element of $\mathrm{GL}(V_1) \circ \mathrm{GL}(V_2)$.

We shall suppose that V_1 and V_2 have dimensions n_1 and n_2 respectively, with $n_1 \leq n_2$. Then since $c > n/3$, it is not hard to see that we have $n_1 = 2$.

Suppose that $g \in M$, and let $g_1 \in \mathrm{GL}(V_1)$ and $g_2 \in \mathrm{GL}(V_2)$ be such that $g = (g_1, g_2)$. Let $h = g^{q^2-1}$. Since the order of g is coprime with q, we see that the element $g_1^{q^2-1}$ is the identity on V_1, and so $h = (1, h_2)$ for some $h_2 \in \mathrm{GL}(V)$.

The largest dimension of an irreducible $\langle h \rangle$-subspace of V is c, and there are at most 2 such subspaces. We obtain the $\langle h \rangle$-subspace decomposition of V up to isomorphism by taking two copies of each summand of the $\langle h_2 \rangle$-subspace decomposition of V_2. It follows that there must be at least two summands of dimension c, and hence that $b = c$ and that $a < b$. It follows also that the a dimensional summand of g splits into two summands as an $\mathbf{F}_q\langle h \rangle$-module. But it is not hard to see that this can occur only if $a = 2$, and so we have $a = 2$ and $b = c = (n-2)/2$. This is the second exceptional case of the lemma.

Example 6. *Subfield examples.* These cannot occur, since g is built up using Singer cycles, which do not preserve any proper subfield structure.

Example 7. *Symplectic type examples.* This class of groups exists only in prime-power dimension, and cannot occur in the cases we are considering since we have assumed that $\nu \geq 3$.

Example 8.(a) *Permutation module examples.* In this case S is an alternating group A_m for some $m \geq 5$. Then it is known that the order of an element in M is at most $(q-1) \cdot e^{\vartheta\sqrt{m \log m}}$ where $\vartheta = 1.05314$, by a result of Massias [15]. Here $n = m - 1$ or $m - 2$. But a routine calculation shows that the inequality $e^{\vartheta\sqrt{(n+2)\log(n+2)}} < (q^{n/3} - 1)/(q-1)$ holds for all $q \geq 2$, and for all $n \geq 98$. (This inequality fails when $q = 2$ and $n = 97$.)

Example 11. *Cross-characteristic groups of Lie type.* The examples not yet ruled out are contained in Table 1.10 of [10]. But the order of an element of M is less than n^3, which is less than $q^{n/3} - 1$ for $n \geq 98$. □

We are now in a position to complete the proof of Theorem 1.4.

Proof. Define a set Ω of classes of G by

$$\Omega = \{[\Gamma_n^{\alpha+q-1}]\} \cup \{[g_\lambda] : \lambda \in P(n)\}.$$

Let $\mathcal{C}$ be a set of conjugacy classes of subgroups of G which covers Ω, of the smallest size such that this is possible. Then clearly $|\mathcal{C}| \leq \gamma(G)$. By the theorem of Kantor [13] mentioned in the proof of Lemma 3.1 above, and by Lemma 5.3, we see that $\mathcal{C}$ must contain a single class of extension field subgroups. If $n \geq 98$ and $\nu \geq 3$

then each remaining elements of $\mathcal{C}$ is either a class of subspace stabilizers, or else one of the classes of subgroups mentioned in the exceptional cases of Lemma 3.1. Each subspace stabilizer contains at most $n/2$ of the elements g_λ, and each of the exceptional classes contains at most $n/4$. Now, using Lemma 5.2, we see that

$$\gamma(G) \ \geq \ |\mathcal{C}| \ \geq \ 1 + \frac{2g(n)}{n} \ > \ \frac{n}{\pi^2},$$

as required for the theorem.

To remove the conditions that $n \geq 98$ and that $\nu \geq 3$, it is enough to observe that the lower bound for $\kappa(G)$ given by Theorem 1.2 is larger than n/π^2 in any case where either of these conditions fails. □

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