# ON PARTIAL AUGMENTATIONS OF ELEMENTS IN INTEGRAL GROUP RINGS

VICTOR BOVDI AND ATTILA MARÓTI

Abstract. Inner relations are derived between partial augmentations of certain elements (units or idempotents) in group rings.

## 1. Introduction

Let $KG$ be the group ring of a group $G$ over a commutative ring $K$ with identity. Let $U(KG)$ be the group of units of $KG$. The subgroup

$$V(KG) = \Big\{ \sum_{g \in G} \alpha_g g \in U(KG) \quad | \quad \alpha_g \in K, \quad \sum_{g \in G} \alpha_g = 1 \Big\}$$

of $U(KG)$ is called the *normalized group of units* of $KG$. It is easy to see that if $U(K)$ denotes the group of units of the ring $K$, then

$$U(KG) = V(KG) \times U(K)$$

and that $G$ is a subgroup of $V(KG)$.

For $g \in G$ let $g^G$ denote the conjugacy class of $g$ in $G$. Let $u = \sum_{g \in G} \alpha_g g \in KG$. For $y \in G$ let $\nu_y(u) = \sum_{g \in y^G} \alpha_g$ be the *partial augmentation* of $u$ with respect to $y$. Observe that $\nu_x(u)$ is the same for all $x \in y^G$.

The element $\mathrm{Tr}^{(n)}(u) = \sum_{g \in G\{n\}} \alpha_g \in K$ is called the $n^{th}$ *generalized trace* of the element $u$ (see [2, p. 2932]), where $G\{n\}$ is the set of elements of order $p^n$ of $G$ where $n$ is a non-negative integer and $p$ is a prime. Clearly, $\mathrm{Tr}^{(0)}(u)$ coincides with $\nu_1(u) = \alpha_1$ of $u \in KG$.

Let $K = \mathbb{Z}$, the ring of integers. Let $u = \sum_{g \in G} \alpha_g g \in V(\mathbb{Z}G)$ be a torsion unit, that is, an element of finite order $|u|$. There are several connections between $|u|$, the partial augmentations $\nu_g(u)$ ($g \in G$) and $\mathrm{Tr}^{(i)}(u)$ for $i = 0, 1, \ldots, |u|$. Such a relationship was first obtained by Higman and Berman (see [2, p. 2932] or [14]), namely that $\nu_1(u) = 0$ for a finite group $G$. More generally, it is also a consequence of the Higman-Berman Theorem that $\nu_g(u) = 0$ for every central element $g$ of a finite group $G$. The Higman-Berman Theorem was extended for arbitrary groups $G$ by Bass and Bovdi (see [2, Fact 1.2, p. 2932], [3, Proposition 8.14, p. 185], [4]). Note that it is still an open question whether $\nu_g(u) = 0$ for every central element $g$ of an arbitrary group $G$?

The *spectrum* of a group is the set of orders of its torsion elements. A main unsolved problem in the theory of integral group rings is the *Spectrum Problem* (SP) which says that the spectra of $G$ and $V(\mathbb{Z}G)$ coincide. A stronger version of SP was the *Zassenhaus Conjecture* (ZC), which says that for a finite group $G$ each torsion unit of $V(\mathbb{Z}G)$ is rationally conjugate to an element of $G$. The ZC can also be reformulated in terms of conditions on $\nu_g(u)$ for each torsion unit $u \in V(\mathbb{Z}G)$. A historical overview of this topic may be found in the survey [13].

For certain finite groups $G$, the cornerstone for solving the ZC is the so-called *Luthar-Passi method* introduced in [12]. Together with results such as [9, Proposition 5], [10, Proposition 3.1], [11, Proposition 2.2], [7] and $(p, q)$-character theory from [6], the Luthar-Passi method provides ZC for certain groups $G$ (see [13]) as well as a counterexample to ZC (see [8]).

After the negative solution of the ZC a question asked by Bovdi (see [2, Fact 1.5, p. 2932]) is gaining relevance. Is it true that if $u$ is a torsion unit of $\mathbb{Z}G$ of order $p^n$ where $p$ is a prime and $n$ is a positive integer, then $\mathrm{Tr}^{(i)}(u) = 0$ for all $i < n$ and $\mathrm{Tr}^{(n)}(u) = 1$?

Note that the above methods work exclusively only when $G$ is finite. With the exception of the Bass-Bovdi Theorem, there is no result which gives a restriction for $\nu_g(u)$ where $G$ is an infinite group and $u$ is a torsion unit.

Recall that the Möbius function $\mu$ is defined on the set of positive integers as follows: $\mu(1) = 1$, $\mu(n) = 0$ if $n$ is divisible by the square of a prime, and $\mu(n) = (-1)^\ell$ if $n = \prod_{i=1}^{\ell} p_i$ where $p_1, \dots, p_\ell$ are distinct primes.

Our first result is a new relation between partial augmentations of a torsion unit of $\mathbb{Z}G$ where $G$ is a finite group.

**Theorem 1.** *Let $u \in V(\mathbb{Z}G)$ be a torsion unit of the integral group ring $\mathbb{Z}G$ of a finite group $G$. Let $k$, $n$ be positive integers such that $k$ is coprime to the exponent of $G$. If $n$ and $k$ are both congruent to $1$ modulo $|u|$, then for every $s \in G$ we have*

$$\nu_s(u) = \sum_{r|t|n} \mu(r) \cdot \Big( \sum_{\substack{x^G,\ \exists y \in G: \\ y^{(knr)/t} = x^k \sim s}} \nu_x(u) \Big). \tag{1}$$

Formula (9), which is part of the proof of Theorem 1, may be of independent interest. The proof of Theorem 1 also depends on the following result in which $G$ is not necessarily a finite group and $u$ is not necessarily a unit.

**Theorem 2.** *Let $u$ be an element of the integral group ring $\mathbb{Z}G$ of a group $G$. Let $p$ be a prime and $q = q' \cdot m$ a positive integer such that $m$ is the $p$-part of $q$ and $q'$ is not divisible by $p$. For every $s \in G$ we have*

$$\nu_s(u^q) \equiv \sum_{r|t|q'} \mu(r) \cdot \Big( \sum_{\substack{x^G,\ \exists y \in G: \\ y^{\frac{qr}{t}} = x^m \sim s}} \nu_x(u^{q'}) \Big) \pmod{p}. \tag{2}$$

In the special case when $G$ is a finite group and $u \in \mathbb{Z}G$ is a torsion unit the main result of Wagner (see [15]) could be compared with our Theorem 2.

Note that Theorem 2 may be applied to the case when $u$ is a nilpotent element of $\mathbb{Z}G$ with nilpotency index larger than $q'$.

Let $G$ be a finite group. Let $\mathbb{Q}$ and $\mathbb{C}$ be the fields of rational and complex numbers respectively. If $e$ is an idempotent of $\mathbb{C}G$, then $\nu_1(e) \in \mathbb{Q}$ and $0 < \nu_1(e) < 1$ unless $e \in \{0, 1\}$ (see [17]). Furthermore, $|\nu_g(e)|^2 \leq |g^G| \cdot \nu_1(e)$ (see [16, Theorem 2, p. 208]) and $\sum_{i=1}^m |\nu_i(e)|^2/|a_i^G| \leq 1$, where $\{a_1, \dots, a_m\}$ is a set of representatives of the conjugacy classes of $G$ (see [9, Corollary 2.6, p. 2330]).

A consequence of Theorem 2 is a new relation between the partial augmentations of an idempotent in $\mathbb{Q}G$ where $G$ is an arbitrary group.

**Corollary 1.** *Let $e$ be an idempotent of $\mathbb{Q}G$ of a group $G$. Let $\beta \in \mathbb{Z}$ such that $u = \beta e \in \mathbb{Z}G$. Let $p$ be a prime and $q = q' \cdot m$ a positive integer such that $m$ is the $p$-part of $q$ and $q'$ is not divisible by $p$. If $p$ does not divide $\beta$, then for every $s \in G$ we have*

$$\nu_s(u) \equiv \sum_{r|t|q'} \mu(r) \cdot \Big( \sum_{\substack{x^G,\ \exists y \in G: \\ y^{\frac{qr}{t}} = x^m \sim s}} \nu_x(u) \Big) \pmod{p}. \tag{3}$$

*Moreover, if $G$ is finite and $p > 4q' \cdot |\beta| \cdot |G|^{3/2}$, then in (3) equality holds.*

## 2. Proofs

Proof of Theorem 2. For elements $x$ and $y$ in $G$ we write $x \sim y$ if $x$ is conjugate to $y$.

Let $s \in G$. We wish to give an expression for $\nu_s(u^q)$. We need some notation.

Consider the set $\mathcal{K} = \{(g_1, \dots, g_q) \in G^q \mid g_1 \cdots g_q \sim s\}$. There is a permutation $\pi$ acting on $\mathcal{K}$ by sending $(g_1, g_2, \dots, g_q) \in \mathcal{K}$ to $(g_2, \dots, g_q, g_1) \in \mathcal{K}$. Let $t$ be a positive divisor of $q$. Let the union of those $\langle\pi\rangle$-orbits on $\mathcal{K}$ which have lengths dividing $t$ be denoted by

$$\mathcal{K}_t = \{(g_1, \dots, g_q) \in \mathcal{K} \mid g_{i+t} = g_i \text{ for every } i \text{ with } 1 \leq i \leq q - t\}$$

and let the union of orbits length $t$ on $\mathcal{K}$ be $\mathcal{K}_t^*$. Observe that $\mathcal{K} = \mathcal{K}_q$ and $\mathcal{K}_t = \cup_{r|t} \mathcal{K}_r^*$.

Write $u=\sum_{g\in G}\alpha_g g\in \mathbb{Z}G$. It is easy to see that

$$\nu_s(u^q)=\sum_{(g_1,\dots,g_q)\in\mathcal{K}}\prod_{j=1}^{q}\alpha_{g_j}=\sum_{t|q}\sum_{(g_1,\dots,g_q)\in\mathcal{K}_t^*}\prod_{j=1}^{q}\alpha_{g_j}. \tag{4}$$

Since $\mathcal{K}_t^*$ is the union of all $\langle\pi\rangle$-orbits of length exactly $t$, the multiplicity of each summand in the sum $\sum_{(g_1,\dots,g_q)\in\mathcal{K}_t^*}\prod_{j=1}^{q}\alpha_{g_j}$ is divisible by $t$. Thus (4) provides

$$\begin{aligned}\nu_s(u^q)&\equiv\sum_{\substack{t|q\\ p\nmid t}}\sum_{(g_1,\dots,g_q)\in\mathcal{K}_t^*}\prod_{j=1}^{q}\alpha_{g_j}\equiv\sum_{t|q'}\sum_{(g_1,\dots,g_q)\in\mathcal{K}_t^*}\prod_{j=1}^{q}\alpha_{g_j}\\ &\equiv\sum_{t|q'}\sum_{(g_1,\dots,g_q)\in\mathcal{K}_t^*}\Big(\prod_{j=1}^{q'}\alpha_{g_j}\Big)^m\equiv\sum_{t|q'}\sum_{(g_1,\dots,g_q)\in\mathcal{K}_t^*}\prod_{j=1}^{q'}\alpha_{g_j}\pmod p.\end{aligned} \tag{5}$$

If $f_1$ and $f_2$ are two functions from $\mathbb{Z}$ to $\mathbb{Z}$ such that $f_1(t)=\sum_{r|t}f_2(r)$, then $f_2(t)=\sum_{r|t}\mu(r)f_1(t/r)$. This is the Möbius inversion formula (see [1, Theorem 2.9,p. 32]).

For positive integers $t$ and $r$, put

$$f_1(t)=\sum_{(g_1,\dots,g_q)\in\mathcal{K}_t}\prod_{j=1}^{q'}\alpha_{g_j}\quad\text{and}\quad f_2(r)=\sum_{(g_1,\dots,g_q)\in\mathcal{K}_r^*}\prod_{j=1}^{q'}\alpha_{g_j}.$$

The Möbius inversion formula then yields

$$\sum_{(g_1,\dots,g_q)\in\mathcal{K}_t^*}\prod_{j=1}^{q'}\alpha_{g_j}=\sum_{r|t}\mu(r)\sum_{(g_1,\dots,g_q)\in\mathcal{K}_{t/r}}\prod_{j=1}^{q'}\alpha_{g_j}. \tag{6}$$

Formulas (5) and (6) yield

$$\begin{aligned}&\nu_s(u^q)\equiv\\ &\equiv\sum_{t|q'}\Big(\sum_{r|t}\mu(r)\sum_{(g_1,\dots,g_q)\in\mathcal{K}_{t/r}}\prod_{j=1}^{q'}\alpha_{g_j}\Big)\equiv\sum_{r|t|q'}\mu(r)\cdot\Big(\sum_{(g_1,\dots,g_q)\in\mathcal{K}_{t/r}}\prod_{j=1}^{q'}\alpha_{g_j}\Big)\\ &\equiv\sum_{r|t|q'}\mu(r)\cdot\Big(\sum_{\substack{(g_1,\dots,g_{q'})\in G^{q'}\\ (g_1\cdots g_{t/r})^{qr/t}=(g_1\cdots g_{q'})^m\sim s}}\prod_{j=1}^{q'}\alpha_{g_j}\Big)\\ &\equiv\sum_{r|t|q'}\mu(r)\cdot\Big(\sum_{\substack{x^G,\ \exists y\in G:\\ y^{qr/t}=x^m\sim s}}\nu_x(u^{q'})\Big)\pmod p.\end{aligned}$$

□

*Proof of Theorem 1.* Let $s, p, q, q'$ and $m$ be as in Theorem 2. Let $n = q'$ and $m = p$. By (2) of Theorem 2 we have

$$\nu_s(u^{np}) \equiv \sum_{r|t|n} \mu(r) \cdot \Big( \sum_{\substack{x^G,\ \exists y \in G:\\ y^{npr/t} = x^p \sim s}} \nu_x(u^n) \Big) \pmod{p}. \tag{7}$$

Let $k$ be a positive integer coprime to the exponent $e$ of $G$. Choose $p$ such that $p \equiv k \pmod{e}$. There are infinitely many such primes by Dirichlet's theorem on arithmetic progressions [1, Chapter 7].

Since $p \equiv k \pmod{e}$, in (7) we have $y^{\frac{npr}{t}} = y^{\frac{nkr}{t}}$ and $x^p = x^k$. Moreover, $u^{np} = u^{nk}$ by the Cohn-Livingstone Theorem [7, Corollary 4.1]. This yields

$$\nu_s(u^{nk}) \equiv \sum_{r|t|n} \mu(r) \cdot \Big( \sum_{\substack{x^G,\ \exists y \in G:\\ y^{nkr/t} = x^k \sim s}} \nu_x(u^n) \Big) \pmod{p}. \tag{8}$$

The absolute value of every partial augmentation of $G$ is at most $\sqrt{|G|}$ (really $\nu_y(x)^2 \leq |y^G|$) by [9, Corollary 2.3, p. 2329] or [5]. The number of summands on the right-hand side of (8) is at most $(2\sqrt{n})^2 \cdot |G|$. Choose $p$ such that $p > (2\sqrt{n})^2 \cdot |G|^{3/2}$. Since both sides of the congruence (8) have absolute value less than $p$,

$$\nu_s(u^{nk}) = \sum_{r|t|n} \mu(r) \cdot \Big( \sum_{\substack{x^G,\ \exists y \in G:\\ y^{(knr)/t} = x^k \sim s}} \nu_x(u^n) \Big). \tag{9}$$

If $k$ and $n$ are both congruent to 1 modulo $|u|$, then we get (1). □

*Proof of Corollary 1.* Let $s, p, q, q'$ and $m$ be as in Theorem 2. Since $u^r = \beta^{r-1} u$, we get $\nu_s(u^r) = \nu_s(\beta^{r-1} u) = \beta^{r-1} \nu_s(u)$, where $r \in \{q, q'\}$. Theorem 2 gives

$$\beta^{q-q'} \nu_s(u) \equiv \sum_{r|t|q'} \mu(r) \cdot \Big( \sum_{\substack{x^G,\ \exists y \in G:\\ y^{qr/t} = x^m \sim s}} \nu_x(u) \Big) \pmod{p}.$$

Congruence (3) follows by observing that $\beta^{q-q'} = (\beta^m)^{q'} \beta^{-q'} \equiv 1 \pmod{p}$ since $m$ is a $p$-power.

The absolute value of the left-hand side of (3) is at most $|\beta| \cdot \sqrt{|G|}$ and the absolute value of the right-hand side of (3) is at most $(2\sqrt{q'})^2 \cdot |G| \cdot |\beta| \cdot \sqrt{|G|}$, by [16, Theorem 2, p. 208]. If $p > 4q' \cdot |\beta| \cdot |G|^{3/2}$, then equality in (3) holds. □

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Department of Mathematical Sciences, UAEU, Al-Ain, United Arab Emirates
*Email address*: vbovdi@gmail.com

Alfréd Rényi Institute of Mathematics, Eötvös Loránd Research Network, P.O. Box 127, H-1364, Budapest, Hungary
*Email address*: maroti.attila@renyi.hu