FINITE GROUPS HAVE MORE CONJUGACY CLASSES

BARBARA BAUMEISTER, ATTILA MARÓTI, AND HUNG P. TONG VIET

ABSTRACT. We prove that for every $\epsilon > 0$ there exists a $\delta > 0$ such that every group of order $n \geq 3$ has at least $\delta \log_2 n/(\log_2 \log_2 n)^{3+\epsilon}$ conjugacy classes. This sharpens earlier results of Pyber and Keller. Bertram speculates whether it is true that every finite group of order n has more than $\log_3 n$ conjugacy classes. We answer Bertram's question in the affirmative for groups with a trivial solvable radical.

1. INTRODUCTION

For a finite group G let $k(G)$ denote the number of conjugacy classes of G. Answering a question of Frobenius, Landau [16] proved in 1903 that for a given k there are only finitely many groups having k conjugacy classes. Making this result explicit, we have $\log \log |G| < k(G)$ for any non-trivial finite group G (see Brauer [5], Erdős and Turán [10], Newman [20]). (Here and throughout the paper the base of the logarithms will always be 2 unless otherwise stated.) Problem 3 of Brauer's list of problems [5] is to give a substantially better lower bound for $k(G)$ than this.

Pyber [21] proved that there exists a constant $\epsilon > 0$ such that for every finite group G of order at least 3 we have $\epsilon \log |G|/(\log \log |G|)^8 < k(G)$. Almost 20 years later Keller [15] replaced the 8 in the previous bound by 7. Our first result gives a further improvement to Pyber's theorem.

Theorem 1.1. *For every $\epsilon > 0$ there exists a $\delta > 0$ such that for every finite group G of order at least* 3 *we have $\delta \log |G|/(\log \log |G|)^{3+\epsilon} < k(G)$.*

There are many lower bounds for $k(G)$ in terms of $|G|$ for the various classes of finite groups G. For example, Jaikin-Zapirain [14] gave a better than logarithmic lower bound for $k(G)$ when G is a nilpotent group. For supersolvable G Cartwright [7] showed $(3/5) \log |G| <$

Date: 16 February 2016.

2010 *Mathematics Subject Classification.* 20E45, 20D06, 20P99.

Key words and phrases. finite groups, number of conjugacy classes, simple groups.

The second author was supported by an Alexander von Humboldt Fellowship for Experienced Researchers, by MTA Rényi "Lendület" Groups and Graphs Research Group, and by OTKA grants K84233 and K115799. Tong-Viet's work is based on the research supported in part by the National Research Foundation of South Africa (Grant Number 93408). Part of the work was done while the last author held a position at the CRC 701 within the project C13 'The geometry and combinatorics of groups'. The first and second authors also wish to thank the CRC 701 for its support.

$k(G)$. For solvable groups the best bound to date is a bit worse than logarithmic and is due to Keller [15].

The conjecture whether there exists a universal constant $c > 0$ such that $c \log |G| < k(G)$ for any finite group G has been intensively studied by many mathematicians including Bertram, see for instance [3]. Bertram observed that $k(G) = \lceil \log_3(|G|) \rceil$ when $G = \mathrm{PSL}_3(4)$ or M_{22} and checked the proposed bound for certain small groups [2, p. 96]. He then speculates whether $\log_3 |G| < k(G)$ is true for every finite group G. In our second result we answer Bertram's question in the affirmative for groups with a trivial solvable radical.

Theorem 1.2. *Let G be a finite group with a trivial solvable radical. Then $\log_3 |G| < k(G)$.*

The paper is structured as follows. We prove Theorem 1.1 in Section 2. This is done by first improving [21, Lemma 4.7] which gives the lower bound for $\log k(G)$ in terms of $\log |G|$ for finite groups with a trivial solvable radical and then applying the argument in [21] and [15] to get the required result for arbitrary finite groups. In Section 3, we compute explicitly the constant c_2 arising from Lemma 2.3. In Section 4 we verify Theorem 1.2 for some almost simple groups whose automorphism groups have a bounded number of orbits on their socles and finally the full proof of Theorem 1.2 is carried out in Section 5.

2. Asymptotics

In this section we first improve [21, Lemma 4.7].

Theorem 2.1. *For every $\epsilon > 0$ there exists $\delta > 0$ such that for every non-trivial finite group G with trivial solvable radical we have $\delta \cdot (\log |G|)^{1/(3+\epsilon)} < \log k(G)$.*

We will prove Theorem 2.1 in this section. Let G be a non-trivial finite group with trivial solvable radical. Suppose that G has r minimal normal subgroups $M_1, \ldots, M_r$. Then each M_i with $1 \leq i \leq r$ is equal to a direct product $T_{i,1} \times \cdots \times T_{i,n_i}$ of n_i isomorphic non-abelian simple groups $T_{i,j}$ with $1 \leq j \leq n_i$. Put $n = \sum_{i=1}^{r} n_i$, and let N be the socle of G, that is, $M_1 \times \cdots \times M_r$.

The group G permutes the simple direct factors of each M_i for $1 \leq i \leq r$. Let B be the kernel of the action of G on the set of n simple direct factors of N. Then B contains N and B/N embeds in the direct product of the outer automorphism groups of the n simple direct factors of N. Furthermore G/B is a subgroup of $\mathrm{S}_{n_1} \times \mathrm{S}_{n_2} \times \cdots \times \mathrm{S}_{n_r} \leq \mathrm{S}_n$.

For a non-abelian finite simple group T let $k^*(T)$ denote the number of $\mathrm{Aut}(T)$-orbits on T. By Burnside's theorem, $|T|$ has at least 3 different prime divisors, so $k^*(T) \geq 4$ by Cauchy's theorem. Further, [21, Lemma 2.5] and [21, Lemma 4.4] yield the following.

Lemma 2.2. *There exists a universal constant $c_1 > 0$ such that whenever G is a finite group with a composition factor isomorphic to a non-abelian simple group T, then*

$$\log k(G) \geq \log k^*(T) > c_1 (\log a / \log \log a)^{1/2}$$

where $a = |\mathrm{Aut}(T)|$.

From this we may derive the following inequality.

Lemma 2.3. *There exists a universal constant $c_2 > 0$ such that whenever T is a non-abelian finite simple group then* $\log|\mathrm{Aut}(T)| < c_2(\log k^*(T))^2 \log\log k^*(T)$.

Proof. From Lemma 2.2 we have $\log|\mathrm{Aut}(T)| < (1/{c_1}^2)(\log k^*(T))^2 \log\log|\mathrm{Aut}(T)|$. From Lemma 2.2 we also have that $2\log\log k^*(T) > 2\log c_1 + \log\log|\mathrm{Aut}(T)| - \log\log\log|\mathrm{Aut}(T)|$. Notice that this lower bound is non-positive for only at most finitely many T's and it tends to infinity as $|\mathrm{Aut}(T)|$ tends to infinity. Thus $2\log\log k^*(T) > c_3 \log\log|\mathrm{Aut}(T)|$ for some universal constant $c_3 > 0$. From these the lemma follows. □

In the next section, we show that c_2 can be chosen to be 1.954.

To slightly simplify notation, for every i with $1 \le i \le r$, put $k_i = k^*(T_{i,j})$ for every j with $1 \le j \le n_i$. We may now give an upper bound for $\log|G|$.

Lemma 2.4. *Let c_2 be as above. Then* $\log|G| < n\log n + c_2 \sum_{i=1}^{r} n_i (\log k_i)^2 (\log\log k_i)$.

Proof. Clearly Lemma 2.3 implies $\log|G| < \sum_{i=1}^{r}(n_i \log n_i + c_2 n_i (\log k_i)^2(\log\log k_i))$. □

The following lemma will also be useful.

Lemma 2.5. *For every i with $1 \le i \le r$ the number of conjugacy classes of G lying inside M_i is larger than $(k_i/n_i)^{n_i}$.*

Proof. Fix an index i. Observe that M_i has at least $k_i^{n_i}$ conjugacy classes and that these are non-trivially permuted by a certain factor group of size at most $n_i! < n_i^{n_i}$. □

For a permutation group H let $s(H)$ be the number of orbits on the power set of the underlying set. The following is [1, Theorem 1].

Lemma 2.6. *Let H be a permutation group of degree n. If H has no composition factor isomorphic to A_m for $m > t \ge 5$, then $s(H) \ge 2^{c_4(n/t)}$ for some absolute constant $c_4 > 0$.*

Let $t \ge 5$ be the largest integer such that A_t is a composition factor of G/B. If no such t exists then set $t = 4$. By Lemma 2.2 we have $\log k(G) \ge \log k^*(\mathrm{A}_t)$, provided that $t \ge 5$. If $t \ge 5$ this is at least $c_5\sqrt{t}$ by [21, Lemma 4.3] for some absolute constant $c_5 > 0$. Thus in all cases we have $\log k(G) \ge c_6\sqrt{t}$ for some other absolute constant $c_6 > 0$.

If $t > (\delta^2/{c_6}^2) \cdot (\log|G|)^{2/(3+\epsilon)}$ then we are finished. Choose $\delta^2 < {c_6}^2$ and assume that $t < (\log|G|)^{2/(3+\epsilon)}$.

By Lemma 2.6 we see that $\log k(G) > c_4(n/t) > c_4(n/(\log|G|)^{2/(3+\epsilon)})$. If this is at least $\delta(\log|G|)^{1/(3+\epsilon)}$, then we are finished. So assume that $(c_4/\delta)n < (\log|G|)^{3/(3+\epsilon)}$. We may choose δ smaller than c_4 so we assume that $n^{1+(\epsilon/3)} < \log|G|$.

Lemma 2.7. *Under our assumptions there exists a constant c_7 such that*

$$n^{1+(\epsilon/3)} < c_7 \sum_{i=1}^{r} n_i (\log k_i)^2 (\log\log k_i).$$

Proof. Notice that if n is bounded then we are finished. So assume that $n \to \infty$. By our assumption and Lemma 2.4 we have

$$n^{1+(\epsilon/3)} < n\log n + c_2 \sum_{i=1}^{r} n_i (\log k_i)^2 (\log\log k_i).$$

Since $(n\log n)/n^{1+(\epsilon/3)} \to 0$ as $n \to \infty$, there exists a constant $c_7 > 0$ such that

$$(c_2/c_7)n^{1+(\epsilon/3)} < n^{1+(\epsilon/3)} - n\log n < c_2 \sum_{i=1}^{r} n_i (\log k_i)^2 (\log\log k_i)$$

for large enough n. Therefore the proof is complete. □

Set $N(\epsilon)$ to be a large enough integer such that $(N(\epsilon)/c_7)^{1/3} > 2\log N(\epsilon) \geq 1$ and $m^{\epsilon/18} > 2\log m$ for all m with $m \geq N(\epsilon)$.

Let J be the set of those i's with $1 \leq i \leq r$ such that $N(\epsilon)\cdot n^{\epsilon/6} < c_7(\log k_i)^2(\log\log k_i)$. We may assume that J is non-empty. Otherwise n is bounded by Lemma 2.7 and so all the k_i's are bounded. This means that $|G|$ is bounded and thus $k(G)$ is bounded. We may set δ small enough such that the theorem holds for these finitely many groups G.

Lemma 2.8. *We may assume that there exists a constant c_8 such that*

$$\log|G| < c_8 \sum_{i\in J} n_i (\log k_i)^2 (\log\log k_i).$$

Proof. By our discussion about J above, our assumption, and Lemma 2.4, we get

$$n^{1+(\epsilon/3)} < \log|G| < n\log n + (c_2N(\epsilon)/c_7)n^{1+(\epsilon/6)} + c_2 \sum_{j\in J} n_i (\log k_i)^2 (\log\log k_i).$$

Let $K(\epsilon)$ be an integer such that whenever $n \geq K(\epsilon)$ then

$$\log|G| - n\log n - (c_2N(\epsilon)/c_7)n^{1+(\epsilon/6)} > 0.$$

Then there exists a constant $c_8 > 0$ such that

$$(c_2/c_8)\log|G| < \log|G| - n\log n - (c_2N(\epsilon)/c_7)n^{1+(\epsilon/6)}$$

whenever $n \geq K(\epsilon)$. Thus we may assume that $n < K(\epsilon)$. Then there exists a positive constant $M(\epsilon)$ such that

$$\log|G| < M(\epsilon) + c_2 \sum_{j\in J} n_i (\log k_i)^2 (\log\log k_i).$$

If the second summand on the right-hand side of the previous inequality is larger than $M(\epsilon)$ then the claim follows. This means that $|G|$ is bounded. But since $J \neq \emptyset$ we can certainly choose (in this case) a suitable c_8 to satisfy the statement of the lemma. □

Lemma 2.9. *We can assume that for all $i \in J$ we have $\log k_i - \log n_i > (\log k_i)/2$.*

Proof. Since $i \in J$, we have $N(\epsilon) \cdot n^{\epsilon/6} < c_7(\log k_i)^3$. From this it follows that

$$(N(\epsilon)/c_7)^{1/3} n^{\epsilon/18} < \log k_i.$$

Finally, $(N(\epsilon)/c_7)^{1/3} n^{\epsilon/18} > 2\log n \geq 2\log n_i$ by our choice of $N(\epsilon)$. □

Finally, by Lemmas 2.8, 2.9 and 2.5, we have

$$\delta^3 \log|G| < \delta^3 c_8 \cdot \sum_{i\in J} n_i(\log k_i)^2(\log\log k_i) < \Big((1/2)\sum_{i\in J} n_i \log k_i\Big)^3 <$$
$$< \Big(\sum_{i\in J} n_i(\log k_i - \log n_i)\Big)^3 < (\log k(G))^3$$

whenever δ satisfies $\delta^3 c_8 < 1/8$. This proves Theorem 2.1.

Proof of Theorem 1.1. The proof of Theorem 1.1 depends on Theorem 2.1. Indeed, in the proof of [15, Corollary 3.3], which is an improved version of the argument on [21, page 248], we can replace 7 by $3+\epsilon$. Notice that the δ's in the statements of Theorems 2.1 and 1.1 are different. □

3. Computing c_2

Now we turn our attention to Bertram's question aiming to give a specific logarithmic lower bound for $k(G)$ in terms of $|G|$ where G is an arbitrary finite group. In order to prove Theorem 1.2, we need to compute specific values of c_2 in Lemma 2.3.

We first fix some notation. Let T be a non-abelian simple group, let $A := \mathrm{Aut}(T)$ and $k := k^*(T)$. We have

(1) $$k \geq k(T)/|\mathrm{Out}(T)|.$$

Denote by $\Gamma = \{x_i\}_{i=1}^m$ a representative set for all conjugacy classes of A, i.e., $A = \cup_{i=1}^m x_i^A$. By definition, we see that

(2) $$k = |\{i \in \Gamma : x_i^A \cap T \neq \emptyset\}|.$$

Notice that $k = k(T)$ when $\mathrm{Out}(T) = 1$. It follows from Lemma 2.3 that

(3) $$\gamma := \gamma(T) := \frac{\log|A|}{(\log k)^2 \log\log k} < c_2.$$

The following lemma is used frequently, whose proof is straightforward and is omitted.

Lemma 3.1. *Let $q = p^f \geq 2$ be a power of a prime p, where $f \geq 1$ is an integer and let $2 \leq a \leq b$ be integers. Then*

(1) $(q^a - 1)(q^b - 1) \leq q^{a+b}$;
(2) $(q^a - 1)(q^b + 1) \leq q^{a+b}$;

(3) *$q \geq 2f$ and if $q \geq 16$, then $q \geq 3f$;*
(4) *If $f \neq 3$, then $2 \log f \leq f$.*

Theorem 3.2. *Let T be a non-abelian simple group. Then $\gamma(T) < 1.613$ unless $T \cong \mathrm{A}_5$ or $\mathrm{PSL}_3(4)$. For the exceptions, we have $\gamma(\mathrm{A}_5) \leq 1.727$ and $\gamma(\mathrm{PSL}_3(4)) \leq 1.954$. Therefore, we can choose $c_2 = 1.954$ in all cases. Furthermore, $k \geq 5$ unless $T \cong \mathrm{A}_5$.*

For brevity, let $c := 1.613$. Using [8, Page xvi], we can easily obtain Table 1, where $q = p^f$ and p is the defining characteristic of T.

TABLE 1. The finite simple groups of Lie type

T	d	$\lvert \mathrm{Out}(T) \rvert$	$\lvert \mathrm{Aut}(T) \rvert \leq$
$\mathrm{PSL}_n(q)$	$\gcd(n, q-1)$	$2df, n \geq 3$	$2fq^{n^2-1}$
		$df, n = 2$	fq^3
$\mathrm{PSU}_n(q)$	$\gcd(n, q+1)$	$2df, n \geq 3$	$2fq^{n^2-1}$
$\mathrm{PSp}_{2n}(q)$	$\gcd(2, q-1)$	$df, n \geq 3$	fq^{2n^2+n}
		$2f, n = 2$	$2fq^{10}$
$\Omega_{2n+1}(q), q$ odd	2	$2f$	fq^{2n^2+n}
$\mathrm{P}\Omega_8^+(q)$	$\gcd(4, q^4-1)$	$6df$	$2fq^{28}$
$\mathrm{P}\Omega_{2n}^+(q)$	$\gcd(4, q^n-1)$	$2df, n \neq 4$	$2fq^{2n^2-n}$
$\mathrm{P}\Omega_{2n}^-(q), n \geq 4$	$\gcd(4, q^n+1)$	$2df$	$2fq^{2n^2-n}$
${}^2\mathrm{B}_2(q^2), q^2 = 2^{2m+1}$	1	$2m+1$	$(2m+1)2^{5(2m+1)}$
${}^2\mathrm{G}_2(q^2), q^2 = 3^{2m+1}$	1	$2m+1$	$(2m+1)3^{7(2m+1)}$
${}^2\mathrm{F}_2(q^2), q^2 = 2^{2m+1}$	1	$2m+1$	$(2m+1)2^{26(2m+1)}$
${}^3\mathrm{D}_4(q)$	1	$3f$	$6fq^{28}$
${}^2\mathrm{E}_6(q)$	$\gcd(3, q+1)$	$2df$	$2fq^{78}$
$\mathrm{G}_2(q), q \geq 3$	1	f, if $p \neq 3$	fq^{14}
		$2f$, if $p = 3$	$2fq^{14}$
$\mathrm{F}_4(q)$	1	$\gcd(2,p)f$	$\gcd(2,p)fq^{52}$
$\mathrm{E}_6(q)$	$\gcd(3, q-1)$	$2df$	$2fq^{78}$
$\mathrm{E}_7(q)$	$\gcd(2, q-1)$	df	fq^{133}
$\mathrm{E}_8(q)$	1	f	fq^{248}

Let $\mathcal{S}$ be the set consisting of all 26 sporadic simple groups, the alternating groups A_n with degree $5 \leq n \leq 22$ and the following nonabelian simple groups of Lie type, where q denotes a prime power:

$$\mathrm{PSL}_2(q)(q \leq 169), \mathrm{PSL}_3(q)(q \leq 9), \mathrm{PSU}_3(q)(q \leq 9), \mathrm{PSU}_4(q)(q \leq 7), \mathrm{PSU}_5(q)(q \leq 4),$$
$$\mathrm{PSU}_n(2)(4 \leq n \leq 7), \mathrm{PSp}_4(q)(q \leq 8), \mathrm{PSp}_6(q)(q \leq 3), \Omega_7(3), \mathrm{P}\Omega_8^{\pm}(q)(q \leq 3), \mathrm{P}\Omega_{10}^+(2),$$
$${}^2\mathrm{B}_2(q^2), (q^2 = 8, 32), {}^2\mathrm{G}_2(3^3), {}^2\mathrm{F}_4(8), {}^3\mathrm{D}_4(2), {}^2\mathrm{E}_6(2), \mathrm{G}_2(q)(q \leq 5), {}^2\mathrm{F}_4(2)', \mathrm{F}_4(2).$$

Let T be any simple group in $\mathcal{S}$. The number $k = k^*(T)$ can be computed using [13] via the 'fusions' of conjugacy classes of T onto that of $\mathrm{Aut}(T)$. Notice that the character tables of T and almost all the character tables of $A = \mathrm{Aut}(T)$ are available in [13, 8]. In the case when the character table of $\mathrm{Aut}(T)$ is not available, we can use the obvious lower bound for $k^*(T)$ which is the number of distinct element orders of T, i.e.,

$$k = k^*(T) \geq e(T) := |\{|x| \,:\, x \in T\}|,$$

where $|x|$ denotes the order of the element $x \in T$.

As an example, let $T = \mathrm{PSL}_3(4)$. Then $A = \mathrm{PSL}_3(4){\cdot}\mathrm{D}_{12}$. We first compute the character tables of A and T.

>t:=CharacterTable("L3(4).D12");; s:=CharacterTable("L3(4)");;

Here t and s are the character tables of A and T respectively.

Next, we compute the fusion of conjugacy classes of T onto that of A.

>fus:=FusionConjugacyClasses(s,t);;

Now we can easily obtain $k = k^*(T)$ via:

>k:=Size(Set(fus));;

We can also obtain both $k(T)$ and $e(T)$ from [13] by first obtaining the conjugacy class names.

>cl:=ClassName(s);; k(T):=Size(cl);;

To obtain $e(T)$, we count the number of classes with name 'ia' where $i = 1, 2, \ldots$. For example, if $T = \mathrm{M}_{12}$, then $cl := [\text{"}1a\text{"}, \text{"}2a\text{"}, \text{"}3a\text{"}, \text{"}5a\text{"}, \text{"}5b\text{"}]$ so $e(T) = 4$.

Next, we obtain $|A|$ via the GAP command

>a:=Size(t);;

Finally, we can easily compute γ using Equation (3).

For sporadic and alternating simple groups of small degrees, $\gamma(T)$ and $k^*(T)$ are given in Table 2.

Lemma 3.3. *If T is a sporadic simple group, the Tits group or the alternating group of degree $n \geq 6$, then $\gamma(T) < c$ while $c < \gamma(\mathrm{A}_5) \leq 1.727$. Moreover, $k \geq 5$ unless $T = \mathrm{A}_5$.*

Proof. (i) Assume first that T is a sporadic simple group or the Tits group. From Table 2, we see that $10 \leq k^*(T) \leq k^*(\mathrm{M}) = 194$ and $\gamma(T) \leq \gamma(\mathrm{M}) < 1.06 < c$. So the lemma holds in this case.

(ii) Assume that $T = \mathrm{A}_n$ with $5 \leq n \leq 21$. From Table 2, if $6 \leq n \leq 21$, then $\gamma(T) < 1 < c$ and $k \geq 5$ while $c < \gamma(\mathrm{A}_5) < 1.727$ and $k^*(\mathrm{A}_5) = 4$.

(iii) Assume that $T = \mathrm{A}_n$ with $n \geq 22$. Since $|\mathrm{S}_n : \mathrm{A}_n| = 2$, Clifford's theorem gives that $k(\mathrm{S}_n) \leq 2k(\mathrm{A}_n)$ and thus by (1) we have $k \geq k(\mathrm{A}_n)/2 \geq k(\mathrm{S}_n)/4 = p(n)/4$, where $p(n)$ is the number of partitions of n. By [19, Corollary 3.1], we have $p(n)/4 \geq e^{2\sqrt{n}}/56$ and so, as $n \geq 22$, we obtain that $k \geq 250$ and $\log k \geq 2\sqrt{n}\log e - \log 56 \geq \sqrt{n}$. Now we can easily

Table 2. Some alternating and sporadic simple groups

T	k	$\gamma <$	T	k	$\gamma <$
M_{11}	10	0.678	M_{12}	12	0.741
M_{22}	11	0.923	M_{23}	17	0.687
M_{24}	26	0.565	J_1	15	0.581
J_2	16	0.632	J_3	17	0.784
HS	21	0.642	Suz	37	0.615
McL	19	0.817	Ru	36	0.586
He	26	0.668	Ly	53	0.673
O'N	25	0.833	Co_1	101	0.511
Co_2	60	0.507	Co_3	42	0.550
Fi_{22}	59	0.530	Fi_{23}	98	0.519
Fi'_{24}	97	0.684	HN	44	0.671
Th	48	0.728	B	184	0.678
M	194	1.06	${}^2F_4(2)'$	17	0.740
A_5	4	1.727	A_6	5	1.602
A_7	8	0.863	A_8	12	0.647
A_9	16	0.578	A_{10}	22	0.509
A_{11}	29	0.470	A_{12}	40	0.423
A_{13}	52	0.399	A_{14}	69	0.374
A_{15}	90	0.355	A_{16}	118	0.336
A_{17}	151	0.324	A_{18}	195	0.310
A_{19}	248	0.300	A_{20}	$\geq 162^*$	0.395
A_{21}	$\geq 204^*$	0.379	A_{22}	$\geq 256^*$	0.365

* We use the bound $k \geq k(A_n)/2$.

check that

$$\gamma \leq \frac{\log n!}{(2\sqrt{n}\log e - \log 56)^2 \log n^{1/2}} < \frac{2n}{(2\sqrt{n}\log e - \log 56)^2} < c.$$

This completes the proof. □

Let $\mathbf{G}$ be a simply connected simple algebraic group of rank $r > 0$ and let F be a Steinberg endomorphism of $\mathbf{G}$ associated to a prime power q. Then $L = \mathbf{G}^F$ is a quasi-simple group and $L/\mathbf{Z}(L) \cong T$ is a finite simple group of Lie type with $d = |\mathbf{Z}(L)|$. From [12, Theorem 3.1] and [12, Lemma 2.1], we have that $k(L) \geq q^r$ and $k(L) \leq k(\mathbf{Z}(L))k(L/\mathbf{Z}(L))$ and thus $k(T) \geq k(L)/k(\mathbf{Z}(L)) \geq q^r/d$ hence by (1), we have

$$k = k^*(T) \geq \max\{e(T), \frac{q^r}{d|\operatorname{Out}(T)|}\}. \tag{4}$$

Denote by $\operatorname{Irr}(H)$ the set of complex irreducible characters of a finite group H. Then it is well-known that $k(H) = |\operatorname{Irr}(H)|$ and by Brauer's permutation lemma, the numbers of $\operatorname{Aut}(H)$-orbits on irreducible characters and on conjugacy classes of H are the same.

Therefore, if we write $\mathrm{cd}(H)$ for the set of character degrees of H, then $k^*(H) \geq |\mathrm{cd}(H)|$. It follows that

$$k^*(T) \geq |\mathrm{cd}(T)|. \tag{5}$$

Lemma 3.4. *Theorem 3.2 holds for finite simple groups of Lie type.*

Proof. For the proof of this lemma, we only give a detailed proof for $T = \mathrm{PSL}_n(q)$ with $n \geq 2$ and $q = p^f$ for some prime p and integer $f \geq 1$, which is the most difficult case. Other families can be dealt with a similar argument.

(i) Assume $T = \mathrm{PSL}_2(q)$ with $q = 2^f$. By Lemma 3.3, we can assume that T is not an alternating group. So $f \geq 3$. In this case, we have that $|A| = q(q^2-1)f \leq f \cdot 2^{3f}$. Now, if $3 \leq f \leq 6$, then k is given in Table 3. For these cases, it is easy to check that $k \geq 5$ and

$$\gamma = \frac{\log |A|}{(\log k)^2 \log\log k} \leq \frac{3f + \log f}{(\log k)^2} < c.$$

Notice that $1.612006 < \gamma(\mathrm{PSL}_2(8)) \leq 1.613 = c$. We now assume that $f \geq 7$. We use the lower bound given in (1) where $|\operatorname{Out}(T)| = f$ and $k(T) = q+1$ (see [9, Theorem 38.2]). So

$$k \geq k(T)/|\operatorname{Out}(T)| = (q+1)/f > 2^f/f > 18.$$

Thus $\gamma \leq (3f + \log f)/(f - \log f)^2$. Direct computation using the previous inequality shows that $\gamma < c$ when $f \leq 16$. So we assume that $f \geq 17$. Then $f \geq f/2 \geq \log f$ and thus $\gamma \leq 4f/(f - f/2)^2 = 16/f < 1$.

(ii) $T = \mathrm{PSL}_2(q)$ with $q = 7$ or $q = p^f \geq 11$ odd. From [9, Theorem 38.1] we derive that $k(T) = (q+5)/2$. Moreover, we have $|A| = q(q^2-1)f$ and $|\operatorname{Out}(T)| = 2f$.

(ii)(a) Assume first that $p = 3$. Then $f \geq 3$. If $f = 3, 4$ or 5, then $k = 7, 15$ or 27. Direct calculation shows that $\gamma < c$. Assume next that $f \geq 6$. We have $k \geq (q+5)/4f \geq 12$ and $\log |A| < \log(fq^3) = 3f \log 3 + \log f \leq 6f$ so $\log k \geq \log(q/4f) = f \log 3 - \log(4f) \geq f - 2$. If $f \geq 10$, then $\gamma < 6f/(f-2)^2 < c$. So assume that $6 \leq f \leq 9$. Then direct calculation using the bound $k \geq (3^f + 5)/4f$ confirms that $\gamma < c$.

(ii)(b) Assume $p \geq 5$ and $f = 1$. Since $\mathrm{PSL}_2(5) \cong \mathrm{A}_5$, we assume that $p \geq 7$. Then $\gamma \leq 3\log p/(\log(p+5) - 2)^2 < 3 \log p/(\log p - 2)^2$. Clearly, $\gamma < c$ whenever $\log p \geq 6$. So assume that $\log p < 6$ or equivalently $p < 2^6 = 64$ and hence $p \leq 61$. Now we can check that $\gamma < c$ by using Table 3. If $7 \leq p \leq 71$, then $k \geq 5$ by Table 3. So assume $p \geq 71$. Then $k \geq (p+5)/4 \geq 19 > 5$.

(ii)(c) Assume $p \geq 5$ and $f = 2$. If $p \leq 13$, then the result follows by using Table 3. So we assume $p \geq 17$. Then $k \geq (p^2+5)/8 \geq 614$ and $\gamma < (6\log p + 1)/(2\log p - 3)^2 < c$ since $\log p \geq 4$.

(ii)(d) Assume $p \geq 5$ and $3 \leq f \leq 4$. Then $k = (q+5)/4f > 10$ and we can use the same argument as in the previous case to show that $\gamma < c$.

(ii)(e) Assume $p \geq 5$ and $f \geq 5$. We have $k \geq (q+5)/4f > 232$ and $t = f \log p \geq 11$. So $\log f \leq f \log p/4 = t/4$ and

$$\gamma < \frac{3f\log p + \log f}{(f\log p - 2 - \log f)^2} \leq \frac{3t + t/4}{(3t/4-2)^2} = \frac{52t}{(3t-8)^2}.$$

Since $t \geq 11$, we see that $52t/(3t-8)^2 < c$ and thus $\gamma < c$ as wanted.

(iii) $T = \mathrm{PSL}_3(q)$ with $q = p^f \geq 3$. Let $d = \gcd(3, q-1)$. Then $|A| < 2fq^8 \leq q^9, |\operatorname{Out}(T)| = 2df$ and $k(T) \geq (q^2+q)/d$ (see [17]) so $k \geq (q^2+q)/2d^2f$.

(iii)(a) Assume first that $d = \gcd(3, q-1) = 3$. We have $q \geq 2f$ so $k \geq q/9$ and thus $\gamma < 9\log q/(\log q - \log 9)^2 \leq 9 \log q/(\log q - 3)^2$. If $\log q \geq 12$, then $9\log q/(\log q-3)^2 < c$ and $k \geq 819$. So assume $\log q < 12$ or $q < 2^{12}$.

Now if $q = 4$, then $\gamma < 1.954$; if $q = 7$, then $\gamma < c$ by direct calculation using Table 3. If $q = 16$, then $k \geq e(T) = 12$ and we get that $\gamma < c$. Assume that $q \not\in \{4, 7, 16\}$. Then $\gamma < c$ by direct calculation using the definition of γ with $k \geq (q^2+q)/18f$ and $|A| \leq 2fq^8$. By Table 3, we see that $k \geq 5$ if $q \leq 9$. Assume $q \geq 11$. If $q/9 > 4$ or $q > 36$ then $k \geq 5$. So we may assume $11 \leq q \leq 35$. Except for $q = 16$, we see that $k \geq (q^2+q)/18f \geq 5$. For $q = 16$, we can see by [13] that $k \geq e(\mathrm{PSL}_3(16)) = 12$.

(iii)(b) Assume $d = 1$. Here, the argument is similar with $k \geq (q^2+q)/2f \geq q+1 > q$ and so $\gamma < (8\log q + \log(2f))/(\log q)^2 \leq 9/\log q$. Clearly if $q \geq 53$, then $9/\log q < c$ and thus $\gamma < c$. For the remaining values of $q > 2$, direct calculation confirms that $\gamma < c$. Now, if $q \geq 4$, then $k \geq q+1 \geq 5$. For the remaining values of q, we see that $k \geq 5$.

(iv) Assume $n \geq 3$ and $q = 2$. Then we may assume that $n \geq 5$ as $\mathrm{PSL}_4(2) \cong \mathrm{A}_8$ and $\mathrm{PSL}_3(2) \cong \mathrm{PSL}_2(7)$. If $n = 5$, then $k = 20$ and $\gamma < c$. So assume $n \geq 6$. We have that $d = (n, q-1) = 1$ and $f = 1$ so $|\operatorname{Out}(T)| = 2$. Hence $k \geq 2^{n-2} \geq 16$ and thus $\gamma < n^2/((n-2)^2\log(n-2))$. Since $n \geq 6$ we see that

$$\frac{n^2}{(n-2)^2\log(n-2)} \leq \frac{9}{8} < c.$$

So we can assume from now on that $n \geq 4$ and $q \geq 3$. Then, we have $k(T) \geq q^{n-1}/d$ (see [12, Corollary 3.7]) and thus $k \geq q^{n-1}/(2d^2f) \geq q^{n-2}/d^2 \geq q^{n-3}/d \geq q^{n-4}$. Therefore

$$\gamma < \frac{(n^2-1)\log q + \log(2f)}{((n-1)\log q - \log(2d^2f))^2 \log((n-1)\log q - \log(2d^2f))} \tag{6}$$

or

$$\gamma < \frac{(n^2-1)\log q + \log(2f)}{((n-2)\log q - 2\log d)^2 \log((n-2)\log q - 2\log d)}. \tag{7}$$

(v) Assume $4 \leq n \leq 7$ and $q \geq 3$. We can use the same argument as in Case (iii) above to obtain the result. As an example, assume that $n = 4$. We deduce from Inequality (7) that

$$\gamma < \frac{15\log q + \log(2f)}{(2\log(q) - 2\log d)^2} \leq \frac{4\log q}{(\log q - \log d)^2} \leq \frac{4 \log q}{(\log q - 2)^2}.$$

Table 3. $\mathrm{PSL}_2(q)$ and $\mathrm{PSL}_3(q)$ with small q

T	k	$\gamma <$	T	k	$\gamma <$
$\mathrm{PSL}_2(8)$	5	1.613	$\mathrm{PSL}_2(16)$	7	1.193
$\mathrm{PSL}_2(32)$	9	1.036	$\mathrm{PSL}_2(64)$	15	0.686
$\mathrm{PSL}_2(7)$	5	1.281	$\mathrm{PSL}_2(11)$	7	0.884
$\mathrm{PSL}_2(13)$	8	0.778	$\mathrm{PSL}_2(17)$	10	0.642
$\mathrm{PSL}_2(19)$	11	0.595	$\mathrm{PSL}_2(23)$	13	0.525
$\mathrm{PSL}_2(25)$	10	0.782	$\mathrm{PSL}_2(27)$	7	1.351
$\mathrm{PSL}_2(29)$	16	0.456	$\mathrm{PSL}_2(31)$	17	0.438
$\mathrm{PSL}_2(37)$	20	0.397	$\mathrm{PSL}_2(41)$	22	0.375
$\mathrm{PSL}_2(43)$	23	0.366	$\mathrm{PSL}_2(47)$	25	0.349
$\mathrm{PSL}_2(49)$	17	0.526	$\mathrm{PSL}_2(53)$	28	0.329
$\mathrm{PSL}_2(59)$	31	0.312	$\mathrm{PSL}_2(61)$	32	0.307
$\mathrm{PSL}_2(67)$	35	0.294	$\mathrm{PSL}_2(71)$	37	0.286
$\mathrm{PSL}_2(121)$	37	0.337	$\mathrm{PSL}_2(169)$	50	0.292
$\mathrm{PSL}_3(4)$	6	1.954	$\mathrm{PSL}_3(7)$	15	0.781
$\mathrm{PSL}_3(3)$	9	0.805	$\mathrm{PSL}_3(5)$	19	0.518
$\mathrm{PSL}_3(8)$	17	0.783	$\mathrm{PSL}_3(9)$	32	0.471

We see that $4\log q/(\log q-2)^2 < c$ whenever $\log q \geq 6$ and thus $\gamma < c$. For all $q \geq 3$ with $\log q < 6$ or equivalently $q < 2^6 = 64$, direct calculation using Equation (6) shows that $\gamma < c$. Since $k \geq q^2/d^2 \geq q^2/16$, we see that $k \geq 5$ if $q > 8$. For $3 \leq q \leq 8$, we can check directly that $k \geq 5$.

(vi) Assume $n \geq 8$ and $q \geq 3$. Then $k \geq q^{n-4} \geq 81$,

$$\frac{n^2}{(n-4)^2} \leq 4$$

and $\log((n-4)\log q) \geq \log 4 = 2$. From Inequality (6), we have that

$$\gamma < \frac{n^2}{(n-4)^2 \log q \log((n-4)\log q)} \leq \frac{4}{2\log q} < c.$$

This completes the proof. □

The proof of Theorem 3.2 now follows by combining Lemmas 3.3 and 3.4.

4. Almost simple groups

In this section, we prove the following.

Theorem 4.1. *Let G be an almost simple group with a non-abelian simple socle T. Suppose that $k = k^*(T) \leq 153$. Then*

$$\log|G| \leq (\log 3)k(G). \tag{8}$$

We now describe our strategy for the proof of this theorem. We consider the following setup. Let T be a non-abelian simple group and, $A := \mathrm{Aut}(T)$ and $k = k^*(T)$. Let G be an almost simple group with socle T, i.e., $T \trianglelefteq G \leq A$.

Firstly, if T is a sporadic simple group, the Tits group or an alternating group of degree at most 22, then the result follows by direct computation with [13] or [4]. For $T = \mathrm{A}_n$ with $n \geq 23$, it follows from the proof of Lemma 3.3 that $k^*(T) \geq 250 > 153$. So we may assume that T is a finite simple group of Lie type.

Now suppose that T is of Lie rank r and defined over a field of size q. Let d be defined as in Section 3. Then we know that $k(T) \geq q^r/d$ and thus $k = k^*(T) \geq q^r/d|\,\mathrm{Out}(T)|$. We now use the restriction $k \leq 153$ to obtain a finite list $\mathcal{L}$ of all simple groups T with $k \leq 153$. Since $k(G) \geq k^*(T)$ by [21, Lemma 2.5] and $\log|G| \leq \log|A|$, if we can show that

$$\log|A| \leq (\log 3)k \tag{9}$$

then obviously Inequality (8) holds. Finally, for the remaining groups, we can check Inequality (8) directly using the known bound for $k(T)$ or using [4, 13, 17].

For the purpose of computation, the following observation will be useful. Suppose that $A := \mathrm{Aut}(T) = \Gamma\langle\tau\rangle$, where $T \trianglelefteq \Gamma \leq A$ with $|A : \Gamma| = s$ for some integer $s \geq 1$. Now, if we can prove that for every almost simple group G with $T \trianglelefteq G \leq \Gamma$, we have $s \cdot |G| \leq 3^{k(G)/s}$, then we get $|H| \leq 3^{k(H)}$ for all almost simple groups H with socle T. This follows from the fact that if $T \trianglelefteq H \leq A$, then $G := H \cap \Gamma$ has index at most s in H, so $|H| \leq s|G|$ and $k(H) \geq k(G)/s$. Therefore, if $s \cdot |G| \leq 3^{k(G)/s}$, then obviously $3^{k(H)} \geq 3^{k(G)/s} \geq s|G| \geq |H|$ as wanted. This will be useful when we can compute $k(G)$ for all $T \trianglelefteq G \leq \Gamma$. This observation applies when, for example, $T = \mathrm{PSL}_n(q), (n \geq 3)$, $\Gamma = \mathrm{P\Gamma L}_n(q)$ and $A = \Gamma\langle\tau\rangle$, where τ is a graph automorphism of T of order 2.

Lemma 4.2. *Theorem 4.1 holds for $T = \mathrm{PSL}_n(q)$ with $q = p^f$ and $n \geq 2$.*

Proof. (i) Assume that $n = 2$. Suppose first that $q = 2^f$. From [9, Theorem 38.2], we have $k(T) = 2^f + 1$. Using [13], we can check that the result holds for $2 \leq f \leq 7$. Assume $f > 7$. Since $153 \geq k \geq (2^f + 1)/f$, we deduce that $7 < f \leq 11$. We have that $\log|G| \leq \log|\mathrm{Aut}(T)| \leq 3f + \log f$. If $G = T$, then the result is obvious, so we may assume $G \neq T$. We now can use [4] to show that Inequality (8) holds for all almost simple groups G with socle $T = \mathrm{PSL}_2(2^f)$, with $7 < f \leq 11$.

Assume next that $q = p^f \geq 7$ is odd. Then $k(T) = (q+5)/2$ and $k(\mathrm{PGL}_2(q)) = q + 2$, see [9, Theorem 38.1]. Clearly, we can check that the result holds in these cases. So we may assume from now on that $G \ncong \mathrm{PSL}_2(q)$ nor $\mathrm{PGL}_2(q)$. Moreover, if $f \geq 5$ and $p \geq 5$, then $k \geq (p^{2f} + 5)/(4f) \geq (5^{2f} + 5)/(4f) \geq 154$. So we only need to consider the following cases.

If $f = 1$, then $q = p^f = p \geq 5$. Since $153 \geq k \geq (p+5)/4$, we have $p \leq 607$. So $G = \mathrm{PSL}_2(p)$ or $\mathrm{PGL}_2(p)$ with $p \leq 607$ and the result follows using [4].

If $f = 2$, then, arguing as above, we obtain that $p \leq 31$. Similarly, if $f = 3$, then $p \leq 11$ and finally, if $f = 4$, then $p \leq 7$. Now, we can use [4] to verify that Inequality (8) holds in these cases.

(ii) Assume that $q = 2$ and $n \geq 3$. Then $k \geq 2^{n-1}/(2d^2 f) = 2^{n-2}$ as $f = d = 1$. Since $k \leq 153$, we have $n \leq 9$. Now, if $n \geq 7$, then $(\log 3)k \geq (\log 3)2^{n-2} \geq n^2 \geq \log |A|$, hence Inequality (9) holds and so Inequality (8) holds in this case. For $3 \leq n \leq 6$, we can check directly that Inequality (8) holds using [13].

(iii) Assume that $q = 3$ and $n \geq 3$. Then $d = \gcd(n, q-1) = \gcd(n, 2) \leq 2$ and $f = 1$, so $k \geq 3^{n-1}/(2d^2 f) = 3^{n-1}/8$. Since $k \leq 153$, we have $n \leq 7$.

If $n = 7$, then $d = \gcd(7, 2) = 1$ so $(\log 3)k \geq (\log 3)3^{n-1}/2 > n^2 \log 3 > \log |A|$ and thus Inequality (9) holds.

If $n = 6$, then $d = \gcd(6, 2) = 2$ and $k(T) = 204$ by [4]. So $k \geq k(T)/|\operatorname{Out}(T)| \geq 51$. Now we can check that $\log |A| < 36 \log 3 < 51 \log 3 < (\log 3)k$.

If $n = 5$, then $d = \gcd(5, 2) = 1$ and $k(T) = 116$, so $k \geq 116/2 = 58$. Hence Inequality (9) holds.

Finally, if $n = 3, 4$, then the results follow by using [4].

(iv) Assume now that $n = 3$ and $q \geq 4$. We have that $k(T) \geq (q^2 + q)/d$ and thus $k \geq (q^2 + q)/(2d^2 f) \geq (q + 1)/d^2 \geq (q + 1)/9$. Since $k \leq 153$, we have $q \leq 1376$. For these values of q, we can check directly that $\log |A| \leq 9 \log q \leq (\log 3)(q^2 + q)/(2d^2 f) \leq k(\log 3)$ unless $q \in \{4, 7, 8, 13, 16, 19, 25\}$.

The cases when $q = 4, 7, 8$ can be checked directly using [13]. For $q = 16, 25$, we can check that $2|G| \leq 3^{k(G)/2}$ for all $T \trianglelefteq G \leq \Gamma$ and thus the results follow by the observation above.

(v) Assume that $n = 4$ and $q \geq 4$. Since $d = \gcd(n, q-1) \leq n = 4$ and $153 \geq k \geq q^3/(2d^2 f) \geq q^2/16$, we deduce that $4 \leq q \leq 49$. However, we can check that $\log |A| < 36 \log q < (\log 3)q^3/(2d^2 f) \leq (\log 3)k \leq (\log 3)k$ unless $q = 4, 5, 9$. Now we use the observation and [4] to show that Inequality (8) holds for the remaining cases.

(vi) Assume that $n = 5$ and $q \geq 4$. Since $d = \gcd(n, q-1) \leq 5$ and $153 \geq k \geq q^4/(2d^2 f) \geq q^3/25$, we deduce that $q \leq 15$, so $q = 4, 5, 7, 8, 9, 11, 13$. However, we can check with [4] that $\log |A| < 25 \log q < (\log 3)q^4/(2d^2 f) \leq (\log 3)k$, so Inequality (9) holds.

(vii) Assume that $n = 6$ and $q \geq 4$. Since $d = \gcd(n, q-1) \leq 6 = n$ and $153 \geq k \geq q^5/(2d^2 f) \geq q^4/36$, we deduce that $q \leq 8$, so $q = 4, 5, 7, 8$. However, we can check that $\log |A| < 36 \log q < (\log 3)q^5/(2d^2 f) \leq (\log 3)k$ unless $q = 4$. Now we use the observation together with [4] to verify (8) for this case.

(viii) Assume that $n = 7$ and $q \geq 4$. Since $d = \gcd(n, q-1) \leq n = 7$ and $153 \geq k \geq q^6/(2d^2 f) \geq q^5/49$, we deduce that $q \leq 5$, so $q = 4, 5$. However, we can check that $\log |A| < 49 \log q < (\log 3)q^6/(2d^2 f) \leq (\log 3)k$.

(ix) Assume that $n \geq 8$ and $q \geq 4$. We see that $k \geq q^{n-1}/(2d^2 f) \geq q^{n-2}/d^2 \geq q^{n-4} \geq 4^4 = 256 > 153$. So this case cannot occur. □

For almost simple groups with non-abelian simple socle $T = \mathrm{PSU}_n(q)$, the argument is exactly the same. So we skip the proof.

Lemma 4.3. *Theorem 4.1 holds for $T = \mathrm{PSU}_n(q)$ with $n \geq 3$.*

Lemma 4.4. *Theorem 4.1 holds for $T = \mathrm{PSp}_{2n}(q)$ with $n \geq 2$, and for $T = \Omega_{2n+1}(q)$ with $n \geq 3$, q odd.*

Proof. Notice that $\Omega_{2n+1}(q)$ and $\mathrm{PSp}_{2n}(q)$, where $n \geq 3$ and q is odd, have the same order and their full automorphism groups also have the same order. Since $d = 2$, they also have the same lower bound $q^n/d|\mathrm{Out}(T)| = q^n/4f$ for $k = k^*(T)$.

(1) Assume $n = 2$. Then $T \cong \mathrm{PSp}_4(q)$ and $k(T) \geq (q^2 + 5q)/2$ if $q \geq 3$ is odd and $k(T) \geq q^2 + 2q$ if q is even.

Assume first that $q \geq 3$ is odd. We can assume $q \geq 5$ as $\mathrm{PSp}_4(3) \cong \mathrm{PSU}_4(2)$. Since $k \leq 153$, we have $153 \geq k \geq q(q+5)/(4f) \geq q$. Now for odd q with $5 \leq q \leq 153$, we see that $\log|G| < 10\log q + \log f < (\log 3)(q^2 + 5q)/(4f) \leq (\log 3)k \leq (\log 3)k(G)$, unless $q = 5, 9$. For the exceptions, we can check with [13] that the results hold.

For even q, since $153 \geq k \geq (q^2 + 2q)/2f \geq q + 2$, we have $q \leq 151$. We see that $\log|G| < 10\log q + \log f < (\log 3)(q^2 + 2q)/(2f) \leq (\log 3)k \leq (\log 3)k(G)$, unless $q = 4, 8$. For $q = 4$, the result holds by using [13]. We are left with $T = \mathrm{PSp}_4(8)$. We have that $|\mathrm{Out}(T)| = 6$ and T has a field automorphism of order 3 and a graph automorphism of order 2. Observe that $T \cdot 3 = \mathrm{P\Gamma Sp}_4(8)$. By [4], we have that $k(T \cdot 3) = 57$ and $k(T) = 83$. Now we can check that for $T \trianglelefteq H \leq T \cdot 3$, we have $2|H| < 3^{k(H)/2}$ such that $|G| \leq 3^{k(G)}$ for all almost simple groups G with socle T by the observation above.

(2) Assume that $n \geq 3$ and $q = 2$. Then $T \cong \mathrm{PSp}_{2n}(2)$, $d = 1 = f$ and $k \geq 2^n$. Since $k \leq 153$, we deduce that $3 \leq n \leq 7$. Notice that $|\mathrm{Out}(T)| = 1$ so $G = T$. Now we can check directly that Inequality (8) holds in these cases.

(3) Assume that $n \geq 3$ and $q \geq 4$ is even. Then $T \cong \mathrm{PSp}_{2n}(q)$, $k \geq q^n/(d^2 f) = q^n/f \geq q^{n-1} \geq 4^{n-1}$. As $k \leq 153$, we have $n \leq 4$.

Assume that $n = 4$. Then $153 \geq k \geq q^4/f \geq q^3$ which implies that $q \leq 5$. So we have $q = 4$. Since $|\mathrm{Out}(T)| = |\mathrm{Out}(\mathrm{PSp}_8(4))| = 2$, $G = T$ or $T \cdot 2$. However, we see that $\log|G| \leq \log(|A|) \leq 73 < 128(\log 3) \leq (\log 3)k$.

Assume that $n = 3$. Then $153 \geq k \geq q^3/f \geq q^2$. So $q \leq 11$. Hence $q = 4, 8$. However, we see that $\log(|A|) < (2n^2 + n)\log q + \log f < (\log 3)q^3/f \leq (\log 3)k$ in these cases.

(4) Assume that $n \geq 3$ and $q \geq 3$ is odd. We have that $153 \geq k \geq q^n/(d^2 f) \geq q^{n-1}/2 \geq 3^{n-1}/2$ so $n \leq 6$.

Assume that $n = 3$. Then $153 \geq k \geq q^3/4f \geq q^2/2$, so $3 \leq q \leq 17$ and q is odd. Now we can check that $\log|A| < 21\log q + \log f < (\log 3)q^3/(4f) \leq (\log 3)k$ unless $q = 3$. For $\mathrm{PSp}_6(3)$ or $\Omega_7(3)$, using [13] $k = 50, 52$ respectively and we can check that $\log|A| < (\log 3)k$.

Assume that $n = 4$. Then $153 \geq k \geq q^4/4f \geq q^3/2$, so $q = 3, 5$. As above, we can see that $\log|A| < 21\log q + \log f < (\log 3)q^4/(4f) \leq (\log 3)k$ unless $q = 3$. For $T = \mathrm{PSp}_8(3)$ or $\Omega_9(3)$, we see that $|\mathrm{Out}(T)| = 2$ and $k(T) \geq 218$. Now, we can check that $\log|A| < (\log 3)k$ with $k \geq k(T)/2f \geq 109$.

Assume that $n = 5$. Then $153 \geq k \geq q^5/4f \geq q^4/2$, so $q = 3$ and $T \cong \mathrm{PSp}_{10}(3)$ or $\Omega_{11}(3)$. From [17], we can check that $k(T) \geq 430$ so $k \geq k(T)/2 \geq 215 > 153$.

Finally, assume that $n = 6$. Then $k \geq q^6/4f \geq q^5/2$ which implies that $q = 3$. But then $k \geq 3^6/4f = 3^6/4 > 182 > 153$. □

Lemma 4.5. *Theorem 4.1 holds for $T = \mathrm{P}\Omega_{2n}^+(q)$ with $n \geq 4$.*

Proof. (1) Assume first that $n = 4$. Recall that $|\mathrm{Out}(T)| = 6df$, so $k \geq q^4/6d^2f$. Since $153 \geq k \geq q^4/(6d^2f) \geq q^3/2^6$, we have $q \leq 21$. We see that $\log|A| < 28\log q + \log(6f) < (\log 3)q^4/(6d^2f) \leq (\log 3)k$ unless $q = 2, 3, 4, 5, 7, 9$. If $2 \leq q \leq 4$, then we can use [13] to verify Inequality (8).

Assume that $q = 7$. The character table of T is available in [13] and we can see that T has exactly 91 distinct character degrees. Since the number of orbits of A on the conjugacy classes of T is the same as that of A on $\mathrm{Irr}(T)$, we see that $k^*(T)$ is at least the number of distinct character degrees of T which implies that $k \geq 91$. Now, we can check that Inequality (9) holds.

Using [17], we see that for odd q, we have

$$\begin{aligned} k(\mathrm{PGO}_8^+(q)) &= q^4 + q^3 + 7q^2 + 10q + 18 \\ k(\mathrm{SO}_8^+(q)) &= q^4 + q^3 + 8q^2 + 15q + 25 \end{aligned}$$

and if $q \equiv 1 \pmod 4$, then

$$k(\mathrm{Spin}_8^+(q)) = q^4 + q^3 + 13q^2 + 28q + 45.$$

It follows that $k(T) \geq k(\mathrm{Spin}_8^+(q))/4 \geq (q^4 + q^3 + 13q^2 + 28q + 45)/4$.

Assume that $q = 9$. Then $|A : T| = 48$, $\log|A| \leq 93$ and $k(T) \geq 2160$. Now if $T \trianglelefteq G \lneq A$, then $|G : T| \leq 24$ and we can check that $\log|A| < 93 < 142 \leq (\log 3)k(T)/24 \leq (\log 3)\log|G|$, so the lemma holds whenever $G \neq A$. For $G = A$, we see that $k(A) \geq k(\mathrm{PGO}_8^+(q))/|A : \mathrm{PGO}_8^+(q)| \geq 3982$ and clearly $\log|A| \leq (\log 3)k(A)$ in this case.

Assume that $q = 5$. Then $|A : T| = 24$, $\log|A| \leq 68$ and $k(T) \geq 315$. If $|G : T| \leq 6$, then $\log|G| \leq 66 < 83 \leq (\log 3)k(T)/6 \leq (\log 3)k(G)$. If $\mathrm{PGO}_8^+(5) \leq G \leq A$, then Inequality (8) holds as $262 < (\log 3)k(\mathrm{PGO}_8^+(5))/6 \leq k(G)$. So if G/T induces only graph or diagonal automorphisms, then $|G : T| \leq 6$ and the lemma holds. Hence, we only need to consider the cases $|G : T| = 8, 12$ or 24. In these cases, we observe that G must contain an outer diagonal automorphism of order 2. There are three such extensions G_i of T in $\mathrm{PGO}_8^+(3)$ and they are fused by the graph automorphism of T of order 3, hence they all have the same number of conjugacy classes which is at least $k(\mathrm{SO}_8^+(5))/2 \geq 525$. Now, we see that $(\log 3)k(G) \geq (\log 3)k(G_1)/12 \geq 69 > \log|A| \geq \log|G|$ and the result holds.

(2) Assume that $n \geq 5$ and $q = 2$. Then $f = 1 = d$ and so $153 \geq k \geq q^n/2 = 2^{n-1}$, hence $5 \leq n \leq 8$. Now we see that $\log|A| \leq 2n^2 - n + 1 \leq (\log 3)2^{n-1} \leq (\log 3)k$ unless $n = 5, 6$. For $q = 5, 6$ we have that $k(T) = 97, 271$, respectively and $|\operatorname{Out}(T)| = 2$. We can check in this case that Inequality (9) holds using the bound $k \geq k(T)/2$.

(3) Assume that $n \geq 5$ and $q = 3$. Then $f = 1$ and $d = \gcd(4, 3^n - 1) \leq 4$. Since $153 \geq k \geq 3^n/2d^2f \geq 3^n/2^5$, we have $5 \leq n \leq 7$. If $n = 7$, then Inequality (9) holds using the bound $k \geq 3^n/(2d^2)$. If $n = 5$, then $k(T) = 393$ and Inequality (9) also holds with $k \geq k(T)/|\operatorname{Out}(T)|$. For $n = 6$, from [17], we deduce that $k(T) \geq 692$ and thus $k \geq k(T)/2d \geq 86$, hence $\log|A| \leq (2n^2 - n + 1)\log 3 = 67\log 3 \leq 86\log 3 \leq (\log 3)k$.

(4) Assume $n \geq 5$ and $q \geq 4$. Then $d = \gcd(4, q^n - 1) \leq 4$ and $153 \geq k \geq q^n/(2d^2f) \geq q^{n-1}/2^4$, so $5 \leq n \leq 6$.

Assume that $n = 6$. Then $153 \geq q^6/2d^2f \geq q^5/16$, hence $q = 4$. However, we can check that $\log|G| \leq (2n^2 - n + 1)\log q \leq (\log 3)q^n/(2d^2f) \leq (\log 3)k(G)$.

Assume that $n = 5$. Then $153 \geq q^5/2d^2f \geq q^4/16$, hence $q = 4, 5, 7$. However, we can check that $\log|G| \leq (2n^2 - n + 1)\log q \leq (\log 3)q^n/(2d^2f) \leq (\log 3)k(G)$. □

The proof of the next lemma is similar.

Lemma 4.6. *Theorem 4.1 holds for* $T = \mathrm{P}\Omega_{2n}^-(q)$ *with* $n \geq 4$.

Lemma 4.7. *Theorem 4.1 holds for* T *a finite simple exceptional group of Lie type.*

Proof. (1) Assume $T = {}^2\mathrm{B}_2(q^2)$ with $q^2 = 2^{2m+1}, m \geq 1$. We have that $k(T) = 2^{2m+1} + 3$ and $|\operatorname{Out}(T)| = 2m + 1$. So $|A| < (2m+1)2^{10m+5}$ and $k \geq (2^{2m+1} + 3)/(2m+1)$. Since $153 \geq k \geq (2^{2m+1} + 3)/(2m+1)$, we must have $1 \leq m \leq 5$. If $m = 4, 5$, then $\log|A| \leq (\log 3)(2^{2m+1} + 3)/(2m+1) \leq (\log 3)k$. If $m = 1, 2$, then the results follow by using [13]. For $m = 3$, we see that $k(T) = 2^7 + 3 = 131$ and we can check directly that $\log|A| \leq (\log 3)131/7 \leq (\log 3)k$.

(2) Assume $T = {}^2\mathrm{G}_2(q^2), q^2 = 3^{2m+1}, m \geq 1$. We have $k(T) = 3^{2m+1} + 8$ and $|\operatorname{Out}(T)| = f$. So $|A| < (2m+1)3^{7(2m+1)}$ and $153 \geq k \geq k(T)/f \geq (3^{2m+1} + 8)/(2m+1)$. Hence $m = 1, 2$. If $m = 1$, then the lemma holds by using [13]. For $m = 2$, we can check that $\log|A| \leq (\log 3)(3^5 + 8)/5 \leq (\log 3)k$.

(3) Assume $T = {}^2\mathrm{F}_4(q^2), q^2 = 2^{2m+1}, m \geq 1$. Then $|\operatorname{Out}(T)| = 2m + 1$ and $|A| < (2m+1)2^{26(2m+1)} < 2^{27(2m+1)}$ and $k \geq (q^4 + 4q^2 + 17)/f$. If $m \geq 2$, then $\log|A| \leq (\log 3)(q^4 + 4q^2 + 17)/f \leq (\log 3)k$. If $m = 1$, then the character table of $T = {}^2\mathrm{F}_4(8)$ is available on [13] and we have that $\log|G| \leq 80$ and $k(T) = 113$. So Inequality (8) holds for T. It follows from [6, § 13.9] that T has 21 unipotent characters and they all extend to A by [18, Theorems 2.4, 2.5]. Therefore, we see that $k(A) \geq 21 \cdot 3 + (113 - 21)/3 \geq 93$, hence Inequality (8) holds for A.

(4) Assume $T = {}^3\mathrm{D}_4(q)$. Then $|\operatorname{Out}(T)| = 3f$, $k \geq q^4/(3f)$ and $|A| < 6fq^{28}$. If $q = 2$, then the lemma holds by using [13]. So assume $q \geq 3$. As $153 \geq q^4/3f \geq q^3/3$, we have $q \leq 7$. Now $\log|A| \leq 28\log q + \log(6f) \leq (\log 3)q^4/(3f) \leq (\log 3)k$ unless $q = 3$. Now from [17], we have that $k(T) = 126$ and so $\log|A| \leq (\log 3)126/(3f) \leq (\log 3)k$.

(5) Assume $T = \mathrm{G}_2(q), q \geq 3$. Then $d = 1$, $|\operatorname{Out}(T)| = f$ if $p \neq 3$ and $|\operatorname{Out}(T)| = 2f$ if $p = 3$, $k \geq (q^2 + 2q + 8)/(2f) \geq q + 2$ and $|A| < 2fq^{14} \leq q^{15}$. If $q = 3, 4, 5$ then the results follow by using [13]. So assume $q \geq 7$. Since $153 \geq k \geq q + 2$, we have $q \leq 151$. Then $14\log q + \log(2f) \leq (\log 3)(q^2 + 2q + 8)/|\operatorname{Out}(T)| \leq (\log 3)k(G)$ unless $q = 9$. Now, suppose that $T = \mathrm{G}_2(9)$. Then $k(T) = 9^2 + 2 \cdot 9 + 8 = 107$ and $\log(2|T|) \leq 45 < (\log 3)k(T)/2$, so Inequality (8) holds true for T and $T \cdot 2$. We now need to verify Inequality (8) for $A = \operatorname{Aut}(T)$. Notice that A/T is an abelian group of order 4. From [6, § 13.9], T has 10 unipotent characters and they all extend to $\operatorname{Aut}(T) = A$ by [18, Theorems 2.4, 2.5], as $|A/T| = 4$, we deduce that $k(A) \geq 10 \cdot 4 + (k(T) - 10)/4 \geq 40 + (107 - 10)/4 \geq 64$ and hence we can check that Inequality (8) holds for A.

For the remaining groups, the argument is similar. □

5. More on groups with a trivial solvable radical

This section is devoted to proving Theorem 1.2. We use the notations and assumptions of Section 2. We start with a lemma.

Lemma 5.1. *With the notation and assumption in Section 2, we have*

$$\prod_{i=1}^{r} \binom{n_i + k_i - 1}{k_i - 1} \leq k(G).$$

Proof. It is sufficient to show that the number of orbits of G on the set of conjugacy classes of N is at least $\prod_{i=1}^{r}\binom{n_i+k_i-1}{k_i-1}$. For this it is sufficient to show that the number of orbits of G on the set of conjugacy classes of M_i (for any fixed i with $1 \leq i \leq r$) is at least $\binom{n_i+k_i-1}{k_i-1}$. The number of orbits of G on the set of conjugacy classes of M_i is at least the number of orbits of S_{n_i} (with its natural permutation action on the factors of M_i) on the set of conjugacy classes of M_i. By [11, Lemma 2.6], this latter number is precisely $\binom{n_i+k_i-1}{k_i-1}$. □

We continue with another lemma.

Lemma 5.2. *Let $4 \leq k \in \mathbf{N}$. Then $(\log k)^2 \log\log k \leq k^2/2$.*

Proof. Let $x = \log k \geq 2$. Then $\log\log k \leq \log k$ and hence, it suffices to prove that $4^x \geq 2x^3$ which is always true when $x \geq 5$. For $2 \leq x < 5$ or $4 \leq k < 32$, we can check directly that the inequality in the lemma holds true. □

Consider the inequality

$$n_i \log n_i + c_2 n_i (\log k_i)^2 (\log\log k_i) \leq w_i \cdot (\log 3)\binom{n_i + k_i - 1}{k_i - 1} \tag{10}$$

for a fixed positive number w_i.

Lemma 5.3. *In Inequality* (10)*, let* $n = n_i \geq 1$*,* $k = k_i \geq 4$*,* $c_2 = 1.954$*, and let* $w = w_i$*. Then*

(i) *If* $n = 1$ *and* $k \geq 222$*, then Inequality* (10) *holds with* $w = 1$*.*
(ii) *If* $n = 2$ *and* $k \geq 9$*, then Inequality* (10) *holds with* $w = 1$*.*
(iii) *Inequality* (10) *always holds with* $w = 1$ *if* $n \geq 3$*.*
(iv) *If* $n = 2$ *and* $4 \leq k < 9$*, then Inequality* (10) *holds with* $w = 1.17$*.*
(v) *If* $n = 1$ *and* $k \geq 4$*, then Inequality* (10) *holds with* $w = 2.5$*.*

Proof. (i) Assume that $n = 1$ and $w = 1$. Then Inequality (10) is equivalent to

$$c_2(\log k)^2 \log\log k \leq k \log 3. \tag{11}$$

Since $k \geq 4$, we see that $\log k \leq k$ and so $\log\log k \leq \log k$. Hence $c_2(\log k)^2 \log\log k \leq c_2(\log k)^3$. Thus it suffices to show that $c_2(\log k)^3 \leq (\log 3)k$ or

$$2^x \geq c_2 x^3/\log 3$$

where $x = \log k$. Clearly, we can see that this inequality holds when $x \geq 11$ or equivalently $k \geq 2^{11}$. For $k < 2^{11}$, we can check that Inequality (11) holds provided that $k \geq 222$.

(ii) Assume that $n = 2$ and $w = 1$. Then Inequality (10) is equivalent to

$$2 + 2c_2(\log k)^2 \log\log k \leq (\log 3)k(k+1)/2. \tag{12}$$

Observe that $2 + 2c_2(\log k)^2 \log\log k \leq 2 + 2c_2(\log k)^3$ and $(\log 3)k(k+1)/2 \geq 3k^2/4$. So it suffices to show that $3k^2/4 \geq 2 + 2c_2(\log k)^3$. We can see that this inequality holds true when $k \geq 32$. For $4 \leq k < 31$, we can check that Inequality (12) holds only when $k \geq 9$.

(iii) Assume that $n \geq 3$. Suppose first that $n = 3$. Arguing as in (ii), we see that Inequality (10) is equivalent to

$$3\log 3 + 3c_2(\log k)^2 \log\log k \leq (\log 3)k(k+1)(k+2)/6. \tag{13}$$

Observe that

$$3\log 3 + 3c_2(\log k)^2 \log\log k \leq 6 + 3c_2(\log k)^3$$

and

$$(\log 3)k(k+1)(k+2)/6 \geq k^3/4.$$

So it suffices to show that $k^3/4 \geq 6 + 3c_2(\log k)^3$. Clearly, the latter inequality holds true when $k \geq 8$. For $4 \leq k < 8$, we can check directly that Inequality (12) holds. The same argument can be applied for $n = 4, 5$ to show that Inequality (10) holds.

So assume that $n \geq 6$. Assume next that $k = 4$. Then Inequality (10) is equivalent to

$$n\log n + 4c_2 n \leq (\log 3)(n+1)(n+2)(n+3)/6. \tag{14}$$

Since $n\log n + 4c_2 n \leq n^2 + 8n$ and

$$\binom{n+3}{3}\log 3 \geq (n+3)(n+2)(n+1)/4,$$

to prove Inequality (14), it suffices to show that $4n(n+8) \leq (n+3)(n+2)(n+1)$ which is always true as $n \geq 6$. Therefore, one can assume that $n \geq 6$ and $k \geq 5$.

Since $k-1 \geq 4$, we deduce that

$$\binom{n+k-1}{n} = \binom{n+k-1}{k-1} \geq \binom{n+k-1}{4}.$$

Hence, as $\log 3 \geq 3/2$, we have

$$(\log 3)\binom{n+k-1}{k-1} \geq \frac{3}{2}\binom{n+k-1}{4} = \frac{(n+k-1)(n+k-2)(n+k-3)(n+k-4)}{16}.$$

Since $(\log k)^2 \log\log k \leq k^2/2$ by Lemma 5.2 and $\log n \leq n$, we deduce that

$$n\log n + c_2 n(\log k)^2(\log\log k) \leq n^2 + nk^2.$$

Therefore, it suffices to show that

$$(n-1+k)(n-2+k)(n-3+k)(n+k-4) \geq 16n(n+k^2). \tag{15}$$

Since $n+k-4 \geq n$, to prove (15), it suffices to prove that

$$(n-1+k)(n-2+k)(n-3+k) \geq 16n + 16k^2. \tag{16}$$

We have that

$$(n-1+k)(n-2+k)(n-3+k) = (n-1)(n-2)(n+k-3) + (2n-3)k(n+k-3) + k^2(n+k-3).$$

Since $n+k-3 \geq n \geq 6$, we have

$$(n-1)(n-2)(n+k-3) \geq 5\cdot 4\cdot n = 20n > 16n. \tag{17}$$

Since $n+k-3 \geq k \geq 5$ and $n \geq 6$, we have

$$k(2n-3)(n+k-3) \geq 9k^2 \tag{18}$$

and

$$(n+k-3)k^2 \geq (6+5-3)k^2 = 8k^2. \tag{19}$$

Adding (18) and (19), we obtain that

$$k(2n-3)(n+k-3) + (n+k-3)k^2 \geq 17k^2 > 16k^2. \tag{20}$$

Now (16) follows by adding (17) and (20).

Finally, (iv) and (v) can be checked using a computer. □

Using the information from Lemma 5.3 we define numbers w_i for each i with $1 \leq i \leq r$. If $n_i = 1$ and $4 \leq k_i < 222$, then put $w_i = 2.5$. If $n_i = 2$ and $4 \leq k_i < 9$, then put $w_i = 1.17$. In all other cases put $w_i = 1$. We need another lemma.

Lemma 5.4. *Let r be a positive integer and let $x_1, \ldots, x_r$ be integers which are at least 4. Then the following are true.*

(*i*) *If $r \geq 3$ then $2.5 \cdot \sum_{i=1}^{r} x_i \leq \prod_{i=1}^{r} x_i$.*
(*ii*) *If $r = 2$ then $2.5x_1 + 1.17x_2 \leq x_1x_2$.*

(*iii*) *If* $r = 2$ *and* $x_i \geq 5$, *then* $2.5x_1 + 2.5x_2 \leq x_1 x_2$.

Proof. (i) can be seen by induction on r. (ii) and (iii) are easy computations. □

Proof of Theorem 1.2. By Lemmas 2.4 and 5.3, we have

$$\log|G| < \sum_{i=1}^{r} \Big(n_i \log n_i + c_2 n_i (\log k_i)^2 (\log\log k_i) \Big) \leq (\log 3) \sum_{i=1}^{r} w_i \binom{n_i + k_i - 1}{k_i - 1}.$$

By Lemma 5.4 and the fact that the binomial coefficients we consider are all at least 4 (since $k_i \geq 4$ and $n_i \geq 1$ for every i with $1 \leq i \leq r$), this is at most

$$(\log 3) \prod_{i=1}^{r} \binom{n_i + k_i - 1}{k_i - 1} \leq (\log 3) k(G)$$

where the last inequality follows from Lemma 5.1, unless possibly if one of the following cases holds.

(1) $r = 1$, $n_1 = 1$ and $4 \leq k_1 < 222$;
(2) $r = 1$, $n_1 = 2$ and $4 \leq k_1 < 9$; or
(3) $r = 2$, $n_1 = n_2 = 1$ and $k_1 = k_2 = 4$.

In all cases the group G has a socle which is the product of at most two non-abelian simple groups.

Case $r = 1$ and $n_1 = 2$. Observe that when $n_1 = 2$, then Inequality (10) holds for simple groups T with $\gamma(T) \leq 1.613$ and $w_1 = 1$. So $\log|G| < (\log 3)k(G)$ whenever $\mathrm{Soc}(G) \cong T^2$ and $T \ncong \mathrm{PSL}_3(4), \mathrm{A}_5$. For the remaining cases, we see that

$$\mathrm{Soc}(G) \cong T^2 \trianglelefteq G \leq \mathrm{Aut}(T^2) \cong \mathrm{Aut}(T) \wr \mathrm{Sym}(2).$$

Now using [4], we can check that $\log|G| \leq (\log 3)k(G)$.

Case $r = 2$, $n_1 = n_2 = 1$ and $k_1 = k_2 = 4$. Then $\mathrm{Soc}(G) \cong T_1 \times T_2$ and

$$T_1 \times T_2 \trianglelefteq G \leq \mathrm{Aut}(T_1) \times \mathrm{Aut}(T_2),$$

where T_i is a non-abelian simple group with $k_i = k^*(T_i) = 4$ for $i = 1$ and 2. It follows from Theorem 3.2 that $T_i = \mathrm{A}_5$ with $i = 1, 2$. Hence $\mathrm{A}_5^2 \trianglelefteq G \leq \mathrm{S}_5 \times \mathrm{S}_5$. Using [4] again, it is routine to check that $\log|G| \leq (\log 3)k(G)$.

Therefore, we are left with the case $r = 1, n_1 = 1$ and $4 \leq k_1 \leq 221$. So G is an almost simple group with non-abelian simple socle T and $4 \leq k = k^*(T) \leq 221$. Clearly, $\log|G| \leq (\log 3)k(G)$ if $T \cong \mathrm{A}_5$ or $\mathrm{PSL}_3(4)$. So we may assume that T is not one of those groups. Then $\gamma(T) < 1.613$ by Theorem 3.2 ($\gamma(T)$ is defined in Section 3). We can now bound k_1 by 153 (see the proof of Lemma 5.3(i)). We obtain the inequality $\log_3|G| \leq k(G)$ by applying Theorem 4.1. As by assumption G is not a 3-group, the latter is a strict inequality and the result follows. □

References

[1] L. Babai and L. Pyber, Permutation groups without exponentially many orbits on the power set. *J. Combin. Theory Ser. A* **66** (1994), no. 1, 160–168.

[2] E. A. Bertram, Lower bounds for the number of conjugacy classes in finite groups, in: *Ischia group theory 2004* Contemp. Math. 402 (Amer. Math. Soc., Providence, RI 2006), pp. 95–117.

[3] E. A. Bertram, New reductions and logarithmic lower bounds for the number of conjugacy classes in finite groups. *Bull. Aust. Math. Soc.* **87** (2013), no. 3, 406–424.

[4] W. Bosma, J. Cannon, and C. Playoust, *The* Magma *algebra system I: The user language*, J. Symbolic Comput. **24** (1997), 235–265.

[5] R. Brauer, Representations of finite groups. 1963 Lectures on Modern Mathematics, Vol. I pp. 133--175 Wiley, New York.

[6] R.W. Carter, *Finite groups of Lie type. Conjugacy classes and complex characters,* Pure and Applied Mathematics (New York). A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1985.

[7] M. Cartwright, The number of conjugacy classes of certain finite groups. *Quart. J. Math. Oxford Ser.* (2) **36** (1985), no. 144, 393--404.

[8] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, *An ATLAS of Finite Groups*, Clarendon Press, Oxford, 1985.

[9] L. Dornhoff, *Group representation theory. Part A: Ordinary representation theory. Pure and Applied Mathematics,* 7. Marcel Dekker, Inc., New York, 1971.

[10] P. Erdős and P. Turán, On some problems of a statistical group-theory. IV. *Acta Math. Acad. Sci. Hungar* **19** (1968) 413--435.

[11] D. A. Foulser, Solvable primitive permutation groups of low rank. *Trans. Amer. Math. Soc.* **143** (1969), 1–54.

[12] J. Fulman, R. Guralnick, Bounds on the number and sizes of conjugacy classes in finite Chevalley groups with applications to derangements, *Trans. Amer. Math. Soc.* **364** (2012), no. 6, 3023–3070.

[13] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.7.5*, 2014, (`http://www.gap-system.org`).

[14] A. Jaikin-Zapirain, On the number of conjugacy classes of finite nilpotent groups. *Adv. Math.* **227** (2011), no. 3, 1129–1143.

[15] T. M. Keller, Finite groups have even more conjugacy classes. *Israel J. Math.* **181** (2011), 433-444.

[16] E. Landau, Über die Klassenzahl der binären quadratischen Formen von negativer Discriminante. *Math. Ann.* **56** (1903), no. 4, 671–676.

[17] F. Lübeck, Character dgrees and their multiplicities for some groups of Lie type of rank < 9, `http://www.math.rwth-aachen.de/ Frank.Luebeck/chev/DegMult/index.html`.

[18] G. Malle, Extensions of unipotent characters and the inductive McKay condition, *J. Algebra* **320** (2008), no. 7, 2963–2980.

[19] A. Maróti, On elementary lower bounds for the partition function, *Integers* **3** (2003), A10, 9 pp.

[20] M. Newman, A bound for the number of conjugacy classes in a group. *J. London Math. Soc.* **43** (1968), 108--110.

[21] L. Pyber, Finite groups have many conjugacy classes. *J. London Math. Soc.* (2) **46** (1992), no. 2, 239–249.

Fakultät für Mathematik, Universität Bielefeld, Postfach 10 01 31, D-33501 Bielefeld, Germany

E-mail address: b.baumeister@math.uni-bielefeld.de

Alfréd Rényi Institute of Mathematics, Reáltanoda utca 13-15, H-1053, Budapest, Hungary

E-mail address: maroti.attila@renyi.mta.hu

Department of Mathematical Sciences, Kent State University, Kent, OH 44242, USA

E-mail address: htongvie@math.kent.edu