Ordering orders and quotient rings*

Alexander Guterman[1,2,3], László Márki[4], Pavel Shteyner[1,2,3]

Dedicated to the memory of K.S.S. Nambooripad,
in highest esteem

Abstract

In the present paper, we introduce a general notion of quotient ring which is based on inverses along an element. We show that, on the one hand, this notion encompasses quotient rings constructed using various generalized inverses. On the other hand, such quotient rings can be viewed as Fountain-Gould quotient rings with respect to appropriate subsets. We also investigate the connection between partial order relations on a ring and on its ring of quotients.

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1 Introduction

Let R be an associative, not necessarily commutative ring. The classical notion of a ring of left quotients Q of its subring R is well known. To be a quotient ring, it is necessary for the ring Q to have an identity element. Then all its elements can be written as "left fractions" $a^{-1}b$, where $a, b \in R$ and every element of R which is not a zero divisor in R should be invertible in Q.

Starting from similar investigations in semigroups, Fountain and Gould introduced in [10] a new generalization of classical quotient rings based on the notion of group inverse. These new quotient rings have been described for some special classes of rings in subsequent research. In particular, such quotient rings need not have an identity.

The procedure of assigning inverses to certain elements is called localization. It can be carried out, more generally, by considering other generalized inverses,

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[1]Faculty of Mathematics and Mechanics, Moscow State University, Moscow, GSP-1, 119991, Russia;
[2] Moscow Institute of Physics and Technology, Dolgoprudny, 141701, Russia
[3] Moscow Center for Fundamental and Applied Mathematics, Moscow, 119991, Russia
[4] Alfréd Rényi Institute of Mathematics, Pf.127, Budapest 1364, Hungary

for example, Moore-Penrose, Drazin, and others. In particular, for rings with involution, rings of quotients with respect to Moore-Penrose inverses were studied in [21] and [4]. An important feature of the notion, introduced in [4], is that it is equipped with an additional parameter, namely, one can specify the elements that are required to have inverses. This leads to certain interesting and useful properties of quotient rings and their orders.

Recently the general concept of an inverse along an element which covers and generalizes the notion of outer generalized inverse was introduced and developed in [15], see also [9]. This notion generalizes all classical outer inverses and unifies many classical notions connected to generalized inverses. In particular, partial order relations on semigroups such as Nambooripad order, sharp order, star order and others, can be defined in terms of outer inverses, see [11, 12].

In the present paper, we introduce a general notion of quotient rings which is based on inverses along an element. We show that, on the one hand, this notion encompasses quotient rings constructed using various generalized inverses. On the other hand, these quotient rings can be viewed as Fountain-Gould quotient rings with respect to appropriate subsets (in the sense of [4]).

Our paper is organized as follows. Sections 2, 3 and 4 contain general information on Green's relations, generalized inverses and inverses along an element, and partial orders on semigroups, correspondingly – much of this is a recapitulation of known results. Section 5 deals with quotient rings. In Section 6 we investigate the connection between partial order relations on a ring and on its ring of quotients.

2 Green's relations

Let S be a semigroup. As usual, S^1 denotes the monoid generated by S, and $E(S)$ denotes the set of idempotents of S. Firstly, we recall some results on Green's relations that we need in the sequel. For more information see for instance [14] or [6].

Definition 2.1. For elements a and b of S, relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$ are defined by

1. $a\mathcal{L}b$ if and only if $S^1a = S^1b$.
2. $a\mathcal{R}b$ if and only if $aS^1 = bS^1$.
3. $a\mathcal{H}b$ if and only if $a\mathcal{L}b$ and $a\mathcal{R}b$.

That is, a and b are *$\mathcal{L}$-related* (*$\mathcal{R}$-related*) if they generate the same left (right) principal ideal, and $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$. As is well known, one can rewrite the $\mathcal{L}$- and the $\mathcal{R}$-relation over a monoid S^1 in terms of equations by substituting Item 1. and Item 2., respectively.

Lemma 2.2.

1. *$a\mathcal{L}b$ if and only if there exist $m, n \in S^1$, such that $ma = b$ and $a = nb$.*
2. *$a\mathcal{R}b$ if and only if there exist $m, n \in S^1$, such that $am = b$ and $a = bn$.*

These are equivalence relations on S, and we denote the $\mathcal{L}$-class ($\mathcal{R}$-class, $\mathcal{H}$-class) of an element $a \in S$ by $\mathcal{L}_a(\mathcal{R}_a, \mathcal{H}_a)$. The $\mathcal{L}$ ($\mathcal{R}$) relation is right (left) compatible, that is, for any $c \in S^1$, $a\mathcal{L}b$ implies $ac\mathcal{L}bc$ ($a\mathcal{R}b$ implies $ca\mathcal{R}cb$).

In parallel with these equivalence relations we have the preorder relations:

1. $a \leq_{\mathcal{L}} b$ if and only if $S^1a \subseteq S^1b$;
2. $a \leq_{\mathcal{R}} b$ if and only if $aS^1 \subseteq bS^1$;
3. $a \leq_{\mathcal{H}} b$ if and only if $a \leq_{\mathcal{L}} b$ and $a \leq_{\mathcal{R}} b$.

3 Generalized inverses

We start by recalling several basic notions.

Definition 3.1. Let $a \in S$.

1. We say that a is *(von Neumann) regular* if $a \in aSa$.
2. A particular solution to $axa = a$ is called an *inner inverse* of a and is denoted by a^-.
3. A solution of the equation $xax = x$ is called an *outer inverse* of a and is denoted by $a^=$.
4. An inner inverse of a that is also an outer inverse is called a *reflexive inverse* and is denoted by a^+.

The set of all inner (resp. outer, resp. reflexive) inverses of a is denoted by $a\{1\}$ (resp. $a\{2\}$, resp. $a\{1,2\}$).

Definition 3.2. A semigroup S is *regular* if all its elements are regular.

The definitions of group, Moore-Penrose and Drazin inverses are standard and can be found in the literature (see, for example, [5, 14]). We provide them here for completeness.

Definition 3.3. Let $a \in S$.

1. The element a is *group invertible* if there is $a^\# \in a\{1,2\}$ that commutes with a.
2. The element a has a *Drazin inverse* a^D if a positive power a^n of a is group invertible and $a^D = \left(a^{n+1}\right)^\# a^n$.
3. If $*$ is an involution in S, then a is *Moore-Penrose invertible* if there is $a^\dagger \in a\{1,2\}$ such that $aa^\dagger$ and $a^\dagger a$ are symmetric with respect to $*$.

Each of these inverses is unique if it exists.

3.1 Inverses along an element

In this section, we recall the definition of inverse along an element, which was introduced in [15], and several useful properties of this inverse.

Lemma 3.4. *[15, Lemma 3] Let $a, b, d, \in S$. Then the following are equivalent.*

1. *$b \leq_{\mathcal{H}} d$, and $d = dab = bad$,*
2. *$b = bab$ and $b\mathcal{H}d$.*

Definition 3.5. We say that b is *an inverse of a along d* (denoted as $b = a^{-d}$) if b satisfies the equivalent conditions of Lemma 3.4.

Theorem 3.6. *[15, Theorem 6] Let $a, d \in S$. If a^{-d} exists, then it is unique.*

Theorem 3.7. *[15, Theorem 7] Let $a, d \in S$. Then the following are equivalent:*

1. *a^{-d} exists.*
2. *$ad\mathcal{L}d$ and $\mathcal{H}_{ad}$ is a group.*
3. *$da\mathcal{R}d$ and $\mathcal{H}_{da}$ is a group.*
   In this case $(ad)^\sharp \in S$ exists and $a^{-d} = d(ad)^\sharp = (da)^\sharp d$.

Denote by $C(X) = \{y \in S \mid xy = yx \text{ for all } x \in X\}$ the centralizer of $X \subseteq S$.

Theorem 3.8. *[11, Lemma 3.31] Let $a, d \in S$. If a^{-d} exists then $a^{-d} = C(C(\{a, d\}))$.*

Theorem 3.9. *[15, Theorem 11] Let $a \in S$. Then*

1. *$a^\sharp = a^{-a}$,*
2. *$a^D = a^{-a^m}$ for some integer m,*
3. *in case S is a $*$-semigroup, $a^\dagger = a^{-a^*}$.*

3.2 Properties of the group inverse

The following statements provide some commutativity relations for group invertible elements of a semigroup and belong to folklore. We include them here with proofs for completeness.

Lemma 3.10. *Let S be an arbitrary semigroup, $u, a \in S$, and u be group invertible. Then $ua = au$ if and only if $au^\# = u^\# a$.*

Proof. Let $au = ua$. By Theorem 3.9 $u^\# = u^{-u}$. Then, by Theorem 3.8, $u^\# \in C(C(\{u\})$. Since $au = ua$ we have $a \in C(\{u\})$. It follows that $u^\# \in C(\{a\})$, that is $u^\# a = au^\#$.

Now let $u^\# a = au^\#$. Therefore, by the above, $ua = (u^\#)^\# a = a(u^\#)^\# = au$. ☐

Corollary 3.11. *Let S be an arbitrary semigroup and $u, a \in S$ be group invertible. Then the following statements are equivalent:*

1. $ua = au$.

2. $au^{\#} = u^{\#}a$.

3. $a^{\#}u = ua^{\#}$.

4. $a^{\#}u^{\#} = u^{\#}a^{\#}$.

Proof. Items 1., 2. and 3. are equivalent by Lemma 3.10. If we apply this lemma to $a^{\#}$ and u, we obtain that Items 3. and 4. are also equivalent. □

Lemma 3.12. *Let S be an arbitrary semigroup. Suppose that $a, b \in S$ are group invertible and $ab = ba$. Then $(ab)^{\#} = a^{\#}b^{\#}$.*

Proof. By Corollary 3.11 $a, b, a^{\#}$ and $b^{\#}$ are mutually commutative. Then $ab(a^{\#}b^{\#}) = (a^{\#}b^{\#})ab$. Also $ab(a^{\#}b^{\#})ab = aa^{\#}abb^{\#}b = ab$. Finally,

$$(a^{\#}b^{\#})ab(a^{\#}b^{\#}) = a^{\#}aa^{\#}b^{\#}bb^{\#} = a^{\#}b^{\#}.$$

Thus $(ab)^{\#} = a^{\#}b^{\#}$. □

4 Order relations on general semigroups and their properties

Let S be an arbitrary semigroup, $a, b \in S$. Following Drazin [8], Nambooripad [18], and Petrich [20] we introduce several useful partial orders on S.

Definition 4.1.

- $a <^{-} b$ if and only if $a^{-}a = a^{-}b$ and $aa^{-} = ba^{-}$ for some $a^{-} \in a\{1\}$ (*minus order*);
- $a\mathcal{N}b$ if and only if $a = axa = axb = bxa$ for some $x \in S$ (*Nambooripad order*);
- $a\mathcal{M}b$ if and only if $a = xb = by$ and $xa = a$ for some $x, y \in S^1$ (*Mitsch order*);

Due to the following result, if the semigroup S is regular then the above partial orders coincide.

Lemma 4.2. *[17, Lemma 1] For a regular semigroup S, the following conditions are equivalent:*

1. $a = eb = bf$ *for some* $e, f \in E(S)$;

2. $a = aa'b = ba''a$ *for some* $a', a'' \in a\{1, 2\}$;

3. $a = aa'b = ba'a$ *for some* $a' \in a\{1, 2\}$;

4. *$a'a = a'b$ and $aa' = ba'$ for some $a' \in a\{1,2\}$, see also [13];*

5. *$a = ab'b = bb'a$, $a = ab'a$ for some $b' \in b\{1,2\}$;*

6. *$a = axb = bxa$, $a = axa$, $b = bxb$ for some $x \in S$;*

7. *$a = eb$ and $aS \subseteq bS$ for some idempotent e such that $aS = eS$, see also [18];*

8. *$a = xb = by$, $xa = a$ for some $x, y \in S$.*

The minus order and the Nambooripad order coincide on any semigroup.

Lemma 4.3. *[12, Lemma 3] The minus order $<^-$ and the Nambooripad order $\mathcal{N}$ are equivalent on any semigroup S.*

The *sharp partial order* on a semigroup is defined in [16] in the following way.

Definition 4.4. Let S be a semigroup, $a, b \in S$, and a be group invertible. We put $a <^{\#} b$ if and only if $aa^{\#} = ba^{\#} = a^{\#}b$.

The following lemma provides some useful properties of the sharp order. We present a proof of these properties for completeness.

Lemma 4.5. *Let $a, b \in S$, where S is an arbitrary semigroup.*
1. *Let $a <^{\#} b$. Then $ab = ba$.*
2. *Suppose that b is group invertible. Then $a <^{\#} b$ if and only if $a^{\#} <^{\#} b^{\#}$.*

Proof. 1. By the definition of the sharp order we have $aa^{\#} = ba^{\#} = a^{\#}b$. Then $aa(aa^{\#}) = aa(a^{\#}b)$, that is, $aa = ab$. Also $(aa^{\#})aa = (ba^{\#})aa$, that is, $aa = ba$.

2. By the definition of the sharp order we have $aa^{\#} = ba^{\#} = a^{\#}b$. By Item 1 we have $ab = ba$.

Let us prove that $a^{\#}a = b^{\#}a$. Indeed, $b^{\#}a = b^{\#}(aa^{\#})a = b^{\#}ba^{\#}a = b^{\#}b(aa^{\#}) = b^{\#}b(ba^{\#}) = (b^{\#}bb)a^{\#} = ba^{\#} = aa^{\#}$, therefore, $b^{\#}a = a^{\#}a$. Dually, $ab^{\#} = aa^{\#}$.

Note that $(x^{\#})^{\#} = x$ for any group invertible $x \in S$. Then $b^{\#}a = ab^{\#} = a^{\#}a$ implies that $a^{\#} <^{\#} b^{\#}$.

By the above, $a^{\#} <^{\#} b^{\#}$ implies $(a^{\#})^{\#} <^{\#} (b^{\#})^{\#}$. That is, $a <^{\#} b$. $\square$

The following example shows that $a <^{\#} b$ does not imply $b^{\#} <^{\#} a^{\#}$.

Example 4.6. Let $S = T_4$ be the semigroup which consists of all maps from the set $\{1, 2, 3, 4\}$ to itself with the composition operation. As usual, we represent such a map by two rows, where the first row contains the elements and the second row contains their images.

Let $a = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{smallmatrix}\right)$, $b = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{smallmatrix}\right)$. Then $a^{\#} = a$ and $b^{\#} = b^{-1} = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{smallmatrix}\right)$. Also $a = a^{\#}a = a^{\#}b = ba^{\#}$. Then $a <^{\#} b$. But $b^{\#}(b^{\#})^{\#} = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{smallmatrix}\right) \neq a^{\#}(b^{\#})^{\#} = a$, that is, $b^{\#} \not<^{\#} a^{\#}$.

If in addition we assume that our semigroup is commutative, then the generalized inverses and the orders under consideration have several additional properties.

Let us prove them here for completeness.

Lemma 4.7. *Let S be a commutative semigroup.*

1. Let $a \in S$ be regular. Then a is group invertible and $a^{\#} = a^{-}aa^{-}$ for any $a^{-} \in a\{1\}$.

2. For $a, b \in S$ it holds that $a <^{-} b$ if and only if $a <^{\#} b$.

3. For $a, b \in S$ it holds that $a\mathcal{M}b$ if and only if $a = xa = xb$ for some $x \in S$.

Proof. 1. Let $a^{-} \in a\{1\}$. We prove that $a^{-}aa^{-} \in a\{1,2\}$. Indeed, we have $(a^{-}aa^{-})a(a^{-}aa^{-}) = a^{-}(aa^{-}a)a^{-}aa^{-} = a^{-}(aa^{-}a)a^{-} = a^{-}aa^{-}$. Also $(aa^{-}a)a^{-}a = aa^{-}a = a$. Since S is commutative, we obtain, by the definition of the group inverse, that a is group invertible and $a^{\#} = a^{-}aa^{-}$.

2. Since $a^{\#}$ is an inner inverse of a, we obtain that $a <^{\#} b$ implies $a <^{-} b$ in general. Suppose that $a <^{-} b$, that is, $aa^{-} = ba^{-}$ for some $a^{-} \in \{1\}$. Then by multiplying by aa^{-} on the right we obtain $(aa^{-})(aa^{-}) = a(a^{-}aa^{-}) = b(a^{-}aa^{-})$. By Item 1 a is group invertible and $a^{\#} = a^{-}aa^{-}$. Finally, $aa^{\#} = ba^{\#} = a^{\#}b$.

3. By the definition of the Mitsch order $a\mathcal{M}b$ means that $a = xa = xb = by$ for some $x, y \in S^1$. In particular, $a = xa = xb$.

Now suppose that $a = xa = xb$, therefore $bx = xb$ since S is commutative. Then $a\mathcal{M}b$ by the definition. $\square$

The following example shows that there are different Mitsch-compatible elements even in commutative semigroups.

Example 4.8. Let $S = \mathbb{Z}_+$ with multiplication. Then $0 = 0n = n0 = 00$ for any $n \in S$. That is, $0\mathcal{M}n$.

5 Quotient rings along functions

Let R be an associative ring and $f : R \to R$ be a given function.

Definition 5.1. $a \in R$ is *left f-cancellable* if $\forall x, y \in R \cup \{1\}$, $af(a)x = af(a)y$ implies $f(a)x = f(a)y$.

Right f-cancellable elements are defined dually. *f-cancellable* means both left and right f-cancellable.

The set of all f-cancellable elements of R is denoted by $\mathcal{C}_f(R)$.

For special types of the function f and special classes of rings, f-cancellable elements and corresponding quotient rings (see Definition 5.4) are widely investigated, see for instance [1, 2, 3, 4, 10, 21]. In the following example we collect some of the most useful such functions.

Example 5.2.

- If $f(a) = a^*$, then f-cancellable means $*$-*cancellable.*

- If $f(a) = a$, then f-cancellable means *square-cancellable* and we use the notation $\mathfrak{C}(R)$ for $\mathcal{C}_f(R)$ in this case.
- If $f(a) = d$ for all $a \in R$ and some $d \in R$, then we call f-cancellable elements *d-cancellable*, and use the notation $\mathcal{C}_d(R)$ in this case.

Lemma 5.3. *Let R be an arbitrary ring and let $a, d \in R$. If a^{-d} exists, then $a \in \mathcal{C}_d(R)$.*

Proof. Let us consider $x, y \in R \cup \{1\}$ satisfying $adx = ady$. Then multiplying by a^{-d} on the left we have $a^{-d}adx = a^{-d}ady$, which implies $dx = dy$ since $a^{-d}ad = d$ by Item 1. of Lemma 3.4. Dually, $xda = yda$ implies $xd = dy$, and then $a \in \mathcal{C}_d(R)$. $\square$

Definition 5.4. Let Q be a ring, R be a subring of Q. Let $f : R \to R$ be a function. We say that R is a left *order along f* in Q and Q is a left *quotient ring of R along f* with respect to a subset $\mathcal{C} \subseteq \mathcal{C}_f(R)$ if:

1. every $u \in \mathcal{C}$ is invertible along $f(u)$ in Q,
2. every $q \in Q$ can be written as $q = u^{-f(u)}b$ for some $u \in \mathcal{C}$ and $b \in R$.

Right orders can be defined dually. In the subsequent discussion we shall omit the word "left".

Some well-known quotient rings are examples of the introduced general construction, see [2, 4, 10].

Example 5.5.

1. Let $f(a) = a$. Then a *quotient ring along f* is a *Fountain-Gould left quotient ring.*
2. Let $f(a) = a^*$. Then a *quotient ring along f* is a *Moore-Penrose quotient ring.*

Remark 5.6. Note that in [4] and [21] a slightly different definition of the Moore-Penrose inverse was used. Namely, there b is called a Moore-Penrose inverse of a if $ab^*a = a$, $ba^*b = b$, $a^*b = b^*a$ and $ba^* = ab^*$. Then $a^\dagger = b^*$, where $(a^\dagger)$ is a Moore-Penrose of a in the sense of Definition 3.1 (see [5], [19]). This is more convenient for the considerations below, since $a^\dagger$ is an outer inverse, and as a consequence $a^\dagger$ is a particular case of an inverse of a along an element, and b is not. Let us also mention that both definitions are common and clearly connected. Also in [4] it is assumed that $\mathcal{C} = \mathcal{C}^*$ in the definition of the Moore-Penrose quotient ring with respect to $\mathcal{C}$. In this case $\{x^\dagger \mid x \in C\} = \{(x^\dagger)^* \mid x \in C\}$.

We provide here the definition of a Fountain-Gould quotient ring to be investigated below.

Definition 5.7. Let R be a subring of a ring Q. We say that R is a *Fountain-Gould left order in* Q and Q *is a Fountain-Gould left quotient ring of* R *with respect to a subset* $\mathcal{C}$ *of* $\mathfrak{C}(R)$ if

1. every $u \in \mathcal{C}$ is group invertible in Q,
2. every $q \in Q$ can be written as $q = u^{\#}b$ for some $u \in \mathcal{C}$ and $b \in R$.

The following three statements allow us to construct the same quotient ring using different functions.

Lemma 5.8. *Let $d_1, d_2 \in R$ be such that $d_1 \mathcal{H} d_2$. Then $\mathcal{C}_{d_1}(R) = \mathcal{C}_{d_2}(R)$.*

Proof. Since $d_1 \mathcal{H} d_2$, by Lemma 2.2 we have $d_2 = d_1 u$ for some $u \in R \cup \{1\}$. Consider $a \in \mathcal{C}_{d_1}(R)$ and suppose that, for some $x, y \in S^1$, $ad_2x = ad_2y$. Then $ad_1(ux) = ad_1(uy)$ and as a consequence $d_1ux = d_1uy$, that is, $d_2x = d_2y$.

By dual arguments $xd_2a = yd_2a$ implies $xd_2 = yd_2$. Thus $a \in \mathcal{C}_{d_2}(R)$ and $\mathcal{C}_{d_1}(R) \subseteq \mathcal{C}_{d_2}(R)$. Dually, $\mathcal{C}_{d_2}(R) \subseteq \mathcal{C}_{d_1}(R)$. □

Corollary 5.9. *Let $f_1, f_2 : R \to R$ be such that $f_1(x) \mathcal{H} f_2(x)$ in R for every $x \in R$. Let $\mathcal{C} \subseteq \mathcal{C}_{f_1}(R) \cup \mathcal{C}_{f_2}(R)$. Then R is an order along f_1 in Q with respect to $\mathcal{C}$ if and only if R is an order along f_2 in Q with respect to $\mathcal{C}$.*

Proof. By the same arguments as in Lemma 5.8, we have $\mathcal{C}_{f_1}(R) = \mathcal{C}_{f_2}(R)$. Then $f_1(u) \mathcal{H} f_2(u)$ implies $u^{-f_1(u)} = u^{-f_2(u)}$ for every $u \in \mathcal{C}$ by Item 2. of Lemma 3.4 and the uniqueness of the inverse along an element (Theorem 3.6). □

Corollary 5.10. *Let R be a subring of a ring Q, $f_1, f_2 : R \to R$ be constant functions: $f_1(x) = d_1, f_2(x) = d_2$ for every $x \in R$ such that $d_1 \mathcal{H} d_2$ in R. Let $\mathcal{C}$ be a subset of $\mathcal{C}_{d_1}(R) \cup \mathcal{C}_{d_2}(R)$. Then R is an order along f_1 in Q with respect to $\mathcal{C}$ if and only if R is an order along f_2 in Q with respect to $\mathcal{C}$.*

If f is a constant function, then the quotient ring along f has the following additional properties:

Lemma 5.11. *Let f be a constant function: $f(x) = d$ for every $x \in R$ for some $d \in R$ and let Q be a quotient ring along f of R with respect to a subset $\mathcal{C}$ of $\mathcal{C}_d(R)$. Then*

1. $\{\mathcal{C}\}^{-d} \subseteq \mathcal{H}_d \subseteq Q$.
2. *If $q \in Q$ and $q = a^{-d}x$ for some $a \in \mathcal{C}$ and $x \in R$, then $daq = dx$.*
3. $Q = \mathcal{H}_d R$.

Proof. 1. Follows from Lemma 3.4.

2. It holds that $d = daa^{-d} = a^{-d}ad$. Thus, if $q = a^{-d}x$, then $daq = daa^{-d}x = dx$.

3. Follows from Definition 5.4 and Item 1. □

A quotient ring along a function is a generalization of the notion of a classical quotient ring.

Definition 5.12. Recall that a ring Q is a *(classical) ring of left quotients of its subring* R, or that R is a *(classical) left order in* Q, if the following three conditions are satisfied:

1. Q has an identity element.

2. Every element of R which is not a zero divisor is invertible in Q.

3. Every $q \in Q$ can be written as $q = a^{-1}b$ where $a, b \in R$ and a^{-1} is the inverse of a in Q.

Lemma 5.13. *Let S be an arbitrary semigroup with identity 1, and suppose that $a \in S$ is invertible along 1. Then the inverse of a along 1 is the classical inverse of a.*

Proof. By Lemma 3.4 (Item 1) a is indeed invertible, since $d = 1$ implies $1 = a^{-1}a1 = 1aa^{-1}$. □

Lemma 5.14. *Let $f(x) = 1$ for all $x \in R$ and let Q be a quotient ring along 1 of R with respect to $\mathcal{C} = \mathcal{C}_1(R)$. Then Q is the classical ring of left quotients of R.*

Proof. Since $f : R \to R$, we automatically assume that $1 \in R$. By Lemma 5.13, Definition 5.4 (Item 2) implies Definition 5.12 (Item 3).

By Definition 5.4 (Item 1) every 1-cancellable element is invertible in Q. Thus we only need to show that every element of R which is not a zero divisor is 1-cancellable.

Indeed, let $a \in R$, where a is not a zero divisor. Suppose that $a1x = a1y$ for some $x, y \in R$. Then $ax - ay = a(x - y) = 0$, that is $x = y$. In other words, a is left 1-cancellable. By dual arguments, a is also right 1-cancellable. □

Corollary 5.15. *Let $f(x)\mathcal{H}1$ in R for all $x \in R$ and let Q be a quotient ring along f of R with respect to $\mathcal{C} = \mathcal{C}_f(R)$. Then Q is the classical ring of left quotients of R.*

Proof. This is a direct corollary of Lemma 5.14 and Corollary 5.9. □

The next lemma deals with pairs of elements of a quotient ring along a constant function.

Lemma 5.16. *Let $d \in R$ be fixed and $f(x) = d$ for all $x \in R$. Let Q be a quotient ring of R along f with respect to $\mathcal{C} = \mathcal{C}_f(R)$.*

Then for any $p, q \in Q$ there exist $u, w \in \mathcal{C}$, $a, b, c \in R$ and $x, y \in Q$ such that $p = u^{-d}a$ and $q = u^{-d}xb = yu^{-d}b = u^{-d}w^{-d}c$.

Proof. By Definition 5.4 we have $p = u^{-d}a$ and $q = v^{-d}b$ for some $u, v \in \mathcal{C}$, $a, b \in R$. By Lemma 3.4(Item 2) $v^{-d}\mathcal{H}d$ and $d\mathcal{H}u^{-d}$ in Q. Then by Lemma 2.2 there exist $x, y \in Q$ such that $v^{-d} = u^{-d}x = yu^{-d}$. Then $q = u^{-d}xb = yu^{-d}b$. Since $x \in Q$, there exist $w \in \mathcal{C}$ and $c' \in R$ such that $x = w^{-d}c'$. If $c = c'b$, then $q = u^{-d}w^{-d}c$. $\square$

The next theorem shows that every quotient ring in the sense of Definition 5.4 can be viewed as a Fountain-Gould left quotient ring with respect to some set.

For $f : R \to R$ and $\mathcal{C} \subseteq \mathcal{C}_f(R)$, we introduce the following notation:

- $\mathcal{D}^l_{\mathcal{C}} = \{f(u)u \mid u \in \mathcal{C}\}$
- $\mathcal{D}^{l\infty}_{\mathcal{C}} = \{f(u)u, (f(u)u)^2, \dots \mid u \in \mathcal{C}\}$
- $\mathcal{D}^r_{\mathcal{C}} = \{uf(u) \mid u \in \mathcal{C}\}$
- $\mathcal{D}^{r\infty}_{\mathcal{C}} = \{uf(u), (uf(u))^2, \dots \mid u \in \mathcal{C}\}$
- $\mathcal{D}_{\mathcal{C}} = \mathcal{D}^l_{\mathcal{C}} \cup \mathcal{D}^r_{\mathcal{C}}$
- $\mathcal{D}^{\infty}_{\mathcal{C}} = \mathcal{D}^{l\infty}_{\mathcal{C}} \cup \mathcal{D}^{r\infty}_{\mathcal{C}}$

Theorem 5.17. *Let $f : R \to R$ and let Q be a quotient ring along f of R with respect to a subset $\mathcal{C}$ of $\mathcal{C}_f(R)$. Then Q is a Fountain-Gould left quotient ring of R with respect to any set $\mathcal{D}$ satisfying $\mathcal{D}^l_{\mathcal{C}} \subseteq \mathcal{D} \subseteq \mathcal{D}^{\infty}_{\mathcal{C}}$. In this case $\mathcal{D} \subseteq \mathfrak{C}(R)$.*

Proof. 1. By Definition 5.4 every $u \in \mathcal{C}$ is invertible along $f(u)$ in Q. Then, by Theorem 3.7 $f(u)u$ and $uf(u)$ are group invertible in Q. As a consequence, $(f(u)u)^n$ and $(uf(u))^n$ are also group invertible for any $n \geq 1$. Finally, every $v \in \mathcal{D} \subseteq \mathcal{D}^{\infty}_{\mathcal{C}}$ is group invertible in Q. In addition, $\mathcal{D} \in \mathfrak{C}(R)$ by Lemma 5.3.

2. By Definition 5.4 every $q \in Q$ can be written as $q = u^{-f(u)}b$ for some $u \in \mathcal{C}$ and $b \in R$. Again, by Theorem 3.7, $f(u)u$ is group invertible and $u^{-f(u)} = (f(u)u)^{\#}f(u)$. It follows that $q = u^{-f(u)}b = (f(u)u)^{\#}f(u)b$. Note that $u, f(u), b \in R$ and $f(u)u \in \mathcal{D}^l_{\mathcal{C}} \subseteq \mathcal{D}$. Finally, every $q \in Q$ can be written as $v^{\#}c$ for some $v \in \mathcal{D}$ and $c \in R$.

Thus Q is a Fountain-Gould quotient ring of R with respect to $\mathcal{D}$. $\square$

For right quotient rings along f the following result can be proved dually.

Lemma 5.18. *Let $f : R \to R$ and let Q be a right quotient ring of R along f with respect to a subset $\mathcal{C}$ of $\mathcal{C}_f(R)$. Then Q is a Fountain-Gould right quotient ring of R with respect to any set $\mathcal{D}$ satisfying $\mathcal{D}^r_{\mathcal{C}} \subseteq \mathcal{D} \subseteq \mathcal{D}^{\infty}_{\mathcal{C}}$. In this case $\mathcal{D} \subseteq \mathfrak{C}(R)$.*

5.1 Properties of Fountain-Gould quotient rings

Lemma 5.19. *Let Q be a Fountain-Gould left quotient ring of R with respect to $\mathcal{C} \subseteq \mathfrak{C}(R)$.*

Then Q is a Fountain-Gould left quotient ring of R with respect to some $\mathcal{C}' \subseteq \mathfrak{C}(R)$, $\mathcal{C} \subseteq \mathcal{C}'$, where $\mathcal{C}'$ is such that $u^2 \in \mathcal{C}'$ for any $u \in \mathcal{C}'$.

Proof. Let $C' = \bigcup_{n=1}^{\infty} \{u^n | u \in \mathcal{C}\}$. It is easy to verify that $u^2 \in \mathcal{C}'$ for any $u \in \mathcal{C}'$. Also $\mathcal{C} \subseteq \mathcal{C}'$ and as a consequence every $q \in Q$ can be written as $q = u^{\#}a$ for some $a \in R$ and $u \in \mathcal{C}' \supseteq \mathcal{C}$.

Finally, every element of $\mathcal{C}'$ is group invertible since group invertibility of $u \in \mathcal{C}$ implies group invertibility of u^n for any $n \geq 1$. Thus Q is a Fountain-Gould left quotient ring of R with respect to $\mathcal{C}'$. $\square$

Theorem 5.17 allows us to use some known properties of Fountain-Gould quotient rings (with respect to a subset). In this section, we provide several such properties. They were originally proved in [2] for the variant of the Fountain-Gould quotient rings which does not involve subsets (or with respect to $\mathcal{C} = \mathcal{C}_f(R)$ in the sense of Definition 5.4). They were reformulated in [4] in a form which is more close to Definition 5.4.

In this section when we choose a subset $\mathcal{C}$ of $\mathfrak{C}(R)$, we additionally assume that $u^2 \in \mathcal{C}$ for any $u \in \mathcal{C}$. If this property holds, we write $\mathcal{C} = \mathcal{C}^{(2)}$. This property is important in some of the following statements. Also it is natural for Fountain-Gould quotient rings (see Lemma 5.19) and can be rather useful, as the next lemma illustrates, see also [10, Lemma 2.1].

Lemma 5.20. *Let Q be a Fountain-Gould left quotient ring of R with respect to $\mathcal{C} = \mathcal{C}^{(2)} \subseteq \mathfrak{C}(R)$. Then every $q \in Q$ can be written as $q = u^{\#}a$, where $uu^{\#}a = a$, $a \in R$ and $u \in \mathcal{C}$.*

Let $q \in Q$. By Definition 5.7 we have $q = v^{\#}b$ for some $b \in R$ and $v \in \mathcal{C}$. Then $v^2 \in \mathcal{C}$ as $\mathcal{C} = \mathcal{C}^{(2)}$. Finally, $(v^2)^{\#}vb = v^{\#}b = q$ and $v^2(v^2)^{\#}vb = vb$. Thus putting $u = v^2$ and $a = vb$ gives the result.

Definition 5.21. [22, p. 1] Let R be a subring of a ring Q. We say that Q is an *Utumi left quotient ring of R* if for all $p, q \in Q$ with $p \neq 0$ there is an $x \in R$ such that $xp \neq 0$ and $xq \in R$.

Proposition 5.22. *[4, Proposition 2.1]. Let Q be a Fountain-Gould left quotient ring of R with respect to $\mathcal{C} = \mathcal{C}^{(2)} \subseteq \mathfrak{C}(R)$. Then Q is an Utumi left quotient ring of R.*

Proposition 5.23. *[4, Proposition 2.2 (Common Denominator Theorem)]. Let Q be a Fountain-Gould left quotient ring of R with respect to $\mathcal{C} = \mathcal{C}^{(2)} \subseteq \mathfrak{C}(R)$. Then for any $p, q \in Q$ there exist $u \in \mathcal{C}$ and $a, b \in R$ such that $p = u^{\#}a$ and $q = u^{\#}b$.*

Denote by $l_R(a)$ and $r_R(a)$ the sets $\{x \in R \mid xa = 0\}$ and $\{x \in R \mid ax = 0\}$ of left and right annihilators of $a \in R$, respectively. If there is no danger of confusion, we shall write simply $l(a)$ and $r(a)$.

Proposition 5.24. *[4, Proposition 2.3]. A ring R has a Fountain-Gould left quotient ring with respect to a subset $\mathcal{C} = \mathcal{C}^{(2)}$ of $\mathfrak{C}(R)$ if and only if it satisfies the following conditions:*

1. *For every $a \in R$, there exists $c \in \mathcal{C}$ such that $l(c) \subseteq l(a)$.*
2. *For every $a \in \mathcal{C}$ and $r \in R$, $(l(a) + Ra)r = 0$ implies $r = 0$.*
3. *For every $a, b \in \mathcal{C}$, there exist $c \in \mathcal{C}$ and $x, y \in R$ such that*
$$l(c) \subseteq l(a) \cap l(b),\ ca = xa^2,\ cb = yb^2.$$
4. *For every $a, c \in \mathcal{C}$ and $b \in R$, there exist $u \in \mathcal{C}$ and $v, x \in R$ such that*
$$l(u) \subseteq l(a),\ ua = xa^2,\ xbc = vc^2.$$

Proposition 5.25. *[4, Proposition 2.4] If R is a Fountain-Gould left order in Q and a Fountain-Gould right order in P with respect to the same set $\mathcal{C} = \mathcal{C}^{(2)} \subseteq \mathfrak{C}(R)$, then there is an isomorphism between Q and P which is the identity on R.*

Remark 5.26. Lemma 5.19 allows us to omit the condition $\mathcal{C} = \mathcal{C}^{(2)}$ in Propositions 5.22 and 5.25. In Proposition 5.23 this condition also becomes unnecessary, if we claim that $u \in \mathfrak{C}(R)$ instead of $u \in \mathcal{C}$.

Corollary 5.27. *Let $f : R \to R$ and let Q be a quotient ring of R along f with respect to a subset $\mathcal{C}$ of $\mathcal{C}_f(R)$. Let $\mathcal{D}$ be any set satisfying $\mathcal{D}^{l\infty}_{\mathcal{C}} \subseteq \mathcal{D} \subseteq \mathcal{D}^{\infty}_{\mathcal{C}}$. Then the following statements hold:*

1. *Q is an Utumi left quotient ring of R.*
2. *For any $p, q \in Q$ there exist $u \in \mathcal{D} \subseteq \mathfrak{C}(R)$ and $a, b \in R$ such that $p = u^{\#}a$ and $q = u^{\#}b$.*
3. *R and $\mathcal{D} \subseteq \mathfrak{C}(R))$ satisfy conditions 1.–4. of Proposition 5.24.*
4. *If P is a right quotient ring of R along f with respect to the same set $\mathcal{C} \subseteq \mathcal{C}_f(R)$, then there is an isomorphism between Q and P which is the identity on R.*

Proof. Let $f : R \to R$ and let Q be a quotient ring of R along f with respect to a subset $\mathcal{C}$ of $\mathcal{C}_f(R)$. By Theorem 5.17, Q is a Fountain-Gould quotient ring of R with respect to $\mathcal{D} \subseteq \mathfrak{C}(R)$. Note that $u^2 \in \mathcal{D}$ for any $u \in \mathcal{D}$ by the construction of $\mathcal{D}^{l\infty}_{\mathcal{C}}$ and $\mathcal{D}^{\infty}_{\mathcal{C}}$. Then

1. By Proposition 5.22, Q is an Utumi left quotient ring of R.

2. By Proposition 5.23, for any $p, q \in Q$ there exist $u \in \mathcal{D} \subseteq \mathfrak{C}(R)$ and $a, b \in R$ such that $p = u^{\#}a$ and $q = u^{\#}b$.

3. R indeed has a Fountain-Gould quotient ring and thus $R, \mathcal{D}$ satisfy conditions 1.–4. of Proposition 5.24.

4. We may choose $\mathcal{D} = \mathcal{D}_{\mathcal{C}}^{\infty}$. Then Q is a quotient ring of R with respect to $\mathcal{D}_{\mathcal{C}}^{\infty}$ and, dually, P is a right quotient ring of P with respect to $\mathcal{D}_{\mathcal{C}}^{\infty}$ by Lemma 5.18. Then, by Proposition 5.25, there is an isomorphism between Q and P which is the identity on R.

□

Theorem 5.28. *Let R be a commutative ring, $f : R \to R$, and let Q be a quotient ring of R along f with respect to a subset $\mathcal{C}$ of $\mathcal{C}_f(R)$. Then Q is also commutative.*

Proof. By Corollary 5.27 (Item 2) we obtain that any $p, q \in Q$ can be written as $p = u^{\#}a$ and $q = u^{\#}b$ for some $a, b, u \in R$. Then $u^{\#}$ commutes with every element of R by Lemma 3.10, and thus $pq = qp$. □

6 Order relations on a quotient ring

In this part, we show that certain partial orders on R are connected to the corresponding partial orders on a quotient ring Q of R.

Let $<$ denote any of the partial orders introduced in Definition 4.1 or Definition 4.4.

Lemma 6.1. *Let $f : R \to R$ and let Q be a quotient ring of R along f with respect to a set $\mathcal{C} \subseteq \mathcal{C}_f(R)$. Suppose that $a, b \in R$ and $a < b$ in R. Then $a < b$ in Q.*

Proof. Follows from the fact that R is a subring of Q and from the definitions of the order relations. □

In the statements below we provide assumptions in the form "x or y". When we write this, we mean that x implies y and we may assume only y. But condition x may be more convenient, for example, easier to verify or concerns only the subring R.

6.1 Mitsch order relation

Lemma 6.2. *Let $f : R \to R$ and let Q be a quotient ring along f of R with respect to $\mathcal{C} \subseteq \mathcal{C}_f(R)$. Suppose that $p, q \in Q$ can be written as $p = u^{-f(u)}a$ and $q = v^{-f(v)}b$ for some $u, v \in \mathcal{C}$ and $a, b \in R$ satisfying the following conditions:*

1. *$a\mathcal{M}b$ in R or $a\mathcal{M}b$ in Q.*
2. *$u^{-f(u)}\mathcal{M}v^{-f(v)}$ in Q.*

3. $a \in C(\{u, f(u), v, f(v)\})$ or $a \in C(\{u^{-f(u)}, v^{-f(v)}\})$.

4. $b \in C(\{v, f(v)\})$ or $b \in C(\{v^{-f(v)}\})$.

Then $p\mathcal{M}q$.

Proof. By definition we have to find $z, t \in Q$ such that $p = zq = qt$ and $zp = p$. We denote $d_u = f(u)$ and $d_v = f(v)$. By the definition of $a\mathcal{M}b$ and $u^{-f(u)}\mathcal{M}v^{-f(v)}$ we have $a = xb = xa = by$ and $u^{-d_u} = lu^{-d_u} = lv^{-d_v} = v^{-d_v}m$ for some $x, y, l, m \in Q$.

By Theorem 3.8, $x \in C(\{u, d_u\})$ implies that $u^{-d_u} \in C(\{x\})$. Thus conditions 3. and 4. imply $au^{-d_u} = u^{-d_u}a$, $av^{-d_v} = v^{-d_v}a$ and $bv^{-d_v} = v^{-d_v}b$.

Then $lxp = lxu^{-d_u}a = l(xa)u^{-d_u} = lau^{-d_u} = (lu^{-d_u})a = u^{-d_u}a = p$.

Also $lxq = lxv^{-d_v}b = l(xb)v^{-d_v} = lav^{-d_v} = (lv^{-d_v})a = u^{-d_u}a = p$.

Finally, $qym = v^{-d_v}(by)m = v^{-d_v}am = a(v^{-d_v}m) = au^{-d_u} = u^{-d_u}a = p$.

Thus $p = zq = qt = zp$, where $z = lx$ and $t = ym$. That is, $p\mathcal{M}q$. □

6.2 Nambooripad order relation

Recall that the Nambooripad order and the minus order coincide on any semigroup, see Lemma 4.3.

Lemma 6.3. *Let $f : R \to R$ and let Q be a quotient ring of R along f with respect to $\mathcal{C} \subseteq \mathcal{C}_f(R)$. Suppose that $p, q \in Q$ can be written as $p = u^{-f(u)}a$ and $q = u^{-f(u)}b$ for some $u \in \mathcal{C}$ and $a, b \in R$ satisfying the following conditions:*

1. $a, b \in C(\{u, f(u)\})$ or $a, b \in C(\{u, u^{-f(u)}\})$ or $a, b \in C(\{uu^{-f(u)}\})$.

2. $a <^- b$ in R or $a <^- b$ in Q.

Then $p <^- q$.

Proof. Let $d = f(u)$. By Theorem 3.8 $u^{-d} \in C(C(\{u, d\}))$. Then $x \in C(\{u, d\})$ implies that $x \in C(\{u, u^{-d}\})$. Thus condition 1. states that $a(uu^{-d}) = (uu^{-d})a$ and $b(uu^{-d}) = (uu^{-d})b$.

By the definition of the minus order we have $aa^- = ba^-$ and $a^-a = a^-b$ for some $a^- \in a\{1\} \subseteq Q$.

Let us prove that $a^-u \in p\{1\}$. Indeed, we have $pa^-up = u^{-d}aa^-(uu^{-d})a = u^{-d}(aa^-a)uu^{-d} = u^{-d}a(uu^{-d}) = (u^{-d}uu^{-d})a = u^{-d}a = p$.

Also $pa^-u = u^{-d}(aa^-)u = (u^{-d}b)(a^-u) = qa^-u$ and $a^-up = a^-(uu^{-d})a = (a^-a)uu^{-d} = a^-b(uu^{-d}) = (a^-u)(u^{-d}b) = a^-uq$.

Finally, we obtain that $a^-up = a^-uq$ and $pa^-u = qa^-u$. Thus $p <^- q$.

□

Corollary 6.4. *Let Q be a Fountain-Gould left quotient ring of R with respect to $\mathcal{C} \subseteq \mathfrak{C}(R)$. Suppose that $p, q \in Q$ can be written as $p = u^{\#}a$ and $q = u^{\#}b$ for some $u \in \mathcal{C}$ and $a, b \in R$ satisfying the following conditions:*

1. $au = ua$ and $ub = bu$.

2. $a <^{-} b$ in R or $a <^{-} b$ in Q.

Then $p <^{-} q$.

Proof. By Corollary 3.11, $ua = au$ implies $u^{\#}a = au^{\#}$ and in particular $a(uu^{\#}) = (uu^{\#})a$. Thus $a \in C(\{uu^{-u}\})$. The same holds for b.

Now the corollary follows by Lemma 6.3. $\square$

6.3 Sharp order

Lemma 6.5. *Let $f : R \to R$ and let Q be a quotient ring of R along f with respect to $\mathcal{C} \subseteq \mathcal{C}_f(R)$. Suppose that $p, q \in Q$ can be written as $p = u^{-f(u)}a$ and $q = v^{-f(v)}b$ for some $u, v \in \mathcal{C}$ and $a, b \in R$ satisfying the following conditions:*

1. $a \in C(\{u, f(u)\})$ *or* $a \in C(\{u^{-f(u)}\})$.

2. $a <^{\#} b$ *in* R *or* $a <^{\#} b$ *in* Q.

3. $u^{-f(u)} <^{\#} v^{-f(v)}$ *in* Q.

Then $p <^{\#} q$.

Proof. Let $d_u = f(u)$ and $d_v = f(v)$.

If $a \in C(\{u, d_u\})$, then by Theorem 3.8 $u^{-d_u} \in C(a)$. Thus condition 1. implies that $a \in C(\{u^{-d_u}\})$. Then $a, a^{\#}, u^{-d_u}$ and $(u^{-d_u})^{\#}$ are mutually commutative by Corollary 3.11.

Also, by Lemma 3.12, $p^{\#} = a^{\#}(u^{-d_u})^{\#}$.

By the definition of the sharp order we have $aa^{\#} = ba^{\#} = a^{\#}b$ and

$$u^{-d_u}(u^{-d_u})^{\#} = v^{-d_v}(u^{-d_u})^{\#} = (u^{-d_u})^{\#}v^{-d_v}.$$

To conclude the proof we only need to show that $pp^{\#} = qp^{\#} = p^{\#}q$. To check the first equality we note that $pp^{\#} = u^{-d_u}a(u^{-d_u})^{\#}a^{\#} = u^{-d_u}(u^{-d_u})^{\#}aa^{\#} = v^{-d_v}(u^{-d_u})^{\#}aa^{\#} = v^{-d_v}aa^{\#}(u^{-d_u})^{\#} = v^{-d_v}ba^{\#}(u^{-d_u})^{\#} = qp^{\#}$.

Finally, $pp^{\#} = u^{-d_u}a(u^{-d_u})^{\#}a^{\#} = u^{-d_u}(u^{-d_u})^{\#}a^{\#}a = u^{-d_u}(u^{-d_u})^{\#}a^{\#}b = a^{\#}(u^{-d_u})^{\#}u^{-d_u}b = a^{\#}(u^{-d_u})^{\#}v^{-d_v}b = p^{\#}q$. $\square$

Corollary 6.6. *Let $f : R \to R$ and let Q be a quotient ring of R along f with respect to $\mathcal{C} \subseteq \mathcal{C}_f(R)$. Suppose that $p, q \in Q$ can be written as $p = u^{-f(u)}a$ and $q = u^{-f(u)}b$ for some $u \in \mathcal{C}$ and $a, b \in R$ satisfying the following conditions:*

1. $a \in C(\{u, f(u)\})$ *or* $a \in C(\{u^{-f(u)}\})$.

2. $a <^{\#} b$ *in* R *or* $a <^{\#} b$ *in* Q.

Then $p <^{\#} q$.

Proof. This follows directly from Lemma 6.5. $\square$

Corollary 6.7. *Let Q be a Fountain-Gould left quotient ring of R with respect to $\mathcal{C} \subseteq \mathfrak{C}(R)$. Suppose that $p, q \in Q$ can be written as $p = u^{\#}a$ and $q = v^{\#}b$ for some $u, v \in \mathcal{C}$ and $a, b \in R$ satisfying the following conditions:*

1. $ua = au$.

2. $a <^{\#} b$ in R or $a <^{\#} b$ in Q.

3. $u <^{\#} v$ in R or $u <^{\#} v$ in Q.

Then $p <^{\#} q$.

Proof. By Corollary 3.11, $ua = au$ implies $u^{\#}a = au^{\#}$, that is, $a \in C(\{u^{-u}\})$.

If $u <^{\#} v$ in R then $u <^{\#} v$ in Q and by Lemma 4.5 $u^{\#} <^{\#} v^{\#}$, that is, $u^{-u} <^{\#} v^{-v}$.

The rest follows from Lemma 6.5 □

Recall that any two elements of a Fountain-Gould quotient ring have a common denominator by Proposition 5.23.

Corollary 6.8. *Let Q be a Fountain-Gould left quotient ring of R with respect to $\mathcal{C} \subseteq \mathfrak{C}(R)$. Suppose that $p, q \in Q$ can be written as $p = u^{\#}a$ and $q = u^{\#}b$ for some $u \in \mathcal{C}$ and $a, b \in R$ satisfying the following conditions:*

1. $ua = au$.

2. $a <^{\#} b$ in R or $a <^{\#} b$ in Q.

Then $p <^{\#} q$.

Proof. Follows directly from Corollary 6.7. □

Note that if Q is commutative then Conditions 1. in Lemma 6.5, Corollary 6.6, Corollary 6.7 and Corollary 6.8 are unnecessary. Also in this case the sharp order coincides with the Nambooripad order (minus order) by Lemma 4.7 (Item 2).

We conclude with the following problem.

Problem 6.9. There are many equivalent conditions for the existence of an inverse along an element. What does the existence of such an inverse mean in terms of localizations? Can we use inverses along elements to ensure that a corresponding localization exists?

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