THE HEREDITARY MONOCOREFLECTIVE SUBCATEGORIES OF ABELIAN GROUPS AND R-MODULES

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Abstract. The non-trivial hereditary monocoreflective subcategories of the Abelian groups are the following ones: $\{G \in \mathrm{Ob}\,\mathbf{Ab} \mid G$ is a torsion group, and $\forall g \in G$ the exponent of any prime p in the prime factorization of $o(g)$ is at most $E(p)\}$, where $E(\cdot)$ is an arbitrary function from the prime numbers to $\{0, 1, 2, ..., \infty\}$. ($o(\cdot)$ means the order of an element, and $n \leq \infty$ means $n < \infty$.) This result is dualized to the category of compact Hausdorff Abelian groups (the respective subcategories are $\{G \in \mathrm{Ob}\,\mathbf{CompAb} \mid G$ has a neighbourhood subbase $\{G_\alpha\}$ at 0, consisting of open subgroups, such that G/G_α is cyclic, of order like $o(g)$ above$\}$), and is generalized to categories of unitary R-modules for R an integral domain that is a principal ideal domain. For general rings R with 1, an analogous theorem holds, where the hereditary monocoreflective subcategories of unitary left R-modules are described with the help of filters $\mathcal{L}$ in the lattice of the left ideals of the ring R. These subcategories consist of those left R-modules, for which the annihilators of all elements belong to $\mathcal{L}$. If R is commutative, then this correspondence between these subcategories and these filters $\mathcal{L}$ is bijective.

1. Introduction

Varieties in a given type of universal algebras are characterized by the property of being closed under products, subalgebras, and homomorphic images, by Birkhoff's theorem. Similar theorems hold in other categories as well, that characterize certain subcategories (always supposed to be *non-empty, full and isomorphism closed; we frequently will write a subcategory as its object class*).

Kannan [Ka] seems to have initiated the investigation of simultaneously reflective and coreflective subcategories in certain categories, and proved that in the category of topological spaces, and some related categories there are no such non-trivial

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subcategories. Hušek [Hu73], [Hu76] proved the same statement for the category of uniform spaces (T_0 not included in their definition), and some related categories.

These raise the question for the asymmetric variants of these theorems. Namely, what is the situation in the category of bitopological spaces, i.e., triples $(X, \mathcal{T}, \mathcal{S})$, where $\mathcal{T}$ and $\mathcal{S}$ are topologies on X, with morphisms $f : (X_1, \mathcal{T}_1, \mathcal{S}_1) \to (X_2, \mathcal{T}_2, \mathcal{S}_2)$ characterized by $f^{-1}(\mathcal{T}_2) \subset \mathcal{T}_1$ and $f^{-1}(\mathcal{S}_2) \subset \mathcal{S}_1$. Also one can ask the same question for quasiproximities (X, δ) and quasiuniformities $(X, \mathcal{U})$, that are defined by the same axioms as proximities and uniformities, except the respective symmetry axiom. For all of these we have as a reflective and coreflective subcategory the symmetrical structures (for bitopological spaces $\mathcal{T} = \mathcal{S}$), however, for bitopological spaces, we also have subcategories characterized by $\mathcal{T} \subset \mathcal{S}$, and $\mathcal{T} \supset \mathcal{S}$. Clearly, such additional examples do not exist for quasiproximities and quasiuniformities, since $\mathcal{U} \subset \mathcal{U}^{-1}$ implies $\mathcal{U}^{-1} \subset \mathcal{U}$ etc.

Herrlich [H81] § 3.2 surveyed a large number of closure operations on subcategories of a given category, and the subcategories closed w.r.t. some subsets of these closure operations, for categories occurring in topology. We mention an example: as a close analogue of Birkhoff's theorem, Petz [Pe] characterized in the category of Hausdorff spaces the subcategory of compact Hausdorff spaces as the only non-trivial subcategory that is epireflective — i.e., is closed under products and closed subspaces — and is closed under epi images — i.e., maps with dense image. (*A subcategory* $\mathbf{C}_1$ *of a category* $\mathbf{C}$ *is closed under subobjects, or epi images*, if for a monomorphism, or epimorphism f of $\mathbf{C}$, we have $\operatorname{cod} f \in \operatorname{Ob} \mathbf{C}_1 \Longrightarrow \operatorname{dom} f \in \operatorname{Ob} \mathbf{C}_1$, or $\operatorname{dom} f \in \operatorname{Ob} \mathbf{C}_1 \Longrightarrow \operatorname{cod} f \in \operatorname{Ob} \mathbf{C}_1$, respectively. Analogously we define closedness under surjective images, closed subspaces, etc.) For T_3 spaces even the subcategories closed under products and closed subspaces, and surjective images were characterized, by Kannan-Soundararajan [KS], Theorem, as classes of spaces X, in which for a certain class $\mathbf{U}$ of ultrafilters on some sets, each ultrafilter $\mathcal{U}$ in X, that is an image of some $\mathcal{U}' \in \mathbf{U}$, is convergent.

We still mention Herrlich-Hušek [HH99], dealing with coreflective subcategories in the category of topological groups, and El Bashir-Herrlich-Hušek [EBHH], dealing with simultaneously reflective and coreflective subcategories of the category of Abelian groups, and giving a characterization of these, as well as solving some related problems, and giving some special examples. Herrlich [H84], pp. 239-245, and Herrlich-Lowen [HL] described simultaneously concretely reflective and coreflective subcategories in certain topological categories. For hereditary (i.e., closed under subspaces) monocoreflective subcategories of topological spaces (and its quotient-reflective subcategories) cf., e.g., Činčura [Č01], Sleziak [Sl04] (and Činčura [Č05], Sleziak [Sl08]). We cite a result of [Č01] (pp. 131-134): a hereditary coreflective subcategory of the category of topological spaces either (i): consists of the empty space, or of all discrete spaces, or of all topological sums of indiscrete spaces; or else (ii): it is the coreflective hull of some class of topological spaces, each having exactly one non-isolated point.

Our paper adds to these results some results in algebra.

The subvarieties of a given variety are characterized in [M], Ch. VI, § 14, formula (1), Th. 2, and Corollary 4, in general, cf. also [Sk], Ch. II, § 3, Exercise 7. Concretely, for the category $\mathbf{Ab}$ of Abelian groups, they are characterized as $\{G \in \operatorname{Ob} \mathbf{Ab} \mid \forall g \in G \ \ ng = 0\}$, where $n \in \mathbb{N} = \{0, 1, 2, ...\}$, cf. [F], Ch. 3, § 18, Exercise 7, and [Sk], Ch. II, § 3, Exercise 1. For this last fact cf. also [EBHH], Theorem 3.6.

For algebra, we refer to [R] and [J], for varieties and universal algebra we refer to [Ku] and [Sk], and for category theory, in particular for reflective and coreflective subcategories we refer to [H68], [HS], [AHS] and [HS97]. For Pontrjagin duality, we refer to [Po] and [DPS].

For a *ring* R, *always with* 1 $(\neq 0)$, we denote the *category of left* R*-modules*, always considered as *unitary, by* R-**mod**. We have, that **Ab**, and, more generally, R-**mod**, is balanced, i.e., {monomorphisms} ∩ {epimorphisms } = {isomorphisms}, hence {extremal epimorphisms} = {epimorphisms}, and {extremal monomorphisms} = {monomorphisms}. Also, **Ab**, and, more generally, R-**mod**, is complete, cocomplete, well-powered, and co-well-powered. Hence, monocoreflective, or epireflective subcategories in them are characterized by being closed under coproducts (sums) and epi images defined by extremal epimorphisms, or under products and subobjects defined by extremal monomorphisms, respectively. The first of these statements holds in any variety, observing that the surjective maps are exactly the extremal epimorphisms (and also exactly the regular epimorphisms).

2. Theorems

In what follows, in a variety, a subcategory is called *hereditary*, if it is closed under subalgebras. (It will lead to no misunderstanding that for topological spaces this was meant in another way.)

In the following theorem, $o(g)$ is the *order of an element*, and when writing the prime factorization of $o(g)$, the exponent of a prime p not occurring in the factorization is meant as 0. Of course, when writing that the exponent of a prime in a prime factorization is at most ∞, we mean that it is less than ∞.

Theorem 1. *The hereditary monocoreflective subcategories of* **Ab***, different from* **Ab***, are* $\mathbf{M}(E) := \{G \in \mathrm{Ob}\,\mathbf{Ab} \mid G$ *is a torsion group, and* $\forall g \in G$ *the exponent of any prime* p *in the prime factorization of* $o(g)$ *is at most* $E(p)\}$*, where* $E(\cdot)$ *is an arbitrary function from the prime numbers to* $\{0, 1, 2, ..., \infty\}$*. If we write the hypotheses as being closed under subobjects, epi images, and coproducts, then none of these can be omitted, without invalidating the above conclusion of the theorem. For different* E*'s, the subcategories* $\mathbf{M}(E)$ *are different; more exactly,* $\mathbf{M}(E_1) \subset \mathbf{M}(E_2) \iff E_1 \leq E_2$*.*

Of course, the characterization of epireflective subcategories of **Ab**, closed under epi images, that are just subvarieties, follows from Theorem 1. But we can give a bit stronger statement.

Corollary 2. *The hereditary monocoreflective subcategories of* **Ab***, closed under countably infinite products, and different from* **Ab***, are* $\{G \in \mathrm{Ob}\,\mathbf{Ab} \mid nG = \{0\}\}$*, where* $n \geq 1$ *is an integer. If we write the hypotheses as being closed under subobjects, epi images, coproducts, and countably infinite products, then none of these can be omitted, without invalidating the above conclusion of the corollary. For different* n*'s the associated subcategories are different.*

Now we turn to the opposite category **CompAb** of **Ab**, i.e., to the category of compact Hausdorff topological groups. Observe that the monomorphisms, epimorphisms and products in **CompAb** are given as injective, surjective maps, and the group product with the product topology, but coproducts are given as the Bohr-compactifications (i.e., epireflections to compact Hausdorff groups, cf. [Ke], Ch. 7, Problem T, and [DPS], Exercise 2.10.25) of the coproducts in **Ab**, taken with the inductive (co)limit topology of all finite sub-coproducts (=subproducts).

For $n \geq 1$ an integer we write $\mathbb{Z}(n) := \mathbb{Z}/(n\mathbb{Z})$.

Theorem 3. *The epireflective subcategories of* **CompAb***, closed under epi images, and different from* **CompAb***, are* $\{G \in \mathrm{Ob}\,\mathbf{CompAb} \mid 0$ *has a neighbourhood subbase consisting of open(-and-closed) subgroups* G_α*, such that* $G/G_\alpha \cong \mathbb{Z}(n_\alpha)$ *(where* $n_\alpha \geq 1$ *is an integer), with* n_α*'s having prime factorizations* $\prod p^{e(p)}$*, where* $e(p) \leq E(p)$*, with* E *as in Theorem 1. If we write the hypotheses as being closed under subobjects, epi images, and products, then none of these can be omitted, without invalidating the above conclusion of the theorem. For different* E*'s the associated subcategories are different; more exactly, the subcategory associated to* E_1 *is a subclass of the subcategory associated to* E_2 *if and only if* $E_1 \leq E_2$*.*

Remark 4. The largest of the subcategories in Theorem 3 (i.e., the duals of all torsion commutative groups, or, alternatively, when $E \equiv \infty$) is described alternatively in [Po], §38, Theorem 46, as the class of all totally disconnected commutative compact Hausdorff groups. The equivalence of these two descriptions is given in [Po], §22, Theorems 16, 17, and [DPS], Corollary 3.3.9: a totally disconnected compact (or locally compact) Hausdorff group G has a neighbourhood base of 0 consisting of open(-and-closed) normal subgroups N (or open-and-compact subgroups N, respectively). Then, for G compact Abelian, G/N is finite Abelian, hence is a product of finite cyclic groups. This implies the the subbase property from Theorem 3 (with $E \equiv \infty$).

By the way, it is easy to see directly, that the subcategories in Theorem 3 are closed for epi images. Let G be in such a subcategory, and let H be a closed subgroup of G. Let us consider a set $H + (\cap G_{\alpha_i})$ (the intersection here being a finite one), that is contained in an arbitrary, but prescribed saturated open set containing H, w.r.t. the canonical map $G \to G/H$, for a suitable choice of the α_i's. Then $H + (\cap G_{\alpha_i})$ is a union of some cosets of $\cap G_{\alpha_i}$, hence is open-and-closed. Also, $H + (\cap G_{\alpha_i})$ is a saturated subgroup of G (w.r.t. the canonical map $G \to G/H$), that contains H, hence it corresponds to an open-and-closed subgroup of G/H. Thus G/H is 0-dimensional, and actually belongs to the considered subcategory from Theorem 3.

Corollary 5. *The epireflective subcategories of* **CompAb***, closed under epi images and countably infinite coproducts, and different from* **CompAb***, are* $\{G \in \mathrm{Ob}\,\mathbf{CompAb} \mid nG = \{0\}\}$*, where* $n \geq 1$ *is an integer. If we write the hypotheses as being closed under subobjects, epi images, products, and countably infinite coproducts, then none of these can be omitted, without invalidating the above conclusion of the corollary. For different* n*'s the associated subcategories are different.*

Remark 6. It would be interesting to analyse the subcategories, say, $\mathbf{M}(E)^{\mathrm{op}}$, from Theorem 3, or Corollary 5, respectively. There are two cases, namely, when $\sum_p E(p) = \infty$, or when $\sum_p E(p) < \infty$. The second case is treated in Corollary 5, which is an easy case. For the first case, in the category $\mathbf{M}(E)^{\mathrm{op}}$ standard methods imply, using the respective analogous properties of **CompAb**, that {monomorphisms} = {injections}, and {epimorphisms} = {extremal epimorphisms } = {regular epimorphisms} = {surjections}.

How can one possibly explicitly describe the free objects in $\mathbf{M}(E)^{\mathrm{op}}$ (i.e., the epireflections in $\mathbf{M}(E)^{\mathrm{op}}$ of the free objects considered in **CompAb**)? This is not clear even in the simplest case when $E \equiv \infty$, i.e., for totally disconnected compact Hausdorff groups.

Another question is the following. We know that **CompAb** (as well as **Comp $\mathbf{T_2}$**) is varietal (=monadic), and even the operations, and the identities linking these operations are given in Weaver [W]. (In short: the operations are 1) the group operations, and 2) the limits of ultrafilters on some sets, meant as operations $f_{I,\mathcal{U}} : G^I \to G$, where $I \neq \emptyset$ is a set and $\mathcal{U}$ is an ultrafilter on I. The associated operations for 2) are the limits of ultrafilters on a compact Hausdorff group G that are the images of $\mathcal{U}$ by some map $\mathbf{x} : I \to G$. The identities among these operations $f_{I,\mathcal{U}}$ correspond roughly to: the principal filters $\dot{g}$ have limits g, and one has a restriction axiom, and an iterated limit axiom. Besides these we have the Abelian group axioms, and the continuity of the group operations can be evidently rewritten by the group operations and the above ultrafilter operations.) What are the identities for the same operations for the subcategories $\mathbf{M}(E)^{\text{op}}$? Again, even the case of totally disconnected compact Hausdorff groups is not clear.

We prove Theorem 1 in the following more general form. We let R be an integral domain (commutative ring with 1 and without divisors of 0), which is a principal ideal domain (for integral domains this means that each ideal is principal). Then in R there is a unique prime factorization, for each element different from 0 and not a unit (i.e., having a multiplicative inverse), up to order, and associates (i.e., multiples by units). We will mean primes only up to associates. (As the prime factorization of a unit $u \in R$ we mean the one-element product u.) Thus it makes sense to speak about divisibility in R (we write $r_1 | r_2$ for r_1 divides r_2), and about least common multiples of finitely many elements of R. *When speaking about equality of elements of the multiplicative semigroup of R, we always mean associates.* If R is not a field, then the set of primes of R is not empty.

For each element m of a (left) R-module M we write $\text{ann}\,(m)$ for the annihilator (left) ideal of m. Since this is a principal (left) ideal, it has a generator, which will be denoted by $o(m)$, that is determined up to unit multiples. The same notation will be applied for any ring R with 1 and left R-module M, if each left ideal of R is principal. A *cyclic* left R-module is a one generated by one element.

By a *field* we mean a skew field.

Theorem 7. *Let R be an integral domain, that is a principal ideal domain. The hereditary monocoreflective subcategories of R-**mod**, different from R-**mod**, are $\mathbf{M}(E) := \{M \in \text{Ob}\, R\text{-}\mathbf{mod} \mid \forall m \in M \quad o(m) \neq 0$, and the exponent of any prime p in the prime factorization of $o(m)$ is at most $E(p)\}$, where $E(\cdot)$ is an arbitrary function from a set P of primes of R, containing exactly one prime from each class of associate primes of R, to $\{0, 1, 2, ..., \infty\}$. For any fixed R, if it is not a field, and if we write the hypotheses as being closed under subobjects, epi images, and coproducts, then none of these can be omitted, without invalidating the above conclusion of the theorem. For different E's, the subcategories $\mathbf{M}(E)$ are different; more exactly, $\mathbf{M}(E_1) \subset \mathbf{M}(E_2) \Longleftrightarrow E_1 \leq E_2$.*

Remark 8. For R a field the situation is much simpler. If $\emptyset \neq \mathbf{M} \subset \text{Ob}\, R$-**mod** is closed under either subobjects or epi images, then it is R-**mod**, or $\{M \in \text{Ob}\, R\text{-}\mathbf{mod} \mid \dim M < \alpha\}$, for some cardinal $\alpha > 0$. If $\mathbf{M}$ is closed under coproducts, then it is $\{M \in \text{Ob}\, R\text{-}\mathbf{mod} \mid \dim\ M \in A\}$, where $A \ni 0$ is a class of cardinals, and is one of the following forms: $\{0\}$, or all cardinals which are at least some infinite cardinal or are equal to 0, or it is the union of some additive subsemigroup of $\{0, 1, ...\}$, containing 0 and also some positive integer, and of all infinite cardinals.

Of course, for R like in Theorem 7, the characterization of epireflective subcategories of R-**mod**, closed under epi images, that are just subvarieties, follows from Theorem 7. But we can give a bit stronger statement.

Corollary 9. *Let R be an integral domain, that is a principal ideal domain. The hereditary monocoreflective subcategories of R-**mod**, closed under countably infinite products (or all products, thus being subvarieties), and different from R-**mod**, are $\{M \in \mathrm{Ob}\, R\text{-}\mathbf{mod} \mid rM = \{0\}\}$, where $r \in R \setminus \{0\}$. For any fixed R, if it is not a field, and if we write the hypotheses as being closed under subobjects, epi images, coproducts, and countably infinite products, then none of these can be omitted, without invalidating the above conclusion of the corollary. For $r_1, r_2 \in R \setminus \{0\}$ not associates these subcategories are different; more exactly, $\{M \in \mathrm{Ob}\, R\text{-}\mathbf{mod} \mid r_1 M = \{0\}\} \subset \{M \in \mathrm{Ob}\, R\text{-}\mathbf{mod} \mid r_2 M = \{0\}\} \Longleftrightarrow r_1 | r_2$.*

Remark 10. Again, for R a field the dependence of the hypotheses on each other is simpler. If $\mathbf{M} \subset \mathrm{Ob}\, R$-**mod** is closed under subobjects (equivalently: under epi images), then if it is also closed under coproducts, then it is either $\{M \in \mathrm{Ob}\, R\text{-}\mathbf{mod} \mid \dim M = 0\}$, or R-**mod**, thus it is closed under countably infinite products. Further, $\mathbf{M} = \{M \in \mathrm{Ob}\, R\text{-}\mathbf{mod} \mid \dim M = 0 \text{ or } \dim M \geq \alpha\}$, where $\alpha > 1$ is a cardinal, is closed under coproducts and countably infinite products, but not under subobjects. Also, $\mathbf{M} = \{M \in \mathrm{Ob}\, R\text{-}\mathbf{mod} \mid |M| \leq 2^{|R|+\aleph_0}\}$ $(\ni R)$ is closed under countably infinite products and subobjects (equivalently: and epi images), but not under coproducts.

Now we turn to the case of R-**mod** for any ring R with 1, and even, more generally, to the case of varieties $\mathbf{V}$.

Proposition 11. *Let $\mathbf{V}$ be any variety, and $\emptyset \neq \mathbf{V}_1 \subset \mathbf{V}$ be any subcategory, consisting of algebras in $\mathbf{V}$, generated by one element, or, besides this, let $\mathbf{V}_1$ be also closed under surjective images, respectively. Then $\mathbf{M}(\mathbf{V}_1) := \{M \in \mathrm{Ob}\,\mathbf{V} \mid \forall m \in M$ the subalgebra of M generated by m belongs to $\mathrm{Ob}\,\mathbf{V}_1\}$ is a hereditary subcategory of $\mathbf{V}$, or, besides this, is also closed under surjective images, respectively.*

Here, in the first case, $\mathbf{M}(\mathbf{V}_1)$ may be empty, e.g., for $\mathbf{V} = \{G \in \mathrm{Ob}\,\mathbf{Ab} \mid 2G = \{0\}\}$, and $\mathrm{Ob}\,\mathbf{V}_1 = \{\mathbb{Z}(2)\}$.

It will be always clear from the notations (and context), whether $\mathbf{M}(E)$, or $\mathbf{M}(\mathbf{V}_1)$, or later $\mathbf{M}(\mathcal{L})$ is meant.

Proposition 12. *Let $\mathbf{V}$ be any variety, and $\mathbf{M} \subset \mathbf{V}$ be a hereditary monocoreflective subcategory of $\mathbf{V}$. Then $\mathbf{M}$ is determined by the class of algebras in $\mathbf{V}$, generated by one element, contained in it. More exactly, we have $M \in \mathrm{Ob}\,\mathbf{M}$ if and only if all subalgebras of M, generated by one element, belong to $\mathrm{Ob}\,\mathbf{M}$.*

Remark 13. By Proposition 12, the determination of all hereditary monocoreflective subcategories of $\mathbf{V}$ reduces to a question about algebras generated by one element. Of course, the class of algebras, generated by one element, contained in a hereditary monocoreflective subcategory $\mathbf{M} \subset \mathbf{V}$ has to be also hereditary (restricting ourselves to subalgebras also generated by one element), and closed under surjective images. However, we do not know, how to characterize closedness of $\mathbf{M}$ under coproducts. Even, if we would postulate closedness under coproducts of the subcategory $\mathbf{M}$, given by Proposition 12, we do not know when different classes $\mathbf{V}_1$ of algebras in $\mathbf{V}$, each of these algebras generated by one element, determine the same hereditary monocoreflective subcategory in the way described in Propositions 11 and 12 (i.e., $\mathrm{Ob}\,\mathbf{M} = \mathbf{M}(\mathbf{V}_1)$, cf. Proposition 11).

Observe that Proposition 12 is a certain analogue of the third paragraph following Corollary 3.4, and of Proposition 3.5 in [C01]. However, for R-**mod**, with R commutative, we will be able to characterize exactly the class of left R-modules, generated by one element, occurring in such a characterization, cf. our Theorem 18. An analogous characterization for Theorem 3.8 of [C01] does not seem to be known. A similar statement holds for the Theorem of [KS], for the classes of ultrafilters occurring in that theorem.

Corollary 14. *Let $\mathbf{M}_1, \mathbf{M}_2$ be hereditary monocoreflective subcategories in a variety $\mathbf{V}$. Then we have $\mathbf{M}_1 \subset \mathbf{M}_2$ if and only if the analogous inclusion holds for the classes of subalgebras, generated by one element, of the algebras contained by them.*

The left ideals of a ring R (with 1), partially ordered by inclusion, form a lattice. Thus it makes sense to speak about *filters of left ideals of* R (these always contain R).

Corollary 15. *Let $\mathcal{L}$ be a filter of left ideals of a ring R with 1. Then $\mathbf{M}(\mathcal{L}) := \{M \in \mathrm{Ob}\,R\text{-}\mathbf{mod} \mid \forall m \in M \ \ \mathrm{ann}\,(m) \in \mathcal{L}\}$ is a hereditary monocoreflective subcategory of left R-modules.*

Proposition 16. *Let $\mathbf{M}$ be a hereditary monocoreflective subcategory of R-**mod**, for a ring R with 1. Then there exists a filter $\mathcal{L}$ of left ideals of R, such that $\mathbf{M} = \mathbf{M}(\mathcal{L})$. We may choose $\mathcal{L} = \mathcal{L}(\mathbf{M}) := \{$left ideals L of $R \mid R/L \in \mathrm{Ob}\,\mathbf{M}\}$.*

Proposition 17. *The subcategories $\mathbf{M}(\mathcal{L})$, where $\mathcal{L}$ is a filter of ideals of a commutative ring R with 1, are, for different $\mathcal{L}$'s, different. More exactly, $\mathbf{M}(\mathcal{L}_1) \subset \mathbf{M}(\mathcal{L}_2) \Longleftrightarrow \mathcal{L}_1 \subset \mathcal{L}_2$.*

Theorem 18. *The hereditary monocoreflective subcategories of R-**mod**, for a ring R with 1, are the subcategories $\mathbf{M}(\mathcal{L})$, where $\mathcal{L}$ is a filter in the set of left ideals of R, partially ordered by inclusion. Let R be any fixed ring, that is not a field, and let us write the hypotheses, as being closed under subobjects, epi images, and coproducts. Then we cannot omit closedness under coproducts, and if moreover R is an integral domain, then we cannot omit either closedness under subobjects, or closedness under epi images, without invalidating the above conclusion of the theorem. For R commutative, for different $\mathcal{L}$'s the subcategories $\mathbf{M}(\mathcal{L})$ are different; more exactly, $\mathbf{M}(\mathcal{L}_1) \subset \mathbf{M}(\mathcal{L}_2) \Longleftrightarrow \mathcal{L}_1 \subset \mathcal{L}_2$.*

Corollary 19. *Let R be a commutative ring with 1, and let the intersection of any infinitely many different ideals be the zero ideal. Then the hereditary monocoreflective subcategories of R-**mod**, closed under countably infinite products (or all products, thus being subvarieties), are $\{M \in \mathrm{Ob}\,R\text{-}\mathbf{mod} \mid LM = \{0\}\}$, where $L \subset R$ is an ideal. Let R be any fixed ring, that is not a field, and let us write the hypotheses, as being closed under subobjects, epi images, coproducts, and countably infinite products. Then we cannot omit closedness under coproducts, and if moreover R is an integral domain, then we cannot omit either closedness under subobjects, or closedness under epi images, without invalidating the above conclusion of the corollary. For different L's these subcategories are different; more exactly, $\{M \in \mathrm{Ob}\,R\text{-}\mathbf{mod} \mid L_1 M = \{0\}\} \subset \{M \in \mathrm{Ob}\,R\text{-}\mathbf{mod} \mid L_2 M = \{0\}\} \Longleftrightarrow L_1 \supset L_2$.*

If for a ring R with 1, and $r, r_i \in R$ we have $r = r_1 r_2$, then we say that r_2 is a right divisor of r, and that r is a left multiple of r_2. We say that r is the least common left multiple of $r_1, ..., r_n$, if it is a common left multiple, and it is a right divisor of any other common left multiple of $r_1, ..., r_n$. (Observe that $r = 0$, $r_i = 0$ *are not excluded*.) We mean closedness under right divisors, and least common left multiples of finitely many elements, and the property of containing 1, *up to the following equivalence:* $r_1 \cong r_2 \Longleftrightarrow Rr_1 = Rr_2$.

Corollary 20. *Let each left ideal of a ring R with 1 be a principal left ideal. Then the hereditary monocoreflective subcategories of R-**mod** are $\{M \in \mathrm{Ob}\, R\text{-}\mathbf{mod} \mid \forall m \in M \;\; o(m) \in S\}$, where $S \subset R$ contains 1, and is closed under right divisors, and least common left multiples of finitely many elements (these properties meant up to the equivalence given before this corollary). For R commutative, for different S's these subcategories are different; more exactly, $\{M \in \mathrm{Ob}\, R\text{-}\mathbf{mod} \mid \forall m \in M \;\; o(m) \in S_1\} \subset \{M \in \mathrm{Ob}\, R\text{-}\mathbf{mod} \mid \forall m \in M \;\; o(m) \in S_2\} \Longleftrightarrow S_1 \subset S_2$ ("$o(m) \in S_i$", "different S's", and "$S_1 \subset S_2$" meant up to the equivalence given before this corollary).*

Corollary 21. *Let each ideal of a commutative ring R with 1 be a principal ideal, and let the intersection of any infinitely many different ideals be the zero ideal. Then the hereditary monocoreflective subcategories of R-**mod**, closed under countably infinite products (or all products, thus being subvarieties), are $\{M \in \mathrm{Ob}\, R\text{-}\mathbf{mod} \mid rM = \{0\}\}$, where $r \in R$. For different r's ("different" meant up to the equivalence given before Corollary 20) these subcategories are different; more exactly, $\{M \in \mathrm{Ob}\, R\text{-}\mathbf{mod} \mid r_1 M = \{0\}\} \subset \{M \in \mathrm{Ob}\, R\text{-}\mathbf{mod} \mid r_2 M = \{0\}\} \Longleftrightarrow r_1 | r_2$.*

Remark 22. The referee posed the following question. We begin with the more modest form. Can one explicitly describe the ("lattice" of) subvarieties of R-**mod**, for R a ring with 1, but R being not commutative? A more difficult question would be to ask for the description of the ("lattice" of) hereditary monocoreflective subcategories of R-**mod**, with R as above. Of course, here Proposition 12, Corollary 14, Proposition 16, Theorem 18, and, for a restricted class of rings, Corollary 20 apply. However, as explained in Remark 13, these cannot be considered as solutions of these questions.

By Proposition 12, these "lattices" are small ones, i.e., are sets. They are invariants of the ring R. What are they like? Possibly, how can they be decribed by more usual terms?

3. Proofs

Proof of Proposition 12. We begin with the "only if" part. Let $M \in \mathrm{Ob}\, \mathbf{M}$. Then, by hereditariness, all subalgebras of M, generated by one element, belong to $\mathrm{Ob}\, \mathbf{M}$.

Conversely, for the "if" part, let each subalgebra of M, generated by any element $m \in M$, say, $A(m)$, belong to $\mathrm{Ob}\, \mathbf{M}$. Then their coproduct $\coprod_{m \in M} A(m)$ belongs to $\mathrm{Ob}\, \mathbf{M}$ as well. Mapping $m \in A(m)$ to $m \in M$, we obtain a uniquely determined map $f : \coprod_{m \in M} A(m) \to M$. This f is surjective, hence, since $\coprod_{m \in M} A(m) \in \mathrm{Ob}\, \mathbf{M}$, we have that $M \in \mathrm{Ob}\, \mathbf{M}$. ■

Proof of Corollary 14. Let $\mathbf{M}_1 \subset \mathbf{M}_2$. Then clearly the analogous inclusion holds for the classes of subalgebras, generated by one element, of the algebras contained by them.

If the analogous inclusion holds for the classes of subalgebras, generated by one element, of the algebras contained by them, then by Proposition 12, the inclusion $\mathbf{M}_1 \subset \mathbf{M}_2$ holds as well. ∎

Proof of Corollary 15. Clearly $\{0\} \in \operatorname{Ob} \mathbf{M}(\mathcal{L})$. Thus, by Proposition 11, it remains to show only that $\mathbf{M}(\mathcal{L})$ is closed for coproducts of non-empty families. Let $M_\alpha \in \operatorname{Ob} \mathbf{M}(\mathcal{L})$ for $\alpha \in A$. Let n be a positive integer, and let $m_{\alpha_i} \in M_{\alpha_i}$, where $i = 1, ..., n$. Consider the element $\oplus^n m_{\alpha_i} \in \coprod^n M_{\alpha_i} \subset \coprod_{\alpha \in A} M_\alpha$. Clearly, $\operatorname{ann}(\oplus^n m_{\alpha_i}) = \cap^n \operatorname{ann}(m_{\alpha_i}) \in \mathcal{L}$. ∎

Proof of Proposition 16. We use Proposition 12 and Proposition 11. Let $\mathbf{V}_1$ denote the class of cyclic modules in $\mathbf{M}$; then $\{0\} \in \operatorname{Ob} \mathbf{V}_1 \subset \operatorname{Ob} \mathbf{M}$. In other words, they are the modules R/L (with L a left ideal of R) occurring in $\operatorname{Ob} \mathbf{M}$. Let $\mathcal{L}$ denote the set of left ideals occurring here, i.e.,

$$\mathcal{L} = \mathcal{L}(\mathbf{M}) := \{L \subset R \text{ is a left ideal} \mid R/L \in \operatorname{Ob} \mathbf{M}\};$$

then $R \in \mathcal{L}$.

Then $\mathbf{V}_1$ is closed under surjective images, which is equivalent to saying that $\mathcal{L}$ is upwards closed.

Let $L_1, L_2 \in \mathcal{L}$. Then the R-module $(R/L_1) \coprod (R/L_2)$ is in $\operatorname{Ob} \mathbf{M}$, and — by hereditariness of $\operatorname{Ob} \mathbf{M}$ — its cyclic submodule generated by its element $(1 + L_1) \oplus (1 + L_2)$ also is in $\operatorname{Ob} \mathbf{M}$. However, the annihilator of this generating element is $(\operatorname{ann}\,(1 + L_1)) \cap (\operatorname{ann}\,(1 + L_2)) = L_1 \cap L_2$. Thus $L_1 \cap L_2 \in \mathcal{L}$. Hence, $\mathcal{L}$ is a filter.

Lastly, $\mathbf{M}(\mathcal{L}) = \mathbf{M}(\mathcal{L}(\mathbf{M})) = \{M \in \operatorname{Ob} R\text{-}\mathbf{mod} \mid \forall m \in M \quad \operatorname{ann}(m) \in \mathcal{L}\} = \{M \in \operatorname{Ob} R\text{-}\mathbf{mod} \mid \forall m \in M \quad Rm \cong R/\operatorname{ann}(m) \in \operatorname{Ob} \mathbf{M}\} = \mathbf{M}$, by the definition of $\mathcal{L}$, and with the last equality following from Proposition 12. ∎

Proof of Proposition 17. Let L be a (left) ideal of R, such that $L \in \mathcal{L}$. Then we have for R/L that $\operatorname{ann}(1+L) = L$, and, for each $r \in R$, that $\operatorname{ann}(r+L) \supset \operatorname{ann}(1+L) = L$, since $r_1 \in L$ implies $r_1(r + L) = r_1 r + L = L$ by commutativity. Hence, for each $r + L \in R/L$, we have $\operatorname{ann}(r + L) \in \mathcal{L}$. Therefore, we have $R/L \in \operatorname{Ob} \mathbf{M}(\mathcal{L})$.

Conversely, let L be a (left) ideal of R, such that $R/L \in \operatorname{Ob} \mathbf{M}(\mathcal{L})$. Then for each element $r + L \in R/L$ we have $\operatorname{ann}(r + L) \in \mathcal{L}$, hence, in particular, $L = \operatorname{ann}(1 + L) \in \mathcal{L}$.

Therefore, $L \in \mathcal{L}$ holds if and only if $R/L \in \operatorname{Ob} \mathbf{M}(\mathcal{L})$. (We can write this in the form $\mathcal{L}(\mathbf{M}(\mathcal{L})) = \{L \subset R \text{ is an ideal} \mid R/L \in \operatorname{Ob} \mathbf{M}(\mathcal{L})\} = \{L \subset R \text{ is an ideal} \mid L \in \mathcal{L}\} = \mathcal{L}$.) This shows that $\mathcal{L}$ and $\mathbf{M}(\mathcal{L})$ uniquely determine each other. Therefore, the subcategories $\mathbf{M}(\mathcal{L})$, for different $\mathcal{L}$'s, are different. Thus we have proved the first statement of the proposition.

For the second statement of the proposition we only need to show that $\mathbf{M}(\mathcal{L}_1) \subset \mathbf{M}(\mathcal{L}_2)$ implies $\mathcal{L}_1 \subset \mathcal{L}_2$. In fact, supposing $\mathbf{M}(\mathcal{L}_1) \subset \mathbf{M}(\mathcal{L}_2)$, we have for a cyclic module R/L, that $R/L \in \operatorname{Ob} \mathbf{M}(\mathcal{L}_1)$ implies $R/L \in \operatorname{Ob} \mathbf{M}(\mathcal{L}_2)$, i.e., $L \in \mathcal{L}_1$ implies $L \in \mathcal{L}_2$. ∎

Proof of Theorem 18. Except for the stated independence of the hypotheses (second statement of the theorem), this follows from Corollary 15, Proposition 16, and Proposition 17.

For the independence of the hypotheses we have to give three examples. A subcategory closed under subobjects and epi images, but not under coproducts, is $\{M \in \operatorname{Ob} R\text{-}\mathbf{mod} \mid |M| \leq |R|\}$ $(\ni R)$. Now let R be an integral domain. Observe that since R is not a field, therefore it has an ideal I and a quotient field Q, such that $\{0\} \subsetneqq I \subsetneqq R \subsetneqq Q$. A subcategory closed under subobjects and coproducts, but not under epi images, is $\{M \in \mathrm{Ob}\, R\text{-}\mathbf{mod} \mid \forall m \in M \ \ \mathrm{ann}\,(m)$ is $\{0\}$ or $R\}$ ($\ni R, \not\ni R/I$). A subcategory closed under epi images and coproducts, but not under subobjects, is $\{M \in \mathrm{Ob}\, R\text{-}\mathbf{mod} \mid \forall m \in M \ \ \forall r \in R \setminus \{0\} \ \ \exists\, m_1 \in M$, such that $rm_1 = m\}$ ($\ni Q, \not\ni R$). ■

Proof of Corollary 19. Clearly $\{M \in \mathrm{Ob}\, R\text{-}\mathbf{mod} \mid LM = \{0\}\}$, where $L \subset R$ is a left ideal, is hereditary, monocoreflective, and is closed under all products (without commutativity of R).

We turn to the other direction. A hereditary monocoreflective subcategory $\mathbf{M}$ is of the form $\mathbf{M}(\mathcal{L})$, where $\mathcal{L}$ is a filter of ideals of R. We distinguish two cases.

If $\mathcal{L}$ is finite, then $\mathbf{M}(\mathcal{L}) = \{M \in \mathrm{Ob}\, R\text{-}\mathbf{mod} \mid (\cap\{L \mid L \in \mathcal{L}\})M = \{0\}\}$.

If $\mathcal{L}$ is infinite, then let $\{L_i \mid i = 1, 2, ...\} \subset \mathcal{L}$ be different. Then, by commutativity of R, and by the proof of Proposition 17, $R/L_i \in \mathrm{Ob}\, \mathbf{M}(\mathcal{L})$, and $\mathrm{ann}\,(1 + L_i) = L_i$. Therefore, $\prod^\infty (R/L_i) \in \mathrm{Ob}\, \mathbf{M}(\mathcal{L})$, and $\mathrm{ann}\,(\langle 1 + L_i \rangle) = \cap^\infty \mathrm{ann}\,(1 + L_i) = \cap^\infty L_i = \{0\}$. Hence $\{0\} \in \mathcal{L}$, and, once more, $\mathcal{L}$ is a principal filter, generated by $\{0\}$, and $\mathbf{M}(\mathcal{L}) = \{M \in \mathrm{Ob}\, R\text{-}\mathbf{mod} \mid (\cap\{L \mid L \in \mathcal{L}\})M = \{0\}\} = \{M \in \mathrm{Ob}\, R\text{-}\mathbf{mod} \mid \{0\}M = \{0\}\} = R\text{-}\mathbf{mod}$. These prove the first statement of the corollary.

Thus, in both cases, $\mathcal{L}$ is a principal filter, generated by $L(\mathcal{L})$, say, i.e., $\mathcal{L} = \{L \subset R$ is an ideal $\mid L \supset L(\mathcal{L})\}$. Then $\{M \in \mathrm{Ob}\, R\text{-}\mathbf{mod} \mid (\cap\{L \mid L \in \mathcal{L}_1\})M = \{0\}\} \subset \{M \in \mathrm{Ob}\, R\text{-}\mathbf{mod} \mid (\cap\{L \mid L \in \mathcal{L}_2\})M = \{0\}\} \iff \mathbf{M}(\mathcal{L}_1) \subset \mathbf{M}(\mathcal{L}_2) \iff \mathcal{L}_1 \subset \mathcal{L}_2 \iff L(\mathcal{L}_1) \supset L(\mathcal{L}_2)$. This proves the third statement of the corollary.

For the second statement of the corollary we have to give three examples, for R not a field, or, additionally, being an integral domain. These are as in the proof of Theorem 18, except that in the first example there we take the last example from Remark 10. (The second and third examples from the proof of Theorem 18 are closed even under arbitrary products.) ■

Proof of Corollary 20. We use Theorem 18, taking in account the following.

All left ideals of R are principal. The generator (not unique) of a left ideal $L \in \mathcal{L}$ will be denoted by $l(L)$; thus, $L = Rl(L)$.

A hereditary monocoreflective subcategory of R-**mod** is of the form $\mathbf{M}(\mathcal{L})$, where $\mathcal{L}$ is a filter in the left ideals of R.

Observe that for two left ideals of R, say, Rr_1, Rr_2, we have $Rr_1 \subset Rr_2$ if and only if r_2 is a right divisor of r_1. Further, for finitely many left ideals of R, say, $Rr_1, ..., Rr_n$, we have $\cap^n (Rr_i) = Rr$ if and only if r is the least common left multiple of $r_1, ..., r_n$.

Thus we can rewrite the property that $\mathcal{L}$ is a filter of ideals in R in the following way: $S := \{l(L) \mid L \in \mathcal{L}\}$ contains 1, and is closed under right divisors, and least common left multiples of finitely many elements (meant up to the equivalence given before Corollary 20). Hence, $\mathbf{M}(\mathcal{L}) = \{M \in \mathrm{Ob}\, R\text{-}\mathbf{mod} \mid \forall m \in M \ \ \mathrm{ann}(m) \in \mathcal{L}\} = \{M \in \mathrm{Ob}\, R\text{-}\mathbf{mod} \mid \forall m \in M \ \ o(m) \in S\}$ ("$o(m) \in S$" meant up to the above equivalence). This proves the first statement of the corollary.

We have, for R commutative, $S_1 \subset S_2 \iff \mathcal{L}_1 \subset \mathcal{L}_2 \iff \mathbf{M}(\mathcal{L}_1) \subset \mathbf{M}(\mathcal{L}_2) \iff \{M \in \mathrm{Ob}\, R\text{-}\mathbf{mod} \mid \forall m \in M \ \ o(m) \in S_1\} \subset \{M \in \mathrm{Ob}\, R\text{-}\mathbf{mod} \mid \forall m \in M \ \ o(m) \in S_2\}$ by Theorem 18 ("$S_1 \subset S_2$", and "$o(m) \in S_i$" meant up to the above equivalence). This proves the second statement of the corollary. ■

Proof of Corollary 21. We use Corollary 19, taking in account the following.

We have $\{M \in \mathrm{Ob}\, R\text{-}\mathbf{mod} \mid LM = \{0\}\} = \{M \in \mathrm{Ob}\, R\text{-}\mathbf{mod} \mid l(L)M = \{0\}\}$, for $L \subset R$ an ideal. This proves the first statement of the corollary.

Hence, also $\{M \in \mathrm{Ob}\, R\text{-}\mathbf{mod} \mid l(L_1)M = \{0\}\} \subset \{M \in \mathrm{Ob}\, R\text{-}\mathbf{mod} \mid l(L_2)M = \{0\}\} \iff \{M \in \mathrm{Ob}\, R\text{-}\mathbf{mod} \mid L_1 M = \{0\}\} \subset \{M \in \mathrm{Ob}\, R\text{-}\mathbf{mod} \mid L_2 M = \{0\}\} \iff L_1 \supset L_2 \iff l(L_1)|l(L_2)$ by Corollary 19. This proves the second statement of the corollary. ■

Proof of Theorem 7. On the one hand, each subcategory $\mathbf{M}(E)$ from the theorem is hereditary and monocoreflective.

We turn to the other implication. We apply Corollary 20. A hereditary monocoreflective subcategory $\mathbf{M}$ is of the form $\{M \in \mathrm{Ob}\, R\text{-}\mathbf{mod} \mid \forall m \in M \quad o(m) \in S\}$, where $S \subset R$ contains 1, and is closed under (right) divisors, and least common (left) multiples of finitely many elements (up to the equivalence given before Corollary 20, that is now the same, as up to associates).

First suppose that $0 \in S$. Then $S = R$ (up to the above equivalence), and $\mathbf{M} = R\text{-}\mathbf{mod}$.

From now on we suppose $0 \notin S$.

Let P be a set of primes of R, containing exactly one element from each equivalence class of primes, under the equivalence relation of being associates. Now let us write each $r \in S$, as a product of finitely many primes in P, and still of a unit u, i.e., $r = u(r) \prod p^{e(r,p)}$, where $e(r,p) \in \mathbb{N} = \{1, 2, ...\}$ (for a unit $u \in S$ this is the one-element product u). We will extend this factorization as a product over P, letting $e(r,p) = 0$ for primes in P not occurring in the factorization. Then $e(r,p) \in \mathbb{Z}_+ = \{0, 1, ...\}$.

Then the fact, that S contains 1, is closed under divisors, and least common multiples of finitely many elements (up to associates), means exactly that the set of functions $\mathcal{E}(S) := \{e(r,p) : P \to \mathbb{Z}_+ \mid r \in S\}$ contains the identically 0 function, is downwards closed, and is closed under finite maxima. Observe that each of these functions is 0 except for finitely many elements $p \in P$. Therefore the last but one sentence is equivalent to the fact that $\mathcal{E}(S)$ is of the form

$$\{e(p) : P \to \mathbb{Z}_+ \big| \sum_{p \in P} e(p) < \infty \text{ and } e(p) \leq E(p)\},$$

where $E(p) : P \to \{0, 1, ..., \infty\}$ is some function.

Therefore, we have $\mathbf{M} = \{M \in \mathrm{Ob}\, R\text{-}\mathbf{mod} \mid \forall m \in M \quad o(m) \in S\} = \{M \in \mathrm{Ob}\, R\text{-}\mathbf{mod} \mid \forall m \in M \quad o(m) \neq 0$, and the exponent of each prime $p \in P$ of R in the prime factorization of $o(m)$ is at most $E(p)\} = \mathbf{M}(E)$ ("$o(m) \in S$" meant up to associates). Thus we have proved the first statement of the theorem.

Moreover, by Corollary 20 we have $\mathbf{M}(E_1) \subset \mathbf{M}(E_2) \iff \{M \in \mathrm{Ob}\, R\text{-}\mathbf{mod} \mid \forall m \in M \quad o(m) \in S_1\} \subset \{M \in \mathrm{Ob}\, R\text{-}\mathbf{mod} \mid \forall m \in M \quad o(m) \in S_2\} \iff S_1 \subset S_2 \iff \mathcal{E}(S_1) \subset \mathcal{E}(S_2) \iff E_1 \leq E_2$ ("$o(m) \in S_i$" and "$S_1 \subset S_2$" meant up to associates). Thus we have proved the third statement of the theorem.

For the second statement of the theorem we have to give three examples. However, these are contained in Theorem 18. ■

Proof of Corollary 9. We show that the hypotheses of Corollary 21 are satisfied.

Let $\{L_i \mid i = 1, 2, ...\}$ be different (left) ideals of R, generated by $l(L_i)$, respectively. Then the $l(L_i)$'s are pairwise non-associates. If $\cap_i L_i$ would not be $\{0\}$, then we could consider $l(\cap_i L_i) \in R \setminus \{0\}$. Then each $l(L_i)$ would be a divisor of $l(\cap_i L_i)$. However, the number of pairwise non-associate divisors of $l(\cap_i L_i)$ is finite, so we have obtained a contradiction.

Thus the conclusion of Corollary 21 holds. This implies the first and third statements of Corollary 9.

For the second statement of Corollary 9 we have to give four examples, for R not a field. Now R has at least one non-invertible non-zero element, hence, also one non-trivial prime element. Hence P from the proof of Theorem 7 is not empty. Hence a subcategory closed under subobjects, epi images, coproducts, but not under countably infinite products, is any of the subcategories described in Theorem 7, with $\sum_{p \in P} E(p) = \infty$. The other three examples are contained in the proof of Corollary 19, observing that the second and third examples from the proof of Theorem 18 are closed under (countably infinite) products. ■

Proofs of Theorem 1 and Corollary 2. These follow from Theorem 7 and Corollary 9 (letting $R = \mathbb{Z}$). ■

Proofs of Theorem 3 and Corollary 5. These will follow by Pontrjagin duality from the proofs of Theorems 1 and 6 and Corollaries 2 and 9, as described below.

For Theorem 3, we only have to describe the Pontrjagin duals (opposites) of the subcategories $\mathbf{M}(E)$ from Theorem 1. Observe that $\mathrm{Ob}\,\mathbf{M}(E)$ consists of the surjective (continuous homomorphic) images of coproducts $\coprod_{\alpha \in A} \mathbb{Z}(n_\alpha)$, where each n_α has a prime factorization as in Theorem 1 (cf. the proofs of Theorems 18 and 6).

Their duals are closed subgroups G of products $\prod_{\alpha \in A} \mathbb{Z}(n_\alpha)$, with canonical projections π_α. Such a subgroup G is also a subgroup of the product of its own canonical projections in the $\mathbb{Z}(n_\alpha)$'s. That is, G is a closed subgroup of a product $\prod_{\alpha \in A} \mathbb{Z}(n'_\alpha)$, where $n'_\alpha | n_\alpha$, so also each n'_α has a prime factorization as in Theorem 1. That is, we may assume that, for each $\alpha \in A$, we have $\pi_\alpha G = \mathbb{Z}(n_\alpha)$. Then a required neighbourhood subbase of 0 in G is $\{\pi_\alpha^{-1}(0) \cap G \mid \alpha \in A\}$.

Conversely, if, for some $G \in \mathrm{Ob}\,\mathbf{CompAb}$, the element 0 has a neighbourhood subbase $\{G_\alpha \mid \alpha \in A\}$ as in the theorem, then via the canonical maps $G \to G/G_\alpha$ we obtain a continuous homomorphism $G \to \prod_{\alpha \in A}(G/G_\alpha) \cong \prod_{\alpha \in A} \mathbb{Z}(n_\alpha)$, that is a homeomorphic embedding onto a closed subgroup of this product.

These imply the first statement of Theorem 3; the second and third ones follow immediately from Theorem 1.

For Corollary 5, observe that, for a (discrete) Abelian group G, we have $nG = \{0\}$ if and only if for its dual compact Abelian group $\hat{G}$ we have $n\hat{G} = \{0\}$, where $n \geq 1$ is an integer.

This implies the first statement of Corollary 5; the second and third ones follow immediately from Corollary 2. ■

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