Limit theorems and infinitely divisible distributions. Part II

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Summary: In the second part of this work we deal with the question when the normalized partial sums of independent random variables, or more generally the sums of the random variables in the same row of a triangular array converge in distribution. We present a result which gives a necessary and sufficient condition for the existence of a limit distribution if the sequence of random variables or the triangular array satisfies the uniform smallness condition, and also describe the limit distribution. It turns out that the limit is always an infinitely divisible distribution. The hard part of the problem is to show that the sufficient condition given for the existence of a limit distribution is at the same time a necessary condition. We discuss the content of this condition in more detail and also show how the most important classical limit theorems can be obtained as special cases of the result discussed in this work.

We also try to explain the main ideas of the proof. An important step in it is the introduction of the so-called associated distributions of the distribution functions of the summands and to show that the convergence of independent random variables with these associated distributions is closely related to the original limit problem. The associate distributions are infinitely divisible. So to understand limit theorems for sums of independent random variables it is useful to study the special problem when the sums of independent random variables with infinitely divisible distributions have a limit.

1. Formulation of the basic results.

We present the proof of a result which gives a necessary and sufficient condition for the existence of a limit distribution for the (normalized) sums of the random variables in the same row of a triangular array if they satisfy the uniform smallness condition. Furthermore, the limit distribution will be also described. It turns out that it is always an infinitely divisible distribution. This means that if the uniform smallness condition holds, then the limit distribution of the normalized sums of independent random variables is infinitely divisible in the most general case. Let us emphasize that we did not assume that the summands are identically distributed.

To study the limit problem we are interested in it is useful to associate to all terms in the triangular array we are working with an infinitely divisible random variable in an appropriate way, We associate independent random variables to random variables in the same row. It can be achieved that the sums of these associated random variables from a row of the triangular array converge in distribution if and only if the original sums we are investigating converge. Moreover, the original sums and the sums made from the associated random variables have the same limit distribution. Since the sum of independent and infinitely divisible random variables is again infinitely divisible, the introduction of these associated random variables leads to the problem when a sequence of infinitely divisible distributions have a limit.

Naturally, the results proved for general triangular arrays also hold in the special case when the elements in a row are not only independent but also identically distributed random variables. This particular case deserves special attention, because in this case the following heuristically "obvious" statement holds which nevertheless demands a special proof: If the sums of the random variables from distinct rows have a limit distribution and the random variables in a row of this triangular array are not only independent but also identically distributed, then the triangular array satisfies the uniform smallness condition. This result will be exploited in the proof of the Lévy–Hinchin formula i.e. in the proof of the result that the construction in Part I describes all possible infinitely divisible distributions, and it describes them in a unique way.

Before formulating the results we are interested in let us recall some important notions and let us introduce some notations. Let $\xi_{k,j}$, $k=1,2,\dots$, $j=1,\dots,n_k$, be a triangular array of random variables, i.e. let us assume that for a fixed number k the random variables $\xi_{k,j}$, $1\le j\le n_k$, are independent. This triangular array satisfies the condition of uniform smallness if for all numbers $\varepsilon>0$ $\lim\limits_{k\to\infty}\sup\limits_{1\le j\le n_k} P(|\xi_{k,j}|>\varepsilon)=0$. Let us also recall the notion of canonical measures introduced in Part I. A measure M on the real line is called a canonical measure if for all finite intervals $[a,b]\subset\mathbf{R}^1$ the measure $M\{[a,b]\}$ is finite, and for an arbitrary number $a>0$

$$\int_a^\infty \frac{1}{x^2}M(\,dx)<\infty,\quad\text{and}\quad \int_{-\infty}^{-a}\frac{1}{x^2}M(\,dx)<\infty.$$

Let $\xi_{k,j}$, $k=1,2,\dots$, $j=1,\dots,n_k$, be a triangular array satisfying the uniform smallness condition, and let $F_{k,j}$ denote the distribution of the random variable $\xi_{k,j}$. Let us introduce the σ-finite measures

$$M_k(\,dx)=\sum_{j=1}^{n_k}x^2F_{k,j}(\,dx) \tag{1.1}$$

and the functions

$$\begin{aligned} M_k^+(x)&=\sum_{j=1}^{n_k}(1-F_{k,j}(x))=\int_x^\infty\frac{1}{u^2}M_k(\,du),\\ M_k^-(x)&=\sum_{j=1}^{n_k}F_{k,j}(-x)=\int_{-\infty}^{-x}\frac{1}{u^2}M_k(\,du),\end{aligned}\qquad k=1,2,\dots,\quad x>0 \tag{1.2}$$

on the real line for all $k=1,2,\dots$. Let us introduce the random sums $S_k=\sum\limits_{j=1}^{n_k}\xi_{k,j}$. The theorem formulated below gives a necessary and sufficient condition for the convergence in distribution of the normalized sums $S_k - b_k$ with appropriate norming constants b_k. This condition is expressed by means of the above introduced measures M_k and functions $M_k^{\pm}$.

Theorem 1. *Let $\xi_{k,j}$, $k = 1, 2, \dots$, $j = 1, \dots, n_k$, be a triangular array satisfying the uniform smallness condition. Let $F_{k,j}$, $k = 1, 2, \dots$, $1 \le j \le n_k$, denote the distribution function of the random variable $\xi_{k,j}$ and set $S_k = \sum\limits_{j=1}^{n_k} \xi_{k,j}$. Let us assume that the normalized sums $S_k - \bar b_k$ with some appropriate constants $\bar b_k$ converge in distribution as $k \to \infty$. Then there exists a canonical measure M on the real line such that the functions $M_k^{\pm}(x)$ defined in formula (1.2) together with the functions*

$$M^+(x) = \int_x^\infty \frac{1}{u^2} M(\,du), \qquad M^-(x) = \int_{-\infty}^{-x} \frac{1}{u^2} M(\,du) \tag{1.2'}$$

defined by means of the canonical measure M in an analogous way satisfy the relation

$$\lim_{k\to\infty} M_k^+(x) = M^+(x) \quad \textit{and} \quad \lim_{k\to\infty} M_k^-(x) = M^-(x) \tag{1.3}$$

in all such points $x > 0$ where the functions $M^+(\cdot)$ or $M^-(\cdot)$ are continuous. Beside this, the relation (1.6) formulated below also holds. To formulate this relation let us first introduce some notations.

Let us fix a number $a > 0$, and define the function

$$\tau(x) = \tau_a(x) = \begin{cases} x & \textit{if } |x| \le a \\ a & \textit{if } x \ge a \\ -a & \textit{if } x \le -a \end{cases} \tag{1.4}$$

and numbers

$$\beta_{k,j} = \beta_{k,j}(a) = E\tau(\xi_{k,j}), \quad b_k = b_k(a) = \sum_{j=1}^{n_k} \beta_{k,j}, \quad B_k = B_k(a) = \sum_{j=1}^{n_k} \beta_{k,j}^2,$$
$$k = 1, 2, \dots, \quad j = 1, \dots, n_k. \tag{1.5}$$

If the normalized sums $S_k - \bar b_k$ converge in distribution with some appropriate constants $\bar b_k$, then the normalized sums $S_k - b_k$ with the above introduced numbers b_k also converge in distribution, and the canonical measures M_k introduced in formula (1.1) satisfy the relation

$$M_k([-s, s]) - B_k \to M([-s, s]), \quad \textit{if } k \to \infty \tag{1.6}$$

in all points $s > 0$ which are points of continuity of the functions M^+ and M^-.

Conversely, if the relations (1.3) and (1.6) hold with an appropriate canonical measure M and the numbers B_k defined in formula (1.5), then the appropriate normalizations of the sums S_k converge in distribution. More explicitly, in this case the *normalized sums* $S_k - b_k$*,* $k = 1, 2, \dots,$ *with the constants* b_k *defined in formula (1.5) have a limit distribution which we can describe. The characteristic function* $\varphi(t)$ *of this limit distribution has a logarithm which can be given by the formula*

$$\log \varphi(t) = \int_{-\infty}^{\infty} \frac{e^{itu} - 1 - it\tau(u)}{u^2} M(\,du) \tag{1.7}$$

with the canonical measure M *determined by formulas (1.3),* $(1.2')$ *and (1.6) and the function* τ *defined in formula (1.4). (Formula (1.6) is needed to define the measure* $M(\{0\})$ *of the origin.)*

To satisfy the above limit theorem it is sufficient to demand a weakened version of condition (1.6), namely to demand that it hold in a point of (joint) continuity of the functions $M^{\pm}$*. (See Remark 1 formulated below.)*

To give a complete formulation of Theorem 1 or more explicitly of formula (1.7) in it we still have to define the value of the integrand in (1.7) also in the point $u = 0$. By continuity arguments this is defined by the relation

$$\left. \frac{e^{itu} - 1 - it\tau(u)}{u^2} \right|_{u=0} = \lim_{u \to 0} \frac{e^{itu} - 1 - it\tau(u)}{u^2} = -\frac{t^2}{2}$$

here and in subsequent formulas.

Let us consider a triangular array $\xi_{k,j}$, $k = 1, 2, \dots,$ $1 \le j \le n_k$, satisfying the uniform smallness condition. Theorem 1 gives a necessary and sufficient condition for the existence of an appropriate normalizations $S_k - b_k$ of the sums $S_k = \sum\limits_{j=1}^{n_k} \xi_{k,j}$ for which these normalized sums converge in distribution. Beside this, this result also gives a possible normalization and describes the limit distribution belonging to it. This limit distribution is described in formula (1.7). A comparison of this formula with the results of Part I of this work yields that this limit is an infinitely divisible distribution whose Poissonian component is the (regularized) sum of the values of a Poisson process with counting measure $u^{-2}M(du)$ and which has a Gaussian component with expectation zero and variance $M(\{0\})$. The measure M appearing here is the limit of the measures M_k introduced in formula (1.1). The limit procedure leading to this measure M is described through formulas (1.3), (1.5) and (1.6). Let us emphasize that the limit in Theorem 1 is always an infinitely divisible distribution, although we did not assume that the terms in the sums we have considered are identically distributed. (In the first part of this work we only gave a heuristic argument why the limit of the sums of independent and identically distributed random variables have always an infinitely divisible distribution.) A result presented in the later formulated Theorem $2'$ also implies that the limit distribution determines the measure M in formula (1.7).

To understand Theorem 1 better we make some comments. Let us first observe that if the sequence of the random variables $S_k - b_k$ converges in distribution, then the sequence $S_k - \bar b_k$ with another sequence of constants $\bar b_k$ converges in distribution if and only if the finite limit $C = \lim\limits_{k \to \infty} (\bar b_k - b_k)$ exists. Indeed, if the above finite limit exists, and the sequence $S_k - b_k$ converges to a distribution $F(x)$, then the sequence $S_k - \bar b_k$ converges to the distribution $F(x + C)$. Conversely, if the sequence $\bar b_k - b_k$ is non-convergent, then either this sequence is non-bounded and it has a subsequence $\bar b_{k_j} - b_{k_j}$ which tends to plus or minus infinity and the measures of all compact sets tends to zero with respect to the distributions of the subsequences $S_{k_j} - \bar b_{k_j}$ in this case, or this sequence has two subsequences with indices k_j and $\bar k_j$ such that the subsequences $\bar b_{k_j} - b_{k_j}$ and $\bar b_{\bar k_j} - b_{\bar k_j}$ have two different finite limits. In the latter case the sequence $S_k - \bar b_k$ has two subsequences with different limits.

The above observation tells us how many freedom we have in the choice of the norming constant b_k in Theorem 1. In its formulation we fixed a parameter $a > 0$ and the constants b_k depended on this number a through the function $\tau(\cdot) = \tau_a(\cdot)$. If a and a' are two different positive constants, then

$$b_k(a) - b_k(a') = \int \frac{(\tau_a(u) - \tau_{a'}(u))}{u^2} M_k(\,du),$$

and the finite limit $C = \lim\limits_{k\to\infty} (b_k(a) - b_k(a')) = \int \frac{(\tau_a(u) - \tau_{a'}(u))}{u^2} M(\,du)$ exists by formula (1.3). Beside this, the normalizations $b_k(a)$ and $b_k(a')$ supply two different limit distributions which are the shift of each other with the above constant C. Indeed, if we consider the logarithms of the characteristic functions of these limit distributions, then their difference equals $it \int \frac{(\tau_a(u) - \tau_{a'}(u))}{u^2} M(\,du) = itC$. On the other hand, if the characteristic function of a random variable ξ equals $\varphi(t)$, then the characteristic function of the random variable $\xi + C$ equals $e^{itC}\varphi(t)$.

Beside this, we shall show that the limit of the measures M_k in Theorem 1 does not depend on the choice of the parameter a. To prove this we have to understand the content of formula (1.6). This formula defines the measure $M(\{0\}) = \lim\limits_{s\to 0} M([-s, s])$, and it depends on the parameter a through the constant $B_k = B_k(a)$ defined in formula (1.5). Beside this, we want to show that $M(\{0\}) \ge 0$ and want to give the probabilistic content of this quantities. This will be done in Remarks 2 and 3. Before these Remarks we show in Remark 1 that if formula (1.6) holds in such a point $s > 0$ which is a point of continuity of the functions $M^+(\cdot)$ and $M^-(\cdot)$, then this relation also holds in all points of continuity of these functions.

Remark 1. Formula (1.3) determines the restriction of the canonical measure M to the measurable subsets of the set $\mathbf{R}^1 \setminus \{0\}$. Then formula (1.6) determines the measure $M(\{0\})$. If formula (1.6) holds in a point of continuity s of the $M^\pm(\cdot)$ functions, then relation (1.3) implies that it holds in all points of continuity $s' > 0$ of these functions. Indeed,

$$\begin{aligned} M_k([-s', s']) - M_k([-s, s]) &= \int_s^{s'} u^2 M_k^+(\,du) + \int_s^{s'} u^2 M_k^-(\,du) \\ \to \int_s^{s'} M(\,du) + \int_{-s'}^{-s} M(\,du) &= M([-s', s']) - M([-s, s]). \end{aligned}$$

This means that condition (1.6) can be replaced by its weakened version which only states that it holds in one point of continuity of the functions $M^{\pm}$.

Remark 2. Although the function $\tau(\cdot)=\tau_a(\cdot)$, and as a consequence the constants $\beta_k=\beta_k(a)$, the norming constants $b_k=b_k(a)$ and the constants $B_k=B_k(a)$ appearing in formula (1.6) depend on the parameter $a>0$, it can be shown that $\lim\limits_{k\to\infty}(B_k(a')-B_k(a))=0$ for two different constants $a>0$ and $a'>0$. This means that formula (1.6) is meaningful, its content does not depend on the choice of the parameter >0. To prove this statement let us first observe that because of the condition of uniform smallness $\lim\limits_{k\to\infty}\sup\limits_{1\le j\le n_k}|\beta_{k,j}(a')-\beta_{k,j}(a)|=0$, and by relation (1.3) $\sup\limits_k\sum\limits_{j=1}^{n_k}|\beta_{k,j}(a')-\beta_{k,j}(a)|<\infty$. The last relation holds, since for all pairs $a'>a>0$

$$\sum_{j=1}^{n_k}|\beta_{k,j}(a')-\beta_{k,j}(a)|\le\int_a^{a'}u(M_k^+(\,du)+M_k^-(du))+(a'-a)[M_k^+(a')+M_k^-(a')],$$

and the right-hand side of the last expression can be bounded because of formula (1.3) independently of k.

Some difficulty arises in the proof, because the sequence $B_k^*=\sum\limits_{j=1}^{n_k}|\beta_{k,j}|$ may be unbounded. To overcome this difficulty we can show that because of the uniform smallness condition $\lim\limits_{k\to\infty}\sup\limits_{1\le j\le n_k}|\beta_{k,j}(a)|=0$. Indeed, for all numbers $\varepsilon>0$ $|\beta_{k,j}(a)|\le\varepsilon+aP(|\xi_{k,j}>\varepsilon)\le2\varepsilon$ if $k\ge k_0(\varepsilon,a)$ and $1\le j\le n_k$. This relation holds, since $aP\left(|\xi_{k,j}|>\varepsilon\right)\le\varepsilon$ if $k\ge k_0(\varepsilon,a)$. Hence $|B_k(a')-B_k(a)|\le I_k+2II_k$, where

$$I_k=\sum_{j=1}^{n_k}|\beta_{k,j}(a')-\beta_{k,j}(a)|^2\le\sup_{1\le j\le n_k}|\beta_{k,j}(a')-\beta_{k,j}(a)|\sum_{j=1}^{n_k}|\beta_{k,j}(a')-\beta_{k,j}(a)|\to0,$$

if $k\to\infty$, and

$$II_k=\sum_{j=1}^{n_k}|(\beta_{k,j}(a')-\beta_{k,j}(a))\beta_{k,j}(a)|\le\sup_{1\le j\le n_k}|(\beta_{k,j}(a)|\sum_{j=1}^{n_k}|\beta_{k,j}(a')-(\beta_{k,j}(a)|\to0$$

if $k\to\infty$. This implies the statement of Remark 2.

Remark 3. Let us define, similarly to formula (1.4), the function $\tau'(x)=\tau'_a(x)$ as $\tau'(x)=x$ if $|x|\le a$, and $\tau'(x)=0$ if $|x|>a$. Set $\beta'_{k,j}=\beta'_{k,j}(a)=E\tau'(\xi_{k,j})$, $B'_k=B'_{k,j}(a)=\sum\limits_{j=1}^{n_k}{\beta'_{k,j}}^2$. With a natural modification of the argument in Remark 2 we get that $\lim\limits_{k\to\infty}(B_k-B'_k)=0$, and here we can write $B'_k(a')$ with an arbitrary number $a'>0$ instead of the number $B'_k(a)$. By exploiting this fact and carrying out a limiting procedure $\varepsilon\to 0$ through such numbers ε which are points of continuity of both functions M^+ and M^- we get that

$$\begin{aligned} M(\{0\}) &= \lim_{\varepsilon\to 0} M([-\varepsilon,\varepsilon]) = \lim_{\varepsilon\to 0}\lim_{k\to\infty} (M_k([-\varepsilon,\varepsilon]) - B'_k(\varepsilon)) \\ &= \lim_{\varepsilon\to 0}\lim_{k\to\infty}\left(\sum_{j=1}^{n_k}\left(E\tau'_\varepsilon(\xi_{k,j})^2 - (E\tau'_\varepsilon(\xi_{k,j}))^2\right)\right) = \lim_{\varepsilon\to 0}\lim_{k\to\infty}\sum_{j=1}^{n_k}\operatorname{Var}\tau'_\varepsilon(\xi_{k,j}). \end{aligned} \tag{1.8}$$

Formula (1.8) means in particular that $M(\{0\})\ge 0$. The following heuristic content can be given to formula (1.8). The variance of the normal component of the limit distribution, the number $M(\{0\})$ can be obtained in the following way: We truncate the random variables in a fixed row of the triangular array at a level $\varepsilon>0$, we sum up these random variables, and calculate the variance of the sum. Then take their limit as the index of the row k tends to infinity and then the level of the truncation ε tends to zero. In an informal way this means that the normal component of the limit distribution is the "contribution of the inside part" of the summands. The former argument also shows that the expression defining the measure $M([-s,s])$ in formula (1.6) is necessarily non-negative.

Formula (1.6) is equivalent to the relation (1.6′)

$$M(\{0\}) = \lim_{\varepsilon\to 0}\lim_{k\to 0}\left(M_k([-\varepsilon,\varepsilon]) - B'_k(\varepsilon)\right), \tag{1.6′}$$

where such numbers $\varepsilon>0$ are taken in the limit which are points of continuity of the measure M. We have already seen that relation (1.6) implies relation (1.6′). To see the converse statement let us first fix some small number $\varepsilon'>0$ write up the identity

$$\begin{aligned} M_k([-s,s]) - B_k - M([-s,s]) &= M_k([-s,s]) - M_k([-\varepsilon',\varepsilon']) \\ &\quad - (M([-s,s]) - M([-\varepsilon',\varepsilon'])) + (B'_k(\varepsilon') - B_k) + (M_k([-\varepsilon',\varepsilon']) \\ &\quad - B'_k(\varepsilon') - M(\{0\}) - (M([-\varepsilon',\varepsilon']) - M\{0\}), \end{aligned}$$

and estimate the right hand side of this identity if relation (1.6′) holds. Observe that

$$\lim_{k\to\infty}(M_k([-s,s]) - M_k([-\varepsilon',\varepsilon']) - (M([-s,s]) - M([-\varepsilon',\varepsilon'])) = 0$$

and $\lim\limits_{k\to\infty}(B'_k(\varepsilon') - B_k) = 0$. In the proof of these two relations we only need formula (1.3) and do not apply formula (1.6). If we fix a number $\varepsilon>0$ and choose the number $\varepsilon'=\varepsilon'(\varepsilon)>0$ sufficiently small then we also can write $M([-\varepsilon',\varepsilon']) - M\{0\}|\le\varepsilon$ and because of relation (1.6′)

$$\limsup_{k\to\infty}|M_k([-\varepsilon',\varepsilon']) - M_k(\{0\}) - B'_k(\varepsilon')|\le\varepsilon.$$

These relations together imply that if relation (1.6′) holds, then

$$\limsup_{k\to\infty}|M_k([-s,s]) - B_k - M([-s,s])|\le 2\varepsilon.$$

Since the last inequality holds for all $\varepsilon \to 0$, relation (1.6′) implies relation (1.6).

Formula (1.6) can be better applied in the subsequent proofs, and probably it is simpler to check it. On the other hand the heuristic content of formula (1.6′) is more understandable.

An important step in the proof of Theorem 1 is the investigation of the question when a sequence of infinitely divisible distributions described by means of the Lévy–Hinchin formula converges in distribution and what the limit distribution is. To formulate a result in this direction let us first introduce the notion of (weak) convergence of canonical measure. This notion is a natural modification of the convergence of distribution functions.

Definition of convergence of canonical measures. *Let M_n, $n=1,2,\dots$, and M be canonical measures on the real line. We say that the canonical measures M_n converge (weakly) to the canonical measure M if*

$$\begin{aligned}\lim_{n\to\infty} M_n^+(x)&=\lim_{n\to\infty}\int_x^\infty \frac{1}{u^2}M_n(\,du)=M^+(x)=\int_x^\infty \frac{1}{u^2}M(\,du),\\ \lim_{n\to\infty} M_n^-(x)&=\lim_{n\to\infty}\int_{-\infty}^{-x} \frac{1}{u^2}M_n(\,du)=M^-(x)=\int_{-\infty}^{-x} \frac{1}{u^2}M(\,du),\end{aligned}$$

for all such numbers $x>0$ where the function $M^+(\cdot)$ or $M^-(\cdot)$ is continuous, and

$$\lim_{n\to\infty} M_n([a,b])=M([a,b])$$

for all such numbers $-<\infty<a<b<\infty$ where the measure M is continuous. (This continuity of the measure means that $M(\{a\})=M(\{b\})=0$.)

Remark 4. Similarly to the convergence of distribution functions the convergence of canonical measures can be expressed by means of convergence of integrals of an appropriate class of continuous functions. A sequence of canonical measures M_n converges (weakly) to a canonical measure M if and only if $\int f(u)M_n(du)\to\int f(u)M(du)$ for all such continuous functions f for which $\sup\limits_u \dfrac{|f(u)|}{1+u^2}<\infty$. Actually this statement can be deduced from the analogous statement about distribution functions if we observe that in the case when the limit measure is not identically zero a sequence of canonical measures M_n converges weakly to a canonical measure M if and only if the corresponding probability measures $F_n=F_n(M_n)$ defined as $F_n(\,dx)=\dfrac{M_n(\,dx)}{C_n(1+x^2)}$, $C_n=\displaystyle\int\frac{M_n(\,dx)}{1+x^2}$ converge (weakly) to the probability measure $F(\,dx)=\dfrac{M(\,dx)}{C(1+x^2)}$, $C=\displaystyle\int\frac{M(\,dx)}{1+x^2}$. Since we do no need this result we omit the details.

Now we formulate the result about the convergence of infinitely distributions we shall need in the sequel.

Theorem 2. *Let a sequence of infinitely divisible distributions be given with characteristic functions $\varphi_n(t)$, $t \in \mathbf{R}$. Let the logarithms of these characteristic functions be of the form*

$$\log \varphi_n(t) = \int_{-\infty}^{\infty} \frac{e^{itu} - 1 - it\tau(u)}{u^2} M_n(\,du) + iB_n t, \quad n = 1, 2, \dots, \tag{1.9}$$

where M_n, $n = 1, 2, \dots$, is a sequence of canonical measures, and the function $\tau(u) = \tau_a(u)$ is defined in formula (1.4). These infinitely divisible distributions converge in distribution if and only if there exists a canonical measure M such that the canonical measures M_n converge (weakly) to the canonical measure M, and also the limit $B = \lim\limits_{n\to\infty} B_n$ exists. If the limit exists, then its characteristic function has a logarithm which can be given in the form

$$\log \varphi(t) = \int_{-\infty}^{\infty} \frac{e^{itu} - 1 - it\tau(u)}{u^2} M(\,du) + iBt.$$

By means of a simplified version of the proof of Theorem 2 the following Theorem 2′ can also be proved.

Theorem 2′. *Let the logarithm of the characteristic function of an infinitely divisible distribution be of the form*

$$\log \varphi(t) = \int_{-\infty}^{\infty} \frac{e^{itu} - 1 - it\tau(u)}{u^2} M(\,du) + iBt$$

with a canonical measure M and real number B. Then the distribution function or what is equivalent to it the logarithm of its characteristic function $\log \varphi(t)$ determines the canonical measure M and constant B.

Finally, we prove the following statement which is useful in the proof of the Lévy–Hinchin formula.

Theorem 3. *Let $\xi_{k,j}$, $k = 1, 2, \dots$, $1 \le j \le n_k$, be such a triangular array whose rows contain such random variables which are not only independent but also identically distributed. (But we do not assume that the triangular array satisfies the uniform smallness condition.) Furthermore, let us assume that $n_k \to \infty$ if $k \to \infty$, and the sums $S_k = \sum\limits_{j=1}^{n_k} \xi_{k,j}$ converge in distribution. Then this triangular array satisfies the uniform smallness condition.*

The main result of this part of the work, Theorem 1, gave a necessary and sufficient condition for the existence of a limit distribution for the normalized sums of the elements $\xi_{k,j}$, $k = 1, 2, \dots$, $1 \le j \le n_k$, of the rows in a triangular array satisfying the uniform smallness condition. With the help of a result contained in Lemma 2 to be formulated later the general problem considered in Theorem 1 can be reduced to the special case when all elements of the triangular array satisfy the condition $E\tau(\xi_{k,j})=0$, $k=1,2,\dots$, $1\le j\le n_k$. Let us restrict our attention in the following consideration to this special case. Then the relations $E\beta_{k,j}=0$, $b_k=0$, $B_k=0$ hold for all indices $k=1,2,\dots$ and $1\le j\le n_k$ in formula (1.5) and the necessary and sufficient condition for the existence of the limit distribution is that the canonical measures M_k defined in formula (1.1) converge weakly to a canonical measure M. Beside this, we can state that the limit distribution is that infinitely divisible distribution whose Poissonian part is determined by a Poisson process with counting measure $u^{-2}M(du)$.

It is worthwhile to compare the result of Theorem 1 and Theorem 2. Given a triangular array $\xi_{k,j}$, $k=1,2,\dots$, $1\le j\le n_k$, $E\tau(\xi_{k,j})=0$ with distribution functions $F_{k,j}$ define a new triangular array whose elements are infinitely divisible random variables $\eta_{k,j}$ and are determined as the (regularized) sums of the elements of Poisson processes with counting measures $F_{k,j}$. We also demand that the random variables $\eta_{k,j}$ be independent for fixed k. The random variables $\eta_{k,j}$ are called the associated random variables to the random variables $\xi_{k,j}$. Let us compare the random sums $S_k=\sum\limits_{j=1}^{n_k}\xi_{k,j}$ and $T_k=\sum\limits_{j=1}^{n_k}\eta_{k,j}$. Theorems 1 and 2 say that these random sums have a limit distribution at the same time, and their limit agree. This agreement of the limit distribution in these two cases is not a random coincidence. It has a deeper reason.

In Part III we shall give another proof of the sufficiency part of Theorem 1. In that part of the work we shall prove the equiconvergence in distribution of the above defined random sums S_k and T_k in a direct probabilistic way by means of an appropriate coupling. Let us give a short informal explanation for this equiconvergence. Let us split both the random variables $\xi_{k,j}$ and the random variables $\eta_{k,j}$ to their inner and outer part as $\xi_{k,j}=\xi_{k,j}I(|\xi_{k,j}|\le\varepsilon_k)+\xi_{k,j}I(|\xi_{k,j}|>\varepsilon_k)$ and $\eta_{k,j}=\eta_{k,j}I(|\eta_{k,j}|\le\varepsilon_k)+\eta_{k,j}I(|\eta_{k,j}|>\varepsilon_k)$ with some appropriate constants ε_k which tend to zero sufficiently slowly. Then some calculation shows that the sum of the inner parts of the random variables $\xi_{k,j}$ and $\eta_{k,j}$ satisfy the central limit theorem with the same variance. The sum of the outer part of the random variables $\xi_{k,j}$ and $\eta_{k,j}$ behave similarly because of a different reason. In this case the Poisson process which determines the random variable $\eta_{k,j}$ contains no points whose absolute value is larger than ε_k with probability almost one because of the uniform smallness condition. The probability of the event that for a fixed index k one of the Poisson processes determing the random variables $\eta_{k,j}$, $1\le j\le n_k$, contains a point with absolute value larger than ε_k is not negligible, but the probability of the event that one of these Poisson processes contains at least two such points is negligibly small. This property enables us to couple the outer part of the random variables $\xi_{k,j}$ and $\eta_{k,j}$ so that they are so close to each other that even their sums are close.

The details of this rather sketchy argument will be worked out in Part III of this work. Such a study may help to understand better the above results. Beside this the coupling method worked out in Part III enables us to make a useful generalization. We shall prove with the help of this method a functional limit theorem version of Theorem 1.

2. Some interesting consequences of the above results.

Most classical limit theorems of probability which are related to the behaviour of the distribution of sums of independent random variables can be deduced from the above results. We present some interesting applications.

A.) The Lévy–Hinchin formula.

The Lévy–Hinchin formula: *A distribution function is infinitely divisible if and only if its characteristic function $\varphi(t)$, $t\in\mathbf{R}^{(1)}$, has a logarithm which be written in the form*

$$\log\varphi(t)=\int_{-\infty}^{\infty}\frac{e^{itu}-1-it\tau(u)}{u^2}M(\,du)+iBt \tag{2.1}$$

where M is a canonical measure on the real line, B is a real number and the function $\tau(\cdot)$ agrees with the function defined in formula (1.4) (with some fixed number $a>0$). In the representation (2.1) of the characteristic function of an infinitely divisible distribution the canonical measure M and number B is uniquely determined.

The proof of the Lévy–Hinchin formula: In Part I. of this work we have already seen that formula (2.1) really defines the logarithm of a characteristic function. Then it is not difficult to see that it is the characteristic function of an infinitely divisible distribution. Indeed, if the logarithm of the characteristic function $\log\varphi(t)$ of a random variable ξ is given by formula (2.1), then for arbitrary integer k its distribution equals the distribution of the sum of k independent and identically distributed random variables whose characteristic functions have a logarithm of the form $\frac{\log\varphi(t)}{k}$, i.e. it is given by formula (2.1) so that the measure $M(\cdot)$ is replaced by $\frac{M(\cdot)}{k}$ and the constant B by $\frac{B}{k}$.

Conversely, if ξ is an infinitely distributed random variable, i.e. for all integers k there exists k independent and identically distributed random variables $\xi_{k,1},\dots,\xi_{k,k}$ such that the distribution of the sum $S_k=\xi_{k,1}+\cdots+\xi_{k,k}$ agrees with the distribution of the random variable ξ, then by Theorem 3 the triangular array $\xi_{k,j}$, $k=1,2,\dots$, $1\le j\le k$, satisfies the uniform smallness condition. Then we can apply Theorem 1 which says that there exists such a sequence of constant b_k that the sequence $S_k-b_k\stackrel{\Delta}{=}\xi-b_k$, where $\stackrel{\Delta}{=}$ denotes equality in distribution, has a limit which can be given by formula (1.7) by means of an appropriate canonical measure M. Since both sequences S_k and S_k-b_k converge in distribution, the limit $\lim\limits_{k\to\infty}b_k=B$ exists, and formula (1.7) implies relation (2.1).

B.) The central limit theorem.

We show that the results formulated in the first Section imply the most important results about the central limit theorem. The most general, and probably most known version of the central limit theorem states the following result: Let $\xi_{k,j}$, $E\xi_{k,j}=0$, $E\xi_{k,j}^2=\sigma_{k,j}^2$, $k=1,2,\dots$, $1\le j\le n_k$, be a triangular array such that $\lim\limits_{k\to\infty}\sum\limits_{j=1}^{n_k}\sigma_{k,j}^2=1$, and the following so-called Lindeberg condition is satisfied: For all numbers $\varepsilon>0$

$\lim\limits_{k\to\infty}\sum\limits_{j=1}^{n_k} E\xi_{k,j}^2 I(|\xi_{k,j}|>\varepsilon)=0$, where $I(A)$ denotes the indicator function of the set A. Then the sums $S_k=\sum\limits_{j=1}^{n_k}\xi_{k,j}$ converge in distribution to the standard normal distribution as $k\to\infty$.

Also the following reversed statement formulated first by Feller holds. Let $\xi_{k,j}$, $E\xi_{k,j}=0$, $E\xi_{k,j}^2=\sigma_{k,j}^2$, $k=1,2,\dots$, $1\le j\le n_k$, be a triangular array such that $\lim\limits_{k\to\infty}\sum\limits_{j=1}^{n_k}\sigma_{k,j}^2=1$. Let us also demand the validity of the following condition which is slightly stronger than the uniform smallness condition: $\lim\limits_{k\to\infty}\sup\limits_{1\le j\le n_k}\sigma_{k,j}^2=0$. If the random sums $S_k=\sum\limits_{j=1}^{n_k}\xi_{k,j}$ converge in distribution to the standard normal distribution, then this triangular array also satisfies the Lindeberg condition. Moreover, this statement also holds if instead of the convergence of the random sums we only assume that the normalized sums S_k-b_k converge in distribution to the standard normal distribution with some appropriate numbers b_k.

We prove the above results by means of the results formulated Section 1. Actually, we shall prove a slightly stronger result. We prove that for the validity of the Lindeberg condition for a triangular array which satisfies the central limit theorem it is enough to assume the uniform smallness condition instead of the relation $\lim\limits_{k\to\infty}\sup\limits_{1\le j\le n_k}\sigma_{k,j}^2=0$.

First we show that the Lindeberg condition follows from the above conditions. As we assumed the validity of the uniform smallness condition we may apply Theorem 1. This result together with the statement about the unique representation of infinitely divisible distributions formulated in Theorem 2$'$ imply that if the central limit theorem holds, then the canonical measures M_k defined by means of formula (1.1) from the distribution functions $F_{k,j}$ of the random variables $\xi_{k,j}$ satisfy formulas (1.3) and (1.6) with the constants B_k defined in formula (1.5) and the canonical measure M such that $M(\{0\})=1$, $M(\mathbf{R}^1\setminus\{0\})=0$. As $B_k\ge0$, hence by formula (1.6) for all numbers $\varepsilon>0$ $\lim\limits_{k\to0}M_k([-\varepsilon,\varepsilon])-B_k=1$ with some number $B_k\ge0$. This means that for all numbers $\varepsilon>0$ $\liminf\limits_{j\to\infty}\sum\limits_{j=1}^{n_k}E\xi_{k,j}^2I(|\xi_{k,j}|\le\varepsilon)\ge1$. On the other hand, $\lim\limits_{j\to\infty}\sum\limits_{j=1}^{n_k}E\xi_{k,j}^2=1$. Hence $\lim\limits_{j\to\infty}\sum\limits_{j=1}^{n_k}E\xi_{k,j}^2I(|\xi_{k,j}|>\varepsilon)=0$, and this is the Lindeberg condition.

Conversely, we show that under the Lindeberg condition the central limit theorem holds. In this case $\sup\limits_{1\le j\le n_k}P(|\xi_{k,j}|\ge\varepsilon)\le\dfrac{1}{\varepsilon^2}\sum\limits_{j=1}^{n_k}E\xi_{k,j}^2I(|\xi_{k,j}|>\varepsilon)\to0$ if $k\to\infty$. Hence the uniform smallness condition holds, and we can apply Theorem 1. We have to show that the canonical measures M_k constructed with the help of the distribution functions $F_{k,j}$ of the random variables $\xi_{k,j}$, $k=1,2,\dots$, $1\le j\le n_k$, satisfy the conditions (1.3) and (1.6) with the measure M defined by the relation $M\{0\})=1$, $M(\mathbf{R}^1\setminus\{0\})=0$, and $\lim\limits_{k\to\infty}b_k=0$. The Lindeberg condition implies formula (1.3) with the above canonical measure M, because for all numbers $a>0$ $M_k^+(a)=\sum\limits_{j=1}^{n_k}(1-F_{k,j}(a))\le\frac{1}{a^2}\sum\limits_{j=1}^{n_k}E\xi_{k,j}^2I(|\xi_{k,j}|>a)\to0$ if $k\to\infty$, and similarly $\lim\limits_{k\to\infty}M_k^-(a)=0$.

By the Lindeberg condition and the relation $\lim\limits_{k\to\infty}\sum\limits_{j=1}^{n_k}E\xi_{k,j}^2=1$ $\lim\limits_{k\to\infty}M_k([-s,s])=1$ for all $\varepsilon>0$. Hence to prove formula (1.6) it is enough to show that $\lim\limits_{k\to\infty}B_k=0$. This relation holds since by the condition $E\xi_{k,j}=0$

$$B_k=\sum_{j=1}^{n_k}(E(\tau(\xi_{k,j})-\xi_{k,j}))^2\le\sum_{j=1}^{n_k}(E|\xi_{k,j}|I(|\xi_{k,j}|>a))^2\le\sum_{j=1}^{n_k}E\xi_{k,j}^2P^2(|\xi_{k,j}|>a),$$

and this implies that $\lim\limits_{k\to\infty}B_k=0$, since $\lim\limits_{k\to\infty}\sum\limits_{j=1}^{n_k}E\xi_{k,j}^2=1$, and $\lim\limits_{k\to\infty}\sup\limits_{1\le j\le n_k}P(|\xi_{k,j}>a)=0$. Similarly,

$$|b_k|=\left|\sum_{j=1}^{n_k}E\left(\tau(\xi_{k,j})-\xi_{k,j}\right)\right|\le\sum_{j=1}^{n_k}E|\xi_{k,j}|I(|\xi_{k,j}|>a)\le\frac{1}{a}\sum_{j=1}^{n_k}E\xi_{k,j}^2I(|\xi_{k,j}|>a),$$

hence $\lim\limits_{k\to\infty}b_k=0$ by the Lindeberg condition. This means that by Theorem 1 the sums $S_k=\sum\limits_{j=1}^{n_k}\xi_{k,j}$ converge in distribution to a distribution function whose characteristic function has the logarithm of the form $\log\varphi(t)=-\frac{t^2}{2}$. Hence the central limit theorem holds under the above conditions.

C.) The weak law of large numbers.

Let $\xi_1,\xi_2,\dots,$ be a sequence of independent, identically distributed random variables, and consider the partial sums $S_n=\sum\limits_{k=1}^{n}\xi_k$, $n=1,2,\dots,$ made from these random variables. A classical result of the probability deals with the problem when these partial sums satisfy the weak law of large numbers, i.e. under what conditions imposed for the distribution function $F(x)$ of the random variables ξ_k does the relation $\frac{S_n}{n}\Rightarrow a$ hold with some real number a as $n\to\infty$, where $\Rightarrow$ denotes stochastic convergence. As stochastic convergence of random variables to a number is equivalent to the convergence of these random variables in distribution to the probability measure concentrated in the point corresponding to this number the results formulated in the first Section help us to answer this question. The following result can be proved.

The weak law of large numbers. *Let $\xi_1,\xi_2,\dots,$ be a sequence of independent and identically distributed random variables with distribution function $F(x)$. The averages*

$\dfrac{S_n}{n} = \dfrac{1}{n}\sum\limits_{k=1}^{n} \xi_k$ *converge stochastically to a real number a if the following two conditions are satisfied:*

i.) $\lim\limits_{x\to\infty} x[1-F(x)] = 0$, *and* $\lim\limits_{x\to\infty} xF(-x) = 0$

ii.) $\lim\limits_{x\to\infty} \displaystyle\int_{-x}^{x} uF(\,du) = a.$

Proof of the weak law of large numbers. Let us consider the triangular array $\xi_{k,j} = \dfrac{\xi_k}{k}$, $k = 1, 2, \dots,\ j = 1, \dots, k$. It satisfies the condition of uniform smallness. The weak law of large numbers means that the sums of the random variables from a row of this triangular array converge to the probability measure concentrated in the point a. By Theorem 1 this relation holds if and only if the measures $M_k(\,dx) = kx^2F(k\,dx)$, the functions $M_k^+(x) = k(1-F(kx))$ an $M_k^-(x) = kF(-kx)$ together with the numbers

$$b_k = k\left(\int_{-a}^{a} uF(k\,du) + a(1-F(ak)-F(-ak))\right)$$
$$= \int_{-ka}^{ka} uF(\,du) + ak\,(1-F(ak)-F(-ak))\,,$$

and $B_k = \dfrac{b_k^2}{k}$ satisfy the conditions (1.3) and (1.6) with a limit measure M, $M(R^1) = 0$, and $\lim\limits_{k\to\infty} b_k = a$.

Some calculation shows that condition (1.3) in this case is equivalent to Condition i.). If Condition i.) is satisfied, then Condition ii.) is equivalent to the relation $\lim\limits_{k\to\infty} b_k = a$. Finally, under conditions i.) and ii.) also the relation (1.6) holds, since in this case $\lim\limits_{k\to\infty} B_k = 0$, and $\lim\limits_{k\to\infty} M_k([-s,s]) = 0$. Indeed, partial integration yields that $M_k([-s,s]) = \displaystyle\int_{-ks}^{ks} \frac{u^2}{k} F(\,du) = s^2k\,[(1-F(ks)+F(-ks)] - \int_0^{ks} \frac{1-F(u)+F(-u)]}{k}\,du$, and this implies that

$$M_k([-s,s]) \le s^2k\,[(1-F(ks)+F(-ks)] \to 0 \quad \text{if} \quad k\to\infty.$$

D.) A LIMIT THEOREM WITH POISSONIAN LIMIT DISTRIBUTION.

In Part I we have formulated and in its Appendix we have also proved a limit theorem where the limit distribution was Poissonian. Now we show that this result is a simple consequence of Theorem 1 formulated in Section 1. The result is the following statement.

Limit theorem with Poissonian limit distribution. *Let $\xi_{k,j}$, $k = 1, 2, \dots,$ $1 \le j \le n_k$ be a triangular array which satisfies the following conditions:*

1.) The random variables $\xi_{k,j}$ take non-negative integer values.

2.) $P(\xi_{k,j}=1)=\lambda_{k,j}$, $\lim\limits_{k\to\infty}\sum\limits_{j=1}^{n_k}\lambda_{k,j}=\lambda>0$.

3.) $\sup\limits_{1\le j\le n_k}\lambda_{k,j}\to 0$ *if* $k\to\infty$, *and* $\sum\limits_{j=1}^{n_k}P(\xi_{k,j}\ge 2)\to 0$ *if* $k\to\infty$.

Then the random sums $S_k=\sum\limits_{j=1}^{n_k}\xi_{k,j}$ *converge in distribution to a Poisson distribution with parameter* λ *if* $k\to\infty$.

Proof of the limit theorem with Poissonian distribution function. The triangular array $\xi_{k,j}$ satisfies the uniform smallness condition, hence Theorem 1 can be applied for instance with the choice $a=\frac12$ in the definition of the function $\tau_a(\cdot)$. Then the conditions of Theorem 1 hold with the limit canonical measure M of the form $M(\{1\})=\lambda$, $M(\mathbf{R}^1\setminus\{1\})=0$ and $\lim\limits_{k\to\infty}b_k=\frac12$. Hence the random sums S_k-b_k converge to a limit distribution whose characteristic function has a logarithm of the form $\log\varphi(t)=\lambda\left(e^{it}-1-i\frac{t}{2}\right)$. This implies that the random sums S_k converge to the Poisson distribution with parameter λ.

Let us mention still another important and interesting application of the results in Section 1. Let $\xi_1,\xi_2,\dots$ be a sequence of independent and identically distributed random variables with some distribution function $F(x)$, put $S_n=\sum\limits_{k=1}^{n_k}\xi_k$, $n=1,2,\dots$, and consider the normalized sums $\dfrac{S_n-B_n}{A_n}$ with some appropriate norming constants A_n and B_n. The following problems arise in a natural way. For which distribution functions F can the norming constants A_n and B_n be chosen in such a way that the above normalized sums converge in distribution? How should we choose these norming constants? What kind of limit distributions can appear? These questions can be answered completely, and the answers lead to the notion of stable distributions. The solution of these problems is also based on the results formulated in the first Section. The reason we do not go into the details is that a complete solution also requires some knowledge about the so-called slowly varying functions, a subject we do not discuss here.

3. The proof of the results.

To prove the results formulated in Section 1 first we prove two technical lemmas. In the first lemma we reformulate the condition of uniform smallness in the language of characteristic functions. The second lemma makes possible to reduce the proof of Theorem 1 to the special case when the relation $E\tau(\xi_{k,j})=0$ holds for all sufficiently large indices k and all numbers $1\le j\le n_k$. After this we turn to the proof of the results.

A.) The proof of two useful lemmas.

Lemma 1. *Let* $\xi_{k,j}$, $k=1,2,\dots$, $1\le j\le n_k$, *be a triangular array. Let* $F_{k,j}$ *denote the distribution and* $\varphi_{k,j}(t)=Ee^{it\xi_{k,j}}$ *the characteristic function of the random* *variable $\xi_{k,j}$, $k=1,2,\dots$, $1\le j\le n_k$. The above triangular array satisfies the uniform smallness condition if and only if for all numbers $K>0$*

$$\sup_{1\le j\le n_k}\sup_{|t|<K}|1-\varphi_{k,j}(t)|<\varepsilon\quad \text{for } k\ge k_0(\varepsilon,K).$$

The proof of Lemma 1. a.) Let us assume that the uniform smallness condition is satisfied. Then with the choice $\varepsilon'=\frac{\varepsilon}{2K}$

$$\begin{aligned}|1-\varphi_{k,j}(t)|&\le\int|1-e^{itx}|F_{k,j}(\,dx)=\int_{|x|\le\varepsilon'}+\int_{|x|>\varepsilon'}\\ &\le\int_{-\varepsilon'}^{\varepsilon'}|tx|F_{k,j}(\,dx)+2P(|\xi_{k,j}|>\varepsilon')\le K\varepsilon'+2P(|\xi_{k,j}|>\varepsilon')\le\varepsilon,\end{aligned}$$

for $|t|\le K$ if $k\ge k_0(\varepsilon,K)$.

b.) If the condition imposed for the characteristic functions $\varphi_{k,j}$ holds then for all numbers $\varepsilon'>0$ and $K>0$

$$\begin{aligned}\varepsilon'&\ge\frac{1}{2K}\int_{-K}^{K}\text{Re}\,[1-\varphi_{k,j}(t)]\,dt=\frac{1}{2K}\int_{-K}^{K}\int_{-\infty}^{\infty}(1-\cos tx)F_{k,j}(\,dx)\,dt\\ &=\frac{1}{2K}\int_{-\infty}^{\infty}2K\left[1-\frac{\sin Kx}{Kx}\right]F_{k,j}(\,dx)\ge\int_{\{|x|>\frac{2}{K}\}}\ge\frac12P\left(|\xi_{k,j}|>\frac{2}{K}\right)\end{aligned}$$

in the case $k\ge k_0(\varepsilon',K)$. This relation with the choice $\varepsilon'=\frac{\varepsilon}{2}$ and $K=\frac{2}{\varepsilon}$ implies the uniform smallness condition of the triangular array.

Lemma 2. *Let $\xi_{k,j}$, $k=1,2,\dots$, $1\le j\le n_k$, be a triangular array satisfying the uniform smallness condition and let us fix a number $a>0$ which appears in the definition of the function $\tau(\cdot)$ given in formula (1.4). Then there exist such numbers $\vartheta_{k,j}=\vartheta_{k,j}(a)$, $k=1,2,\dots$, $1\le j\le n_k$, for which $\lim\limits_{k\to\infty}\sup\limits_{1\le j\le n_k}|\vartheta_{k,j}|=0$, and the triangular array $\xi'_{k,j}=\xi_{k,j}-\vartheta_{k,j}$ satisfies the condition $E\tau(\xi'_{k,j})=E\tau_a(\xi'_{k,j})=0$ for all indices $k\ge k_0=k_0(a)$, $1\le j\le n_k$, with an appropriate threshold index $k_0(a)$, and the function $\tau(\cdot)=\tau_a(\cdot)$ defined in formula (1.4). Let $F'_{k,j}(x)=F_{k,j}(x+\vartheta_{k,j})$ denote the distribution function of the random variable $\xi'_{k,j}$, and let us define the canonical measures M'_k and functions ${M'}_k^{\pm}(x)$ similarly to the measures M_k and functions $M_k^{\pm}$ by means of formulas (1.1) and (1.2) with the difference that we replace the distribution functions $F_{k,j}$ with the distribution functions $F'_{k,j}$ in these formulas.*

The measures M_k and functions $M_k^{\pm}$ satisfy the relations (1.3) and (1.6) if and only if the measures M'_k and ${M'}_k^{\pm}$ satisfy them (with the same canonical measure M, but with the difference that $B_k=0$ has to be written in formula (1.6) if the measure M_k is replaced by the measure M'_k.) If these relations hold, then the numbers $b_k=\sum\limits_{j=1}^{n_k}\beta_{k,j}$ and $b'_k=\sum\limits_{j=1}^{n_k}\vartheta_{k,j}$ satisfy the relation $\lim\limits_{k\to\infty}(b_k-b'_k)=0$.

Lemma 2 states that the random sums $S'_k = \sum_{j=1}^{n_k} \xi'_j$ defined with the help of the random variables $\xi'_{k,j}$ introduced in Lemma 2 and the normalized sums $S_k - b_k$ with $S_k = \sum_{j=1}^{n_k} \xi_{k,j}$ considered in Theorem 1 converge simultaneously in distribution. Furthermore, the limit distributions of these expressions agree. Let us also remark that the necessary and sufficient condition of the convergence for the new triangular array $\xi'_{k,j}$ formulated in Theorem 1 means that the canonical measures M'_k weakly converge to the canonical measure M. Beside this, because of the property $\lim\limits_{k\to\infty} \sup\limits_{1\le j\le n_k} |\vartheta_{k,j}| = 0$ the uniform smallness condition for the triangular array $\xi_{k,j}$, $k = 1, 2, \dots$, $1 \le j \le n_k$, implies that this condition also holds for the triangular array $\xi'_{k,j}$, $k = 1, 2, \dots$, $1 \le j \le n_k$.

The proof of Lemma 2. Define the functions $f_{k,j}(\vartheta) = E\tau(\xi_{k,j} - \vartheta)$. The uniform smallness property of the triangular array implies that for an arbitrary small number $\varepsilon > 0$ $f_{k,j}(\varepsilon) < 0$ and $f_{k,j}(-\varepsilon) > 0$ if $k > k_0(\varepsilon)$ with some threshold $k_0(\varepsilon)$. Indeed, we may assume that $a > 2\varepsilon$. For all sufficiently large indices k the random variable $\tau(\xi_{k,j} - \varepsilon)$ is less than $-\varepsilon/2$, with probability almost one, and it is less than a with probability one. This implies that $E\tau(\xi_{k,j} - \varepsilon) < 0$ if $k \ge k_0(\varepsilon)$. The inequality $E\tau(\xi_{k,j} + \varepsilon) > 0$ can be proved similarly. Because of these inequalities and the continuity of the functions $f_{k,j}(\cdot)$ there exist numbers $\vartheta_{k,j}$ such that $E\tau(\xi_{k,j} - \vartheta_{k,j}) = 0$ if $k \ge k_0$, and $\lim\limits_{k\to\infty} \sup\limits_{1\le j\le n_k} |\vartheta_{k,j}| = 0$.

As $|\vartheta_{k,j}| < \varepsilon$ holds for all sufficiently large indices k and all numbers $1 \le j \le n_k$, the inequalities $M_k^+(x-\varepsilon) < {M'}_k^+(x) < M_k^+(x+\varepsilon)$ and $M_k^-(x-\varepsilon) < {M'}_k^-(x) < M_k^-(x+\varepsilon)$ hold if $k > k_0(\varepsilon)$. This implies that in all points x of continuity of the function $M^\pm(\cdot)$ the functions $M_k^\pm(x)$ and ${M'}_k^\pm(x)$ simultaneously converge or do not converge to the function $M^\pm(x)$. This means that the functions $M^\pm$ satisfy the relation (1.3) if and only if the function ${M'}^\pm$ satisfies it.

The identity $\tau(\xi'_{k,j}) + \vartheta_{k,j} - \tau(\xi_{k,j}) = 0$ holds on the set $\{\omega\colon |\xi_{k,j}(\omega)| \le a - \varepsilon\}$ if $k \ge k_0(\varepsilon)$. Beside this the inequality $|\tau(\xi'_{k,j}) + \vartheta_{k,j} - \tau(\xi_{k,j})| \le 2\vartheta_{k,j}$ always holds, since $\tau(\cdot)$ is a Lipschitz 1 function. Applying these results and summing up for all indices j we get that

$$
\begin{aligned}
\sum_{j=1}^{n_k} |\vartheta_{k,j} - \beta_{k,j}| &= \sum_{j=1}^{n_k} |E\tau(\xi'_{k,j}) + \vartheta_{k,j} - E\tau(\xi_{k,j})| \\
&= \sum_{j=1}^{n_k} \left|E\left(\tau(\xi'_{k,j}) + \vartheta_{k,j} - \tau(\xi_{k,j})\right) I(|\xi_{k,j}| > a - \varepsilon)\right| \\
&\le 2 \sup_{1\le j\le n_k} |\vartheta_{k,j}| (M_k^+(a-\varepsilon) + M_k^-(a-\varepsilon)),
\end{aligned}
$$

if $k \ge k_0(\varepsilon)$. Then we get, by taking the limit $k \to \infty$ we get that $\lim\limits_{k\to\infty} \sum_{j=1}^{n_k} |\vartheta_{k,j} - \beta_{k,j}| = 0$, and this is a stronger statement then the relation $b_k - b'_k \to 0$ as $k \to \infty$ formulated in Lemma 2.

To complete the proof of Lemma 2 it is enough to show that if relation (1.3) holds, then for all points of continuity $s>0$ of the functions $M^{\pm}(\cdot)$

$$\lim_{k\to\infty}(M_k([-s,s])-M'_k([-s,s])-B_k)=0$$

Indeed, this means that the measures M_k and M'_k simultaneously satisfy relation (1.6) in these points s. (Observe that in formula (1.6) the number $B'_k=0$ has to be taken from the measure M'_k. This is so, because $B'_k=0$ for large k since $\beta'_{k,j}=E\tau(\xi'_{k,j})=0$ for all indices $k\ge k_0$ and $1\le j\le n_k$.)

To prove this relation first we show that for all points of continuity s of the functions $M^{\pm}(\cdot)$

$$\lim_{k\to\infty}\sum_{j=1}^{n_k}\bigl(P(|\xi_{k,j}|>s)-P(|\xi'_{k,j}|>s)\bigr)=0, \tag{3.1}$$

and

$$\lim_{k\to\infty}\sum_{j=1}^{n_k}(E\tau(\xi_{k,j})^2-E\tau(\xi'_{k,j})^2)-B_k=0. \tag{3.2}$$

As we consider a point of continuity s of the functions $M^{\pm}$, hence we get by applying formula (1.3) to the functions $M_k^{\pm}$ and ${M'}_k^{\pm}$ that

$$\begin{aligned}\lim_{k\to\infty}&\sum_{j=1}^{n_k}(P(|\xi_{k,j}|>s)-P(|\xi'_{k,j}|>s))\\ &=\lim_{k\to\infty}\Bigl((M_k^+(s)-{M'}_k^+(s))+(M_k^-(s)-{M'}_k^-(s))\Bigr)=0,\end{aligned}$$

hence relation (3.1) holds. On the other hand, for all sufficiently large indices k and all numbers $1\le j\le n_k$ $\tau(\xi_{k,j})^2-\tau(\xi'_{k,j})^2+\vartheta_{k,j}^2-2\vartheta_{k,j}\tau(\xi_{k,j})=0$ on the set $\{\omega\colon |\xi_{k,j}(\omega)|<a-\varepsilon\}$, and this expression is always smaller than const. $|\vartheta_{k,j}|$. Hence by taking the absolute values of the appropriate expressions and summing them up in the variable j we get that

$$\begin{aligned}&\left|\sum_{j=1}^{n_k}E\left(\tau(\xi_{k,j})^2-E\tau(\xi'_{k,j})^2+\vartheta_{k,j}^2-2\vartheta_{k,j}\beta_{k,j}\right)\right|\\ &\qquad\le \text{const.}\left(2M_k^+(a-\varepsilon)+2M_k^-(a-\varepsilon)\right)\sup_{1\le j\le n_k}|\vartheta_{k,j}|,\end{aligned}$$

and exploiting this inequality we get that

$$\begin{aligned}&\left|\sum_{j=1}^{n_k}E\left(\tau(\xi_{k,j})^2-E\tau(\xi'_{k,j})^2)\right)-B_k\right|=\left|\sum_{j=1}^{n_k}E\left(\tau(\xi_{k,j})^2-E\tau(\xi'_{k,j})^2-\beta_{k,j}^2\right)\right|\\ &\qquad\le \text{const.}\left(2M_k^+(a-\varepsilon)+2M_k^-(a-\varepsilon)\right)\sup_{1\le j\le n_k}|\vartheta_{k,j}|+\sum_{j=1}^{n_k}(\vartheta_{k,j}-\beta_{k,j})^2.\end{aligned}$$

As the sequences $M_k^{\pm}(a-\varepsilon)$, $k=1,2,\dots$, are bounded, and as we have proved $\lim\limits_{k\to\infty}\sum\limits_{j=1}^{n_k}|\vartheta_{k,j}-\beta_{k,j})|=0$, the right-hand side of the above expression tends to zero as $k\to\infty$, hence relation (3.2) holds.

Let us denote by $\tau_s(\cdot)$ the version of the function $\tau(\cdot)=\tau_a(\cdot)$ defined in formula (1.4) if the parameter a in formula (1.4) is replaced by the number s. Then we can write the identity

$$
\begin{aligned}
M_k([-s,s])-M_k'([-s,s])-B_k&=\sum_{j=1}^{n_k}\left(E\tau_s(\xi_{k,j})^2-E\tau_s(\xi'_{k,j})^2\right)-B_k\\
&\qquad -s^2\sum_{j=1}^{n_k}(P(|\xi_{k,j}|>s)-P(|\xi'_{k,j}|>s))\\
=\sum_{j=1}^{n_k}\left(E\tau_s(\xi_{k,j})^2-E\tau(\xi_{k,j})^2\right)&-\sum_{j=1}^{n_k}\left(E\tau_s(\xi'_{k,j})^2-E\tau(\xi'_{k,j})^2\right)\\
+\sum_{j=1}^{n_k}E\left(\tau(\xi_{k,j})^2-E\tau(\xi'_{k,j})^2\right)&-B_k-s^2\sum_{j=1}^{n_k}(P(|\xi_{k,j}|>s)-P(|\xi'_{k,j}|>s)).
\end{aligned}
$$

To complete the proof of Theorem 2 it is enough to show that the expression at the right-hand side of this identity tends to zero as $k\to\infty$. That part of this expression which is contained in the second line tends to zero by formulas (3.1) and (3.2). We still have to understand the contribution of the terms obtained as the function τ_a was replaced by τ_s. Then the desired statement follows from the following estimations.

$$
\begin{aligned}
\sum_{j=1}^{n_k}\left(E\tau_s(\xi_{k,j})^2-E\tau(\xi_{k,j})^2)\right)&=\int\frac{\tau_s(u)^2-\tau(u)^2}{u^2}M_k(du)\\
&=\int_{\min(a,s)}^{\infty}\left(\tau_s(u)^2-\tau(u)^2\right)(M_k^+(\,du)+M_k^-(\,du))\\
&\quad\to\int_{\min(a,s)}^{\infty}\left(\tau_s(u)^2-\tau(u)^2\right)(M^+(\,du)+M^-(\,du)).
\end{aligned}
$$

The expression $\sum\limits_{j=1}^{n_k}\left(E\tau_s(\xi'_{k,j})^2-E\tau(\xi'_{k,j})^2)\right)$ has the same limit. Thus we hove shown that $M_k([-s,s])-M_k'([-s,s])-B_k\to0$ if $k\to\infty$ and completed the proof of Lemma 2.

The proof of the sufficiency part of Theorem 1, the proof of the statement that the limit distribution exists if the conditions of Theorem 1 hold is relatively simple, and it can be directly done. The proof of the necessity part of Theorem 1 is harder and to carry it out we shall need the result of Theorem 2. So we shall prove it. In the proof of Theorem 2 we also prove a result formulated in Lemma 3. This Lemma 3 will be useful also in later considerations. In the next part we prove the above results together with Theorem 2′.

B.) The proof of Theorems 2 and 2′ and of the sufficiency part of Theorem 1.

First we prove the sufficiency part of Theorem 1 i.e. the statement that if conditions (1.3) and (1.6) hold then the normalized sums $S_k - b_k$ converge in distribution to that distribution function whose characteristic function has a logarithm of the form (1.7).

The proof of the sufficiency part of Theorem 1. We have to show that the logarithms of the characteristic functions of the normalized random sums $S_k - b_k$ satisfies the relation

$$\log Ee^{it(S_k-b_k)} = \sum_{j=1}^{n_k} \log \varphi_{k,j}(t) - it\beta_{k,j} \to \log \varphi(t), \quad \text{if } k \to \infty$$

for all $t \in \mathbf{R}^1$ where the function $\log \varphi(t)$ is defined in formula (1.7). Let us observe that because of the uniform smallness condition assumed for the triangular array $\xi_{k,j}$, $k = 1, 2, \dots$, $1 \le j \le n_k$, $|1 - \varphi_{k,j}(t)| \le \varepsilon$ for all $\varepsilon > 0$ if $k \ge k_0(\varepsilon, t)$. Hence the logarithm of the function $\varphi_{k,j}(t)$ is meaningful if $k \ge k(t)$.

Let us first restrict our attention to the case when $\beta_{k,j} = E\tau(\xi_{k,j}) = 0$ for all sufficiently large k and $1 \le j \le n_k$. We shall prove that in this case

$$\limsup_{k\to\infty} \sum_{j=1}^{n_k} |1 - \varphi_{k,j}(t)| < \infty$$

$$\lim_{k\to\infty} \sum_{j=1}^{n_k} (1 - \varphi_{k,j}(t)) = -\log \varphi(t).$$

This two relations imply the limit theorem in the present case, because $|\log z + (1-z)| < 2|z|^2 < 2\varepsilon|z|$ if $|1-z| < \varepsilon$ and $\frac{1}{2} > \varepsilon > 0$, and the first relation together with the uniform smallness condition imply that

$$\lim_{k\to\infty} \sum_{j=1}^{n_k} |\log \varphi_{k,j}(t) + (1 - \varphi_{k,j}(t))| = 0.$$

We get with the help of the relation $E\tau(\xi_{k,j}) = 0$ that

$$\begin{aligned}
\sum_{j=1}^{n_k} |1 - \varphi_{k,j}(t)| &= \sum_{j=1}^{n_k} |1 - \varphi_{k,j}(t) + itE\tau(\xi_{k,j})| \le \sum_{j=1}^{n_k} \int |1 - e^{itx} + it\tau(x)| F_{k,j}(dx) \\
&\le \sum_{j=1}^{n_k} \left(\int_{-a}^{a} \frac{1}{2} t^2 x^2 F_{k,j}(dx) + \int_{\{|x|>a\}} (2 + a|t|) F_{k,j}(\,dx) \right) \\
&= \frac{1}{2} t^2 M_k\{[-a, a]\} + (2 + a|t|)(M_k^+(a) + M_k^-(a)) < \text{const.},
\end{aligned}$$

and this is the first statement we wanted to prove. The second statement can be proved similarly, because

$$\sum_{j=1}^{n_k}(1-\varphi_{k,j}(t)) = \sum_{j=1}^{n_k}(1-\varphi_{k,j}(t)+itE\tau(\xi_{k,j})) = \sum_{j=1}^{n_k}\int(1-e^{itx}+it\tau(x))F_{k,j}(dx)$$
$$= \int_{-\infty}^{\infty}\frac{1-e^{itx}+it\tau(x)}{x^2}M_k(\,dx) = \int_{-K}^{K}+\int_{|x|>K}$$

for arbitrary number $K>0$. Let us fix some number $\varepsilon>0$. If $K=K(\varepsilon)$ is sufficiently large, and the points $\pm K$ are points of continuity of the measure M, then because of relation (1.3) and the boundedness of the function $1-e^{itx}+it\tau(x)$

$$\left|\int_{\{|x|>K\}}\frac{1-e^{itx}+it\tau(x)}{x^2}M_k(\,dx)\right|<\varepsilon$$

if $k>k_0$, and $\left|\int_{\{|x|>K\}}\frac{1-e^{itx}+it\tau(x)}{x^2}M(\,dx)\right|<\varepsilon$ if we replace the measures M_k by the limit measure M. On the other hand, because of the convergence of the canonical measures M_k to the canonical measure M and the continuity of the function $\frac{1}{x^2}(1-e^{itx}+it\tau(x))$

$$\int_{-K}^{K}\frac{1-e^{itx}+it\tau(x)}{x^2}M_k(\,dx)\to\int_{-K}^{K}\frac{1-e^{itx}+it\tau(x)}{x^2}M(\,dx),\quad \text{if } k\to\infty.$$

The above results imply the sufficiency in the special case considered above. The general case when $E\tau(\xi_{k,j})\neq 0$ is also possible can be deduced from the already proven case with the help of Lemma 2.

Indeed, we can apply the already proven part of Theorem 1 for the triangular array $\xi'_{k,j}$, $k=1,2,\dots$, $1\le j\le n_k$, defined in Lemma 2. With the help of this lemma we get that the random sums $S_k-b'_k$ and S_k-b_k, where the numbers b'_k were defined also in Lemma 2 have a limit distribution if the condition of Theorem 1 holds. Beside this the logarithm of the characteristic function of the limit distribution is given by formula (1.7).

The proof of Theorem 2. First we prove the simpler sufficiency part, i.e. the statement that the weak convergence of the canonical measures M_n to the canonical measure M together with the convergence of the numbers $B_n\to B$ imply that the sequence of infinitely divisible distribution defined in Theorem 2 with the help of the canonical measures M_n and constants B_n converge to the infinitely divisible distribution determined by the canonical measure M and constant B. As the convergence of the characteristic functions of distribution functions to the characteristic function of a distribution function imply the convergence of these distribution functions to the distribution function with the limit characteristic function it is enough to show that for all real numbers t

$$\log\varphi_n(t)=\int_{-\infty}^{\infty}\frac{e^{itu}-1-it\tau(u)}{u^2}M_n(\,du)+iB_nt$$
$$\to\log\varphi(t)=\int_{-\infty}^{\infty}\frac{e^{itu}-1-it\tau(u)}{u^2}M(\,du)+iBt,$$

if $n\to\infty$.

As $M_n\to M$, and the integrand in the integrals we have considered satisfy the inequality $\left|\dfrac{e^{itu}-1-it\tau(u)}{u^2}\right|<\dfrac{\text{const.}}{1+u^2}$ for all $\varepsilon>0$ there exist such numbers $K=K(\varepsilon)>0$ for which $\pm K$ are points of continuity of the measure M,

$$\left|\int_{\{|u|>K\}}\frac{e^{itu}-1-it\tau(u)}{u^2}M_n(\,du)\right|\le\varepsilon\quad\text{if } n\ge n_0(\varepsilon),$$

and a similar inequality holds if we replace the measures M_n by the measure M. Beside this, the convergence of the canonical measures M_n to the canonical measure M and the continuity of the integrand also implies that

$$\int_{\{|u|<K\}}\frac{e^{itu}-1-it\tau(u)}{u^2}M_n(\,du)\to\int_{\{|u|<K\}}\frac{e^{itu}-1-it\tau(u)}{u^2}M(\,du)$$

if $n\to\infty$. These estimates and the relation $B_n\to B$ yield the sufficiency part of Theorem 2 with a limiting procedure $\varepsilon\to0$.

To prove the necessity part of Theorem 1 let us first observe that if (the (logarithms of) the characteristic functions of distribution functions converge to a continuous function then the limit function is (the logarithm of) the characteristic function of a distribution function which is the limit of these distribution functions. The main problem of the proof will be to decide when the functions $\log\varphi_n$ defined in the formulation of Theorem 2 converge to a continuous function and to describe the limit function.

To describe the limit of the sequence of functions $\log\varphi_n(t)$ let us define the following "smoothed version" of these functions. Fix some number $h>0$ and put

$$\psi_n(t)=\psi_n^h(t)=\log\varphi_n(t)-\frac{1}{2h}\int_{-h}^{h}\log\varphi_n(t+s)\,ds.$$

Then

$$\begin{aligned}\psi_n(t)&=\int_{-\infty}^{\infty}\left(\frac{e^{itu}-1-it\tau(u)}{u^2}-\frac{1}{2h}\int_{t-h}^{t+h}\frac{e^{isu}-1-is\tau(u)}{u^2}\,ds\right)M_n(\,du)\\&=\int_{-\infty}^{\infty}\left(\frac{e^{itu}}{u^2}-\frac{1}{2h}\int_{t-h}^{t+h}\frac{e^{isu}}{u^2}\,ds\right)M_n(\,du)=\int_{-\infty}^{\infty}e^{itu}K(u)M_n(\,du),\end{aligned}\tag{3.3}$$

where

$$K(u) = K_h(u) = \frac{1}{u^2}\left(1 - \frac{\sin hu}{hu}\right). \tag{3.3'}$$

We shall prove with the help of Lemma 3 formulated below that if the infinitely divisible distribution functions considered in Theorem 2 have a limit, then there exists a continuous function $\bar\varphi(t)$ such that

$$\lim_{n\to\infty} \psi_n(t) \to \psi(t) = \bar\varphi(t) - \frac{1}{2h}\int_{-h}^{h} \bar\varphi(t+u)\, du. \tag{3.4}$$

Indeed, by Lemma 3 if the functions $\log\varphi_n(t)$ are characteristic functions of convergent distribution functions, then they converge uniformly in all finite intervals to a continuous function. Then we get formula (3.4) by taking a limit $n\to\infty$ in the formula defining the function $\psi_n(\cdot)$ because of Lebesgue's dominated convergence theorem and Lemma 3.

We have formulated Lemma 3 in such a way that we could also apply it in the proof of Theorem 1. Before its formulation we make the following remark.

Let $\log\varphi_n(t)$, $n=1,2,\dots$, be a sequence of functions defined in formula (1.9). By the results of Part I we know that they are the logarithms of the characteristic functions of infinitely divisible distribution functions. If these distribution functions converge in distribution then the functions $\log\varphi_n(t)$ converge uniformly to a continuous function $\bar\varphi(t)$ in a small neighbourhood of the origin. But we can state this uniform convergence — at least before a deeper investigation — only in a small neighbourhood of the origin. The above remark implies in particular that if the logarithms of the characteristic functions defined in formula (1.9) belong to convergent distribution functions, then these distribution functions satisfy the conditions of Lemma 3 formulated below.

Lemma 3. *Let* $\log\varphi_n(t)$ *be a sequence of functions defined by formula (1.9) with some canonical measures* M_n *and real numbers* B_n*. Let us assume that these functions* $\varphi_n(t)$ *converge uniformly to a continuous function in a small neighbourhood of the origin. (But we do not demand that the distribution functions related to the functions* $\log\varphi_n(t)$ *should converge in distribution.) Then the functions* M_n *taking part in the definition of the functions* $\log\varphi_n(t)$ *satisfy the inequality*

$$\sup_{1\le n<\infty}\int_{-\infty}^{\infty}\frac{1}{1+u^2}M_n(\,du) < \infty. \tag{3.5}$$

In this case also the inequality

$$\sup_{1\le n<\infty}\ \sup_{|t|<K} |\log\varphi_n(t)| < \infty \tag{3.6}$$

holds for all $K>0$.

If the functions $\log\varphi_n(t)$ *are the logarithms of the characteristic function of a* convergent *sequence of distribution functions, then not only the characteristic functions of* *these distribution functions but also their logarithms, the functions* $\log\varphi_n(t)$*, converge uniformly to a continuous function in all finite intervals.*

The proof of Lemma 3. As the functions $\log\varphi_n(t)$ are the logarithms of characteristic functions of distribution functions, $\varphi_n(0)=1$, and under the conditions of Lemma 3 there exists an appropriate $h>0$ such that the limit $\lim\limits_{n\to\infty}\log\varphi_n(t)=\bar\varphi(t)$ exists in the interval $|t|\le h$, the limit is a continuous function, and the convergence is uniform in this interval. For the sake of simpler notations we fix such a number h and we work with this number h in the proof of Lemma 3 and Theorem 2 (thus for instance in the definition of the already introduced function $\psi_n(\cdot)=\psi_n^h(\cdot)$).

The uniform convergence of the functions $\log\varphi_n(t)$ in the interval $[-h,h]$ together with their representation imply that there exists an appropriate number $K>0$ such that

$$\begin{aligned}\infty>K\ge 2\sup_{n<\infty}\sup_{|t|\le h}|\log\varphi_n(t)|&\ge\sup_{n<\infty}\left|\log\varphi_n(0)-\frac{1}{2h}\int_{-h}^{h}\log\varphi_n(t)\,dt\right|\\ =\sup_{n<\infty}\left|\int_{-\infty}^{\infty}\frac{1}{u^2}\left(1-\frac{\sin hu}{hu}\right)M_n(\,du)\right|&\ge C(h)\sup_{n<\infty}\left|\int_{-\infty}^{\infty}\frac{1}{1+u^2}M_n(\,du)\right|,\end{aligned}$$

because

$$\frac{1}{u^2}\left(1-\frac{\sin hu}{hu}\right)=h^2\frac{1}{(hu)^2}\left(1-\frac{\sin hu}{hu}\right)\ge\text{const.}\,\frac{1}{1+h^2u^2}\ge\text{const.}'\frac{1}{1+u^2}$$

with some appropriate const. and const.$'$ depending on h. Thus we have proved formula (3.5). On the other hand, as

$$\left|\frac{e^{itu}-1-i\tau(u)t}{u^2}\right|\le\begin{cases}\dfrac{t^2}{2}, & \text{if } |u|<a\\[2mm] \dfrac{2+aK}{u^2}, & \text{if } |u|\ge a,\ \text{and } |t|\le K\end{cases}$$

formula (3.5) implies that the integral in the definition of the function $\log\varphi_n(t)$ is uniformly bounded for a fixed finite interval $|t|<K$ and all numbers $n=1,2,\dots$. The sequence B_n has to be finite, since otherwise the sequence $\log\varphi_n(h)$ would be not bounded, thus it would not convergence. The above argument implies relation (3.6). Finally the uniform boundedness of the functions $\log\varphi_n(t)$ in finite intervals imply that the functions $\varphi_n(t)$ and their limit, the function $\varphi(t)$ is separated both from zero and infinity in all finite intervals. Hence if the distributions determined by the functions $\log\varphi_n(t)$ converge in distribution, then not only the characteristic functions but the logarithms of the characteristic functions of these distributions converge uniformly in all finite intervals. Lemma 3 is proved.

Let us turn back to the proof of Theorem 2. Let us introduce the measures $\mu_n(\,du)=K(u)M_n(\,du)$, $n=1,2,\dots$, with the help of the function $K(\cdot)\ge0$ defined in formula (3.3$'$). By rewriting the expression given for the function a $\psi_n(t)$ in formula (3.3) by means of formula (3.4) we get that the Fourier transforms of the measures μ_n, the functions $\psi_n(t) = \int_\infty^\infty e^{itu}\mu_n(\,du)$, converge to a continuous function $\psi(t)$. Furthermore, the inequality

$$\frac{C_1}{1+u^2} \le K(u) = \frac{1}{u^2}\left(1 - \frac{\sin hu}{hu}\right) \le \frac{C_2}{1+u^2}$$

holds with some appropriate constants $C_2 = C_2(h) > C_1 = C_1(h) > 0$. We claim that the following two possibilities can appear. Either $\lim\limits_{n\to\infty} \psi_n(0) = 0$, and in this case the canonical measures M_n weakly converge to the measure $M \equiv 0$, i.e. in this case $M(\mathbf{R}^1) = 0$ or $\lim\limits_{n\to\infty} \psi_n(0) = \psi(0) > 0$, and in this case the probability measures $\bar\mu_n = \dfrac{1}{\psi_n(0)}\mu_n$ converge weakly to a probability measure μ.

Indeed, if $\lim\limits_{n\to\infty} \psi_n(0) = 0$, then by the lower estimate given for the function $K(u)$ we get that $\lim\limits_{n\to\infty} \displaystyle\int \frac{1}{1+u^2} M_n(\,du) = 0$, i.e. the measures M_n converge weakly to the zero measure. If $\lim \psi_n(0) > 0$ (as $\psi_n(0) \ge 0$, only the above two cases are possible), then the above defined measures $\bar\mu_n$ are probability measures and their characteristic functions converge to the continuous function $\dfrac{\psi(t)}{\psi(0)}$. Hence in this case the measures $\bar\mu_n$ converge weakly to a probability measure μ. Let us finally remark that because of the lower bound given for the function $K(u)$ the continuity of the function $\dfrac{K(u)}{1+u^2}$ and the relation $\lim\limits_{n\to\infty} \bar\mu_n = \mu$ the canonical measure $M_n(du) = K^{-1}(u)\mu_n(du)$ converge weakly to the canonical measure $M(du) = K^{-1}(u)\psi(0)\mu(du)$.

Let us finally remark that, as we have already seen in the first part of the proof, the convergence of the canonical measures M_n to the canonical measure M implies that the integral parts of the formulas expressing the functions $\log\varphi_n(t)$ converge to the integral part of the formula expressing the function $\log\varphi(t)$. As the weak convergence of the distributions considered in Theorem 2 implies that the functions $\log\varphi_n(t)$ converge to the function $\log\varphi(t)$, hence the constants B_n in these formulas should converge to the constant B. Theorem 2 is proved.

Theorem $2'$ can be proved similarly to the necessity part of Theorem 2.

The proof of Theorem $2'$. Let us define, similarly to the argument in Theorem 2, the function

$$\psi(t) = \psi^h(t) = \log\varphi(t) - \frac{1}{2h}\int_{-h}^{h} \log\varphi(t+u)\,du.$$

Then

$$\psi(t) = \int_{-\infty}^{\infty} e^{itu} K(u) M(du)$$

with $K(u) = \dfrac{1}{u^2}\left(1 - \dfrac{\sin hu}{hu}\right)$. The function $\varphi(t)$ determines also the function $\psi(t)$. If $\psi(0) = 0$, then the measure M is identically zero. If $\psi(0) > 0$, then $\bar\mu(\,du) =$

$\dfrac{K(u)M(\,du)}{\psi(0)}$ is the uniquely determined probability measure whose characteristic function is $\dfrac{\psi(t)}{\psi(0)}$. Then the formula $M(\,du)=\dfrac{\psi(0)}{K(u)}\bar\mu(\,du)$ also determines the measure M. Finally, the function $\varphi(t)$ and the measure M also determine the constant B in the formula expressing the function $\varphi(t)$.

C.) THE PROOF OF THE NECESSITY PART OF THEOREM 1.

Let us first briefly explain the idea of the proof. Lemma 2 enables us to reduce the problem to the case when the summands $\xi_{k,j}$, $k\ge k_0$, $1\le j\le n_k$, satisfy the identity $E\tau(\xi_{k,j})=0$ with some threshold index k_0. If the sums of the random variables $\xi_{k,j}$ converge in distribution, then their characteristic functions, the products $\prod\limits_{j=1}^{n_k}\varphi_{k,j}(t)$, converge to a characteristic function $\psi(t)$. It is natural to take logarithm in this relation. Then, since $\varphi_{k,j}(t)\sim 1$ because of the uniform smallness condition one expects that the approximation $\log\varphi_{k,j}(t)\sim\varphi_{k,j}(t)-1$ causes a small error. These considerations suggest that $\lim\limits_{k\to\infty}\sum\limits_{j=1}^{n_k}(\varphi_{k,j}(t)-1-itE\tau(\xi_{k,j}))=\lim\limits_{k\to\infty}\sum\limits_{j=1}^{n_k}(\varphi_{k,j}(t)-1)=\log\psi(t)$. Then if we write $\varphi_{k,j}(t)-1-itE\tau(\xi_{k,j})=\displaystyle\int\frac{e^{itx}-1-it\tau(x)}{x^2}x^2F_{k,j}(\,dx)$, sum up these identities for the argument j, then the last relation together with the necessity part of Theorem 2 enable us to prove the desired result.

Nevertheless, all steps of the above argument demand a more detailed justification. This is done in the proof below where we first consider a simpler special case. Then we reduce the general case to it by means of a technique called the symmetrization in the literature.

Proof of the necessity part of Theorem 1. Let us first consider the special case when all random variables have symmetric distribution, i.e. when the distribution functions of the random variables $\xi_{k,j}$ and $-\xi_{k,j}$ agree, and the sequence of random sums S_k, $k=1,2,\dots$, has a limit distribution.

If the sequence of random sums S_k converges in distribution, then the characteristic functions $\varphi_{k,j}(t)$ of the random variables $\xi_{k,j}$ satisfy the relation

$$\lim_{k\to\infty}\psi_k(t)=\lim_{k\to\infty}\prod_{j=1}^{n_k}\varphi_{k,j}(t)=\psi(t) \tag{3.7}$$

with a continuous function $\psi(t)$ which is the characteristic function of the limit distribution. This relation implies that

$$\lim_{k\to\infty}\log\psi_k(t)=\lim_{k\to\infty}\sum_{j=1}^{n_k}\log\varphi_{k,j}(t)=\log\psi(t). \tag{3.8}$$

in an appropriate interval $|t|\le h$. For the time being we cannot prove this statement for all $t\in\mathbf{R}^1$, since we do not know that the function $\psi(\cdot)$ in no points takes the value zero.

Because of the symmetry of the distribution functions of the random variables $\xi_{k,j}$ $\varphi_{k,j}(t) = E\cos(t\xi_{k,j})$ is a real number, and $-1 \le \varphi_{k,j}(t) \le 1$ for all numbers $t \in \mathbf{R}^1$. Hence $1-\varphi_{k,j}(t) = |1-\varphi_{k,j}(t)|$. By Lemma 1 the uniform smallness condition imposed on the triangular array $\xi_{k,j}$, $k=1,2,\dots$, $1 \le j \le n_k$, implies that for all numbers $K>0$ we have $\lim\limits_{k\to\infty} \sup\limits_{|t|<K} \sup\limits_{1\le j\le n_k} |1-\varphi_{k,j}(t)| = 0$. Hence for all numbers $\varepsilon > 0$ $|\log\varphi_{k,j}(t) + (1-\varphi_{k,j}(t))| \le \varepsilon |1-\varphi_{k,j}(t)|$ if $k \ge k_0(\varepsilon)$, and we can write

$$\lim_{k\to\infty} \log\bar\psi_k(t) = \lim_{k\to\infty} \sum_{j=1}^{n_k} (\varphi_{k,j}(t)-1) = \log\psi(t), \tag{3.8$'$}$$

instead of the relation (3.8) with the function $\bar\psi_k(t) = \prod\limits_{j=1}^{n_k} e^{\varphi_{k,j}(t)-1}$ if $|t| \le h$. Furthermore,

$$\begin{aligned} \log\bar\psi_k(t) = \sum_{j=1}^{n_k}(\varphi_{k,j}(t)-1) &= \sum_{j=1}^{n_k} \int (e^{itx}-1-it\tau(x))F_{k,j}(\,dx) \\ &= \int \frac{e^{itx}-1-it\tau(x)}{x^2} M_k(\,dx), \end{aligned} \tag{3.9}$$

because of the symmetric distribution of the random variables $\xi_{k,j}$, where the function $\tau(x)$ and the measure M_k are the quantities defined in Theorem 1. This relation and formula (3.8$'$) imply that Lemma 3 can be applied for the functions $\bar\psi_k(t)$. Hence those versions of formulas (3.5) and (3.6) hold where the measures M_n are replaced by the measures M_k and $\log\bar\psi_k(t)$ is written instead of $\log\varphi_n(t)$. This version of formula (3.6) implies that

$$\sup_{1\le k<\infty} \sup_{|t|\le K} \sum_{j=1}^{n_k} (1-\varphi_{k,j}(t)) = \sup_{1\le k<\infty} \sup_{|t|\le K} \sum_{j=1}^{n_k} |1-\varphi_{k,j}(t)| < \infty,$$

for all numbers $K>0$, and $0 \le \sup\limits_{1\le k<\infty} \sup\limits_{|t|\le K} -\sum\limits_{j=1}^{n_k} \log\varphi_{k,j}(t) < \infty$ because $|\log\varphi_{k,j}(t) + (1-\varphi_{k,j}(t))| < (1-\varphi_{k,j}(t))$ if $|t| \le K$ and $k \ge k_0(K)$. Hence we can take logarithm in formula (3.7) for all numbers $t \in \mathbf{R}^1$, and relations (3.8) and (3.8$'$) hold for all $t \in \mathbf{F}^1$. This means that the functions $\bar\psi_k(t)$ are the characteristic functions of such (infinitely divisible) distributions which converge in distribution. Hence Theorem 2 can be applied for these functions, and it yields together with formula (3.9) that relations (1.3) and (1.6) hold (the latter one with $B_k = 0$ for all indices k) in the above considered case.

In the next step we prove the necessity part of Theorem 1 in the case when $E\tau(\xi_{k,j}) = 0$ for all $k \ge k_0$ and $1 \le j \le n_k$ with an appropriate threshold index k_0, and the normalized sums of the random variables from fixed rows of the triangular array we consider, the random variables $S_k - \bar b_k$, converge in distribution with an appropriate norming sequence $\bar b_k$. We prove that in this case the canonical measures M_k constructed by means of the triangular array $\xi_{k,j}$, $k=1,2,\dots$, $1 \le j \le n_k$, satisfy relations (1.3) and (1.6).

In the proof of this statement we apply symmetrization of the random variables we are working with, a technique useful in several investigations of probability theory. That is, we consider a new triangular array $\bar\xi_{k,j}$, $k=1,2,\dots$, $1\le j\le n_k$, independent of the original triangular array $\xi_{k,j}$, $k=1,2,\dots$, $1\le j\le n_k$, and such that the random variables $\xi_{k,j}$ and $\bar\xi_{k,j}$ have the same distribution. Then we define the random sums $\bar S_k=\sum\limits_{j=1}^{n_k}\bar\xi_{k,j}$ similarly to the random sums S_k and consider the differences $S_k-\bar S_k$. The convergence of the random sums $S_k-\bar b_k$ in distribution implies the same convergence for the expressions $S_k-\bar S_k$, and the latter random variables can be obtained as the sums of the random variables of the triangular array $\eta_{k,j}=\xi_{k,j}-\bar\xi_{k,j}$, $k=1,2,\dots$, $1\le j\le n_k$, in fixed rows. Observe that the random variables $\eta_{k,j}$ are symmetrically distributed, hence we have already proved the necessity part of Theorem 1 for the triangular array consisting of these random variables.

We shall prove with the help of the above symmetrization that

$$\sup_{1\le k<\infty} M_k^\pm(s)<\infty \text{ for all numbers } s>0,\qquad \lim_{K\to\infty}\sup_{1\le k<\infty} M_k^\pm(K)=0,$$

and

$$\sup_{1\le k<\infty} M_k([-a,a])<\infty.$$

Indeed, let us define the measures M_k^0 and functions ${M^0}_k^\pm$ similarly to the measures M_k and functions $M_k^\pm$ introduced to the formulation of Theorem 1 with the difference that now we replace the distribution functions $F_{k,j}$ of the random variables $\xi_{k,j}$ by the distribution function $\bar F_{k,j}=F_{k,j}*F_{k,j}^-$ of the random variables $\eta_{k,j}=\xi_{k,j}-\bar\xi_{k,j}$ in the definition of these quantities, where $*$ denotes convolution, and $F^-_{k,j}(x)=1-F_{k,j}(-x)$ is the distribution function of the random variable $-\xi_{k,j}$. Then relations (1.3) and (1.6) hold (with constant $B_k=0$) with an appropriate canonical measure M^0 if we replace the quantities $M_k^\pm$ and M_k by the quantities ${M^0}_k^\pm$ and M^0. Beside this, for all $\varepsilon>0$ there exists a threshold index $k_0=k_0(\varepsilon)$ such that $1-\bar F_{k,j}(x-\varepsilon)=P(\xi_{k,j}-\bar\xi_{k,j}>x-\varepsilon)>P(\xi_{k,j}>x)P(\bar\xi_{k,j}>-\varepsilon)>(1-\varepsilon)(1-F_{k,j}(x))$ for $x>2\varepsilon$ and $k>k_0(\varepsilon)$ because of the uniform smallness condition. A similar inequality holds for the quantity $\bar F_{k,j}(-x)$. Summing up these estimates for all $j=1,\dots,n_k$ we obtain that $M_k^\pm(x)\le\dfrac{1}{1-\varepsilon}{M^0}_k^\pm(x-\varepsilon)$ if $x\ge2\varepsilon$ and $k\ge k_0(\varepsilon)$. As $\sup\limits_{k<\infty}{M^0}_k^\pm(x)<\infty$ for all numbers $x>0$, and $\lim\limits_{K\to\infty}\sup\limits_{k<\infty}{M^0}_k^\pm(K)=0$, the above inequalities imply the validity of the relations formulated to the functions $M_k^\pm$.

We prove an inequality useful for our purposes to estimate the quantity $M_k([-a,a])$. In its proof we exploit that $E\tau(\xi_{k,j})=E\tau(\bar\xi_{k,j})=0$, and the random variables $\xi_{k,j}$, and $\bar\xi_{k,j}$ are independent. Beside this, the functions

$$v(x)=v_a(x)=\begin{cases} a & \text{ha } x>a\\ 0 & \text{ha } -a\le x\le a\\ -a & \text{ha } x<-a\end{cases}$$

satisfy the following relations: $\tau(x)-v(x)=x$ if $|x|\le a$, and $\tau(x)-v(x)=0$ if $|x|>a$. Furthermore, $\tau(x)v(x)=v^2(x)$. Hence

$$\begin{aligned}\int_{-2a}^{2a} x^2\bar F_{k,j}(dx)&=\int\int_{\{(x,y)\colon\, |x+y|\le 2a\}}(x+y)^2F_{k,j}(\,dx)F^-_{k,j}(\,dy)\\&\ge\int\int_{\{(x,y)\colon\, |x|\le a,\,|y|\le a\}}(x+y)^2F_{k,j}(\,dx)F^-_{k,j}(\,dy)\\&=\int\int_{\{(x,y)\colon\, |x|\le a,\,|y|\le a\}}(x-y)^2F_{k,j}(\,dx)F_{k,j}(\,dy)\\&=E\left(\tau(\xi_{k,j})-v(\xi_{k,j})-(\tau(\bar\xi_{k,j})-v(\bar\xi_{k,j}))\right)^2\\&=2E\tau(\xi_{k,j})^2+2Ev(\xi_{k,j})^2-2(Ev(\xi_{k,j}))^2-4E\tau(\xi_{k,j})v(\xi_{k,j})\\&\ge 2\int_{-a}^{a}x^2F_{k,j}(dx)-4a^2(1-F_{k,j}(a)+F_{k,j}(-a)),\end{aligned}$$

since

$$(Ev(\xi_{k,j}))^2=a^2(1-F_{k,j}(a)+F_{k,j}(-a))^2\le a^2(1-F_{k,j}(a)+F_{k,j}(-a)),$$

and

$$Ev(\xi_{k,j})^2=E\tau(\xi_{k,j})v(\xi_{k,j})=a^2(1-F_{k,j}(a)+F_{k,j}(-a)).$$

By summing up these inequalities for $j=1,\dots,n_k$ we get that

$$M_k^0([-2a,2a])\ge 2M_k([a,a])-4a^2(M_k^+(a)+M_k^-(a)).$$

We know that $\sup\limits_{1\le k<\infty}M_k^0([-2a,2a])<\infty$, (the measures M_k^0 satisfy relation (1.6) with the choice $B_k=0$), and have also seen that $\sup\limits_{1\le k<\infty}M_k^\pm(a)<\infty$, These relations imply that the inequality $\sup\limits_{1\le k<\infty}M_k([-a,a])<\infty$ holds.

With the help of the estimates obtained for the quantities M_k and $M_k^\pm$ we prove that for all numbers $T>0$

$$\sup_{1\le k<\infty}\sum_{j=1}^{n_k}|1-\varphi_{k,j}(t)|\le C(T)\quad\text{if } |t|\le T \tag{3.10}$$

with an appropriate constant $C(T)<\infty$. Indeed, for all numbers $|t|\le T$

$$\begin{aligned}|1-\varphi_{k,j}(t)|&=\left|\int_{-\infty}^{\infty}(1-e^{itx}+it\tau(x))F_{k,j}(\,dx)\right|\le\int_{-a}^{a}|1-e^{itx}+itx|F_{k,j}(\,dx)\\&\qquad+\int_{|x|>a}|1-e^{itx}+ita|F_{k,j}(\,dx)\\&\le\frac{t^2}{2}\int_{-a}^{a}x^2F_{k,j}(\,dx)+(2+|t|a)\left(F_{k,j}(-a)+[1-F_{k,j}(a)]\right).\end{aligned}$$

By summing up these formulas for all $1 \le j \le n_k$ and by exploiting the inequality $|t| \le T$ we get that

$$\sum_{j=1}^{n_k} |1-\varphi_{k,j}(t)| \le \frac{T^2}{2} M_k([-a,a]) + (2+Ta)(M_k^+(a)+M_k^-(a)).$$

This estimate together with the existence of a finite bound for the numbers $M_k^\pm(a)$ and $M_k([-a,a])$ independent of the index k imply relation (3.10).

The uniform smallness condition, imposed for the triangular array $\xi_{k,j}$, $k=1,2,\dots$, $1 \le j \le n_k$, and Lemma 1 imply that $\lim\limits_{k\to\infty} \sup\limits_{|t|\le T} \sup\limits_{1\le j\le n_k} |1-\varphi_{k,j}(t)|=0$ for all numbers $t \in \mathbf{R}^1$. As a consequence, the relation $\lim\limits_{x\to 0} \dfrac{\log(1-x)+x}{x}=0$ with the choice $x = 1-\varphi_{k,j}(t)$ implies that

$$\lim_{k\to\infty} \sup_{|t|\le T} \sup_{1\le j\le n_k} \frac{|\log \varphi_{k,j}(t) + (1-\varphi_{k,j}(t))|}{|1-\varphi_{k,j}(t)|} = 0$$

By this formula and relation (3.10)

$$\lim_{k\to\infty} \sup_{|t|\le T} \sum_{j=1}^{n_k} |\log \varphi_{k,j}(t) + (1-\varphi_{k,j}(t))| \to 0 \tag{3.11}$$

for all numbers $T>0$.

The convergence of the sequence of random variables $S_k - \bar b_k$ in distribution implies that

$$\lim_{k\to\infty} \prod_{j=1}^{n_k} \left(e^{-it\bar b_k/n_k} \varphi_{k,j}(t) \right) = \psi(t), \tag{3.12}$$

where $\psi(t)$ is the characteristic function of the limit distribution. Furthermore, the convergence is uniform in all finite intervals. We claim that we can take logarithm in the above relation, that is

$$\lim_{k\to\infty} \sum_{j=1}^{n_k} \left(\log \varphi_{k,j}(t) - \frac{it\bar b_k}{n_k} \right) = \log \psi(t). \tag{3.13}$$

To prove this formula let us observe that by relations (3.10) and (3.11)

$$\sup_{t\colon\,|t|\le T} \sup_k \sum_{j=1}^{n_k} \log|\varphi_{k,j}(t)| \le C(T) \quad \text{if } k \ge k_0 \tag{3.14}$$

with some appropriate constant $C(T)<\infty$ and threshold index k_0.

Because of relation (3.14) for all numbers $T>0$ there exist such constants $0<C_1<C_2<\infty$ and threshold index k_0 which satisfy the relation $C_1\le\prod\limits_{j=1}^{n_k}|\varphi_{k,j}(t)|\le C_2$ for all numbers $-T\le t\le T$. Hence relation (3.12) implies the following weakened version of formula (3.13): For all numbers $\varepsilon>0$, $-T\le t\le T$ and indices $k\ge k_0(\varepsilon,T)$ where $k_0=k_0(\varepsilon,T)$ is an appropriate threshold index there exists an integer $m=m(k,t)$ such that $\left|\sum\limits_{j=1}^{n_k}\left(\log\varphi_{k,j}(t)-\dfrac{itb_k}{n_k}\right)-\log\psi(t)-i2\pi m(k,t)\right|<\varepsilon$. (In this argument we must be a little careful, because the logarithm is not a one-valued function on the complex plane. This is the reason for the appearance of the integers $m(k,t)$ in the last relation.) But because both functions $\log\psi(t)$ and $\sum\limits_{j=1}^{n_k}\left(\log\varphi_{k,j}(t)-\dfrac{itb_k}{n_k}\right)$ are continuous, $\log\psi(0)=0$, $\sum\limits_{j=1}^{n_k}\log\varphi_{k,j}(0)=0$, which implies that $m(k,0)=0$, hence $m(k,t)=0$ for all numbers $-T\le t\le T$ and indices $k\ge k_0$. This means that relation (3.12) implies formula (3.13) also in its original form.

By relation (3.11) we can replace the functions $\log\varphi_{k,j}(t)$ by the functions $\varphi_{k,j}(t)-1$ in formula (3.13) In such a way we get that

$$\lim_{k\to\infty}\sum_{j=1}^{n_k}(\varphi_{k,j}(t)-1)-it\bar b_k=\log\psi(t).$$

This formula can be rewritten because of the identity $E\tau(\xi_{k,j})=0$ as

$$\lim_{k\to\infty}\int\frac{e^{itx}-1-it\tau(x)}{x^2}M_k(dx)-it\bar b_k=\log\psi(t)$$

with the canonical measure M_k introduced in the formulation of Theorem 1. Then Theorem 2 can be applied, and it yields that the sequence of canonical measures M_k (weakly) converges to a canonical measure M. This means that relations (1.3) and (1.6) hold (with the choice $B_k=0$), and this is the statement we wanted to prove. (Beside this, the relation $\lim\limits_{k\to\infty}b_k=b$ also holds, and this implies that the normalization $\bar b_k=0$ is also applicable, i.e. the non-normalized random sums S_k also have a limit distribution.)

Thus we have proved the necessity part of Theorem 1 in the case when $E\tau(\xi_{k,j})=0$ for all sufficiently large k and $1\le j\le n_k$. The result in the general situation can be deduced from this case by means of Lemma 2. Indeed, by Lemma 2 one can find a number $\vartheta_{j,k}$ for all $k\ge k_0$ and $1\le j\le n_k$ with an appropriate threshold index k_0 such that the random variables $\xi'_{k,j}=\xi_{k,j}-\vartheta_{k,j}$ satisfy the identity $E\tau(\xi'_{k,j})=0$, and $\lim\limits_{k\to\infty}\sup\limits_{1\le j\le n_k}|\vartheta_{k,j}|=0$. Then the necessity part of Theorem 1 holds for the triangular array $\xi'_{k,j}$, $k=1,2,\dots$, $1\le j\le n_k$. (Put $\xi'_{k,j}=\xi_{k,j}$ for $k<k_0$ for the sake of definitiveness.) Beside this, Lemma 2 also states that this result also implies that the original triangular array $\xi_{k,j}$, $k=1,2,\dots$, $1\le j\le n_k$, satisfies relations (1.3) and (1.6),

and the normalized random sums $S_k - b_k$ have a limit distribution, i.e. we can apply the norming constant in the way described in Theorem 1.

In such a way we have proved Theorem 1. We finish Part II of this work with the proof of Theorem 3.

D.) The proof of Theorem 3.

The proof of Theorem 3. It seems to be more appropriate first to translate this problem to the language of characteristic functions and to study it that way. By Lemma 1 the result we want to prove can be expressed in the language of characteristic functions in the following way: If a sequence of characteristic functions $\omega_k(t)$, $t \in \mathbf{R}^1$, $k = 1, 2, \dots$, is given together with a characteristic function $\varphi(t)$ and a sequence of positive integers n_k, $\lim\limits_{k\to\infty} n_k \to \infty$, in such a way that $\lim\limits_{k\to\infty} \omega_k^{n_k}(t) \to \varphi(t)$ for all real numbers t, then $\lim\limits_{k\to\infty} \sup\limits_{|t|\le K} |1-\omega_k(t)| = 0$ for all numbers $K > 0$.

We shall prove this result with the help of the symmetrization technique. Let us consider beside a sequence of independent, identically distributed random variables $\xi_{k,j}$, $1 \le j \le n_k$, with characteristic function $\omega_k(t)$ a new sequence of independent random variables $\bar\xi_{k,j}$, $1 \le j \le n_k$, which have the same distribution as the random variables $\xi_{k,j}$, and let the sequences of the random variables $\xi_{k,j}$ and $\bar\xi_{k,j}$ be independent. Put $\eta_{k,j} = \xi_{k,j} - \bar\xi_{k,j}$. Then the random variables $\eta_{k,j}$ have characteristic function $|\omega(t)|^2$, the sums $\sum\limits_{j=1}^{n_k} \eta_{k,j}$ tend in distribution to a distribution function with characteristic function $|\varphi(t)|^2$. Hence $\lim\limits_{k\to\infty} |\omega_k(t)|^{2n_k} \to |\varphi(t)|^2$.

First we prove the following auxiliary statement: For all finite intervals $[-K, K]$ there exists a number $C = C(K) > 0$ such that $\limsup\limits_{k\to\infty} \sup\limits_{|t|\le K} n_k(1-|\omega_k(t)|^2) \le C$ if $|t| \le K$.

For sufficiently small $K > 0$ this auxiliary statement holds. Indeed, because of the continuity of the function $\varphi(t)$ and the relation $\varphi(0) = 1$ for all $\varepsilon > 0$ there exists a number $K = K(\varepsilon)$ such that $|1-\varphi(t)| \le \varepsilon$ if $|t| \le K$. Then the uniform convergence of the characteristic functions implies that $1 \ge \liminf\limits_{k\to\infty} \inf\limits_{|t|\le K} |\omega_k(t)|^{2n_k} \ge 1 - 2\varepsilon = C_1 > 0$, hence $1 - \dfrac{C_2}{n_k} \le |\omega_k(t)|^2 \le 1$, and $n_k(1-|\omega_k(t)|^2) \le C_3 < \infty$ for all numbers $|t| \le K$ with appropriate constants $C_1 > 0$, $C_2 > 0$ and $C_3 > 0$.

As the auxiliary statement we want to prove holds in a small neighbourhood of the origin it is enough to show that if it holds in an interval $[-K, K]$, then it also holds in the interval a $[-2K, 2K]$. We prove this with the help of the following estimation where G_k denotes that (symmetrical) distribution whose characteristic function is $|\omega_k(t)|^2$. If $|t| \le K$, then

$$n_k(1-|\omega_k(2t)|^2) = n_k \int (1-\cos 2tx) G_k(\,dx) = 2n_k \int \left(1-\cos^2 tx\right) G_k(\,dx)$$

$$\le 4n_k \int (1-\cos tx) G_k(\,dx) = 4n_k(1-|\omega_k(t)|^2) \le 4C,$$

and this implies that if the auxiliary statement holds in the interval $[-K,K]$ with an upper bound C, then it also holds statement in the interval $[-2K,2K]$) with a new constant $C'=4C$.

The auxiliary statement together with the condition $\lim\limits_{k\to\infty} n_k=\infty$ imply that $\lim\limits_{k\to\infty}(1-|\omega_k(t)|)=0$, and the convergence is uniform in all finite intervals. Beside this, it also implies that $\liminf\limits_{k\to\infty} \inf\limits_{|t|\le K} |\omega_k(t)|^{n_k}>0$, and as a consequence $\inf\limits_{|t|\le K}|\varphi(t)|>0$ for all numbers $K>0$. Indeed, the relation $0\le n_k(1-|\omega_k(t)|)<C<\infty$ holds, and it implies that $|\omega_k(t)|^{n_k}>C'>0$ with an appropriate constant $C'=C'(C)$ for all sufficiently large k. Let us write the characteristic functions $\omega_k(t)$ for $k\ge k_0$ with a sufficiently large threshold index $k_0=k_0(K)$ and the characteristic function $\varphi(t)$ in the polar form $\omega_k(t)=|\omega_k(t)|e^{iu_k(t)}$ and $\varphi(t)=|\varphi(t)|e^{iv(t)}$ in an interval $[-K,K]$. Such a representation is possible, because if k_0 is sufficiently large then all these characteristic functions are separated from zero in the interval $[-K,K]$. We may also assume that $u_k(0)=v(0)=0$, and the functions $u_k(t)$ and $v(t)$ are continuous in the interval $[-K,K]$. (The last assumption means that we define the exponent in the polar representation of the characteristic functions in the natural way. We do not deteriorate the nice behaviour of the exponents by adding some unnecessary number $i2\pi r$ with some integer r to the functions $u_k(\cdot)$ or $v(\cdot)$ in some points t.) We complete the proof of Theorem 3 if we show that $\lim\limits_{k\to\infty}\sup\limits_{|t\le K}|u_k(t)|=0$ for all numbers $K>0$. Indeed, this relation implies that $\lim\limits_{k\to\infty}\sup\limits_{|t\le K}|\omega_k(t)-|\omega_k(t)||=0$ which fact together with the auxiliary statement imply the relation $\lim\limits_{k\to\infty}\sup\limits_{|t|\le K}|1-\omega_k(t)|=0$ we wanted to prove.

Let us fix an interval $[-K,K]$. We know that with above the notations $u_k(0)=v(0)=0$, and the functions $u_k(t)$ and $v(t)$ are continuous. Beside this, the convergence of the characteristic functions we consider and their separation from zero in finite intervals imply that $\lim\limits_{k\to\infty} n_ku_k(t)=v(t)$ in the interval $[K,K]$, and the convergence is uniform since the convergence of characteristic functions to a limit characteristic function is uniform in all finite intervals. Hence the value of the function $v(t)$ determines the value $n_kv_k(t)$ with a good accuracy for large indices k. Nevertheless, some problems arise at this step in the proof, because the number $n_ku_k(t)$ determines the number $u_k(t)$ only modulo $\dfrac{2\pi}{n_k}$. It depends also on the behaviour of the function $u_k(\cdot)$ in the interval $[0,t)$ how we have to define the number $u_k(t)$.

This difficulty can be overcome if we exploit the uniform continuity of the function $v(\cdot)$ and the uniform convergence of the functions $n_ku_k(\cdot)$ to the function $v(\cdot)$ in the interval $[-K,K]$. These properties imply that there exists some number $\delta=\delta(K)>0$ such that $\sup\limits_{|t|\le K,|t-s|\le\delta}|v(t)-v(s)|\le\dfrac{\pi}{3}$, and $\sup\limits_{|t|\le K,|t-s|\le\delta} n_k|u_k(t)-u_k(s)|\le\dfrac{\pi}{2}$ for all sufficiently large indices k. We claim that these facts together with the continuity of the functions $u_k(t)$ imply that $|u_k(t)-u_k(s)|\le\frac{\pi}{n_k}$ for all sufficiently large indices k if $|t-s|\le\delta$, $|s|\le K$ and $|t|\le K$.

Indeed, let us consider an interval $[s,t]\subset[-K,K]$ whose length is not greater than δ. By the above facts the map $\mathbf{T}_k(x)=n_ku_k(x)$ defined for $x\in[-K,K]$ maps such an interval $[s,t]$ to some interval J shorter than π. Hence the set $\{u_k(x)\colon x\in[s,t]\}$ is a subset of the union of the intervals $\frac{J}{n_k}+l\frac{2\pi}{n_k}$, $l=1,2,\dots$, and these intervals are disjoint because of their short lengths. Hence the continuity of the function u_k implies that the set $\{u_k(x)\colon x\in[s,t]\}$ is contained in one of the intervals $\frac{J}{n_k}+l\frac{2\pi}{n_k}$, hence the inequality $|u_k(t)-u_k(s)|\le\frac{\pi}{n_k}$ holds under the above conditions.

Hence $\sup\limits_{|t|\le K}|u_k(t)|\le\frac{K\pi}{\delta n_k}$ for all sufficiently large indices k, and since $n_k\to\infty$ as $k\to\infty$ $\lim\limits_{k\to\infty}\sup\limits_{|t|\le K}|u_k(t)|=0$. Theorem 3 is proved.