Limit theorems and infinitely divisible distributions. Part I.

by Péter Major
The Mathematical Institute of the Hungarian Academy of Sciences

Summary: In this work we discuss when the appropriately normalized partial sums of independent random variables or more generally the sums of the random variables in a triangular array have a limit distribution and also describe the limit. The first part is of introductory character. Here we introduce the most important notions, formulate the questions we are interested in and recall some classical results. We show how the so-called infinitely distributed random variables, the random variables whose distributions are the natural (and right) candidates for the limit distribution in these limit theorems, can be constructed as the (regularized) sums of the elements of a Poisson process. We also show that infinitely divisible distributions can be characterized by means of the Lévy–Hinchin formula. At the end, beside the construction of infinitely divisible distribution we also construct infinitely divisible processes with nice trajectories. The first part also has an Appendix which contains some useful results like a simple construction of a Poisson process and limit theorems with Poissonian limit.

The second part of this work contains the proof of the necessary and sufficient condition for the existence of a limit distribution in the problem mentioned above together with some interesting consequences of this result. The third part contains the functional limit theorem version of this result.

1. Introduction.

One of the basic questions of the probability theory is the following problem:

Let $\xi_1, \xi_2, \dots,$ be a sequence of independent random variables, and let $S_n = \sum\limits_{k=1}^{n} \xi_k$, $n = 1, 2, \dots,$ be the sequence of partial sums made from these random variables. Let us consider the normalized partial sums $\frac{S_n - A_n}{B_n}$ with an appropriate normalization. When do the distributions of these normalized partial sums behave for large n similarly to each other, i.e. when do these normalized partial sums have a limit distribution if $n \to \infty$? How should we choose the norming constants A_n and B_n? What kind of distributions appear as a limit distribution?

The same question arises in a natural way if $\xi_1, \xi_2, \dots,$ is a sequence of independent and *identically distributed* random variables. We are interested in the question whether this additional restriction modifies the possible normalizations and the set of limit distributions. A natural modification of the problem is the following question formulated about triangular array. Before its formulation let us first recall the notion of triangular arrays.

The definition of triangular arrays. *The set of random variables*

$$\begin{array}{ccc} \xi_{1,1}, & \dots, & \xi_{1,n_1} \\ \vdots & & \vdots \\ \xi_{k,1}, & \dots, & \xi_{k,n_k} \\ \vdots & & \vdots \end{array}$$

$k \to \infty$, is a triangular array if the random variables $\xi_{k,1}, \dots, \xi_{k,n_k}$ in a row are independent. (We assume nothing about the relation among random variables in different rows.)

Let $S_k = \sum\limits_{j=1}^{n_k} \xi_{k,j}$, where $\xi_{k,j}$, $1 \le j \le n_k$, $k = 1, 2, \dots$ be a triangular array. We are interested in the question what kind of limit theorems can the sums S_k or (normalized) sums $S_k - A_k$ satisfy. What kind of limit theorems can appear if the random variables in a fixed row of the triangular array are not only independent but also identically distributed?

The following relation can be established between the investigation of the limit theorems for normalized partial sums of independent random variables and limit theorems for the sums of the random variables in a row of a triangular array.

Let $\xi_1, \xi_2, \dots$ be a sequence of independent (and possibly identically distributed) random variables. Let us define the triangular array $\xi_{k,j} = \frac{\xi_j}{B_k}$, $1 \le j \le k$, $k = 1, 2, \dots$. (Here the numbers B_k agree with the norming constants B_k of the partial sums, and $n_k = k$.) With the help of this construction the investigation of partial sums of independent random variables can be considered as the investigation of the sums of the random variables in a row of specially chosen triangular arrays.

In the investigation of limit theorems we want to exclude some trivially non-interesting cases. Such a case appears for instance if $\xi_1 = \xi$, and $\xi_k \equiv 0$ if $k \ge 2$. In this case $S_n = \xi$, and $\frac{S_n - 0}{1} \to \xi$ if $n \to \infty$. More generally, we want to exclude the possibility that one or a fixed number of random variables played a dominant role in the limit behaviour of the partial sums. To exclude such possibilities we introduce the notion of uniform smallness of the random variables in partial sums of sequences of independent random variables or in the sums of rows of triangular arrays.

Definition of uniform smallness. *Let $\xi_1, \xi_2, \dots$ be a sequence of independent random variables, and let us consider their normalized partial sums with a norming (dividing) constant B_n. We say that this sequence of random variables (with this norming factor) satisfies the condition of uniform smallness if for all $\varepsilon > 0$*

$$\sup_{1 \le j \le n} P(|\xi_j| > \varepsilon B_n) < \varepsilon, \quad \text{if } n > n_0(\varepsilon).$$

A triangular array $\xi_{k,j}$, $1 \le j \le n_k$, satisfies the uniform smallness condition if for all $\varepsilon > 0$ there exists a threshold index $k_0 = k_0(\varepsilon)$ such that

$$\sup_{1 \le j \le n_k} P(|\xi_{k,j}| > \varepsilon) < \varepsilon, \quad \text{if } k > k_0(\varepsilon)$$

for all $\varepsilon > 0$.

In the sequel we shall investigate limit theorems for partial sums of random variables or triangular arrays which satisfy the uniform smallness condition. We formulate some well-known classical results.

Central limit theorem.
a.) For partial sums of independent random variables: Let $\xi_1, \xi_2, \dots,$ *be a sequence of independent random variables,* $E\xi_j = 0$, $E\xi_j^2 = \sigma_j^2$, $j = 1, 2, \dots$, $D_n^2 = \sum\limits_{j=1}^n \sigma_j^2$, *and let this sequence satisfy the following Lindeberg condition:*

$$\lim_{n\to\infty} \frac{1}{D_n^2} \sum_{j=1}^n E\xi_j^2 I(|\xi_j| > \varepsilon D_n) = 0$$

for all numbers $\varepsilon > 0$. *Let us consider the sequence of partial sums* $S_n = \sum\limits_{j=1}^n \xi_j$. *Their appropriate normalizations, the sequence* $\frac{S_n}{D_n}$ *converges in distribution to the standard normal distribution.*
b.) For triangular arrays: Let $\xi_{k,j}$, $1 \le j \le n_k$,, $k = 1, 2, \dots,$ *be a triangular array such that* $E\xi_{k,j} = 0$, $E\xi_{k,j}^2 = \sigma_{k,j}^2$, $1 \le k \le n_k$, $k = 1, 2, \dots$. *Set* $S_k = \sum\limits_{j=1}^{n_k} \xi_{k,j}$. *Let us assume that* $\lim\limits_{k\to\infty} \sum\limits_{j=1}^{n_k} E\xi_{k,j}^2 = 1$ *and the following Lindeberg condition is satisfied:*

$$\lim_{k\to\infty} \sum_{j=1}^{n_k} E\xi_{k,j}^2 I(|\xi_{k,j}| > \varepsilon) = 0 \quad \textit{for all numbers } \varepsilon > 0.$$

Then the random variables S_k *converge in distribution to the standard normal distribution as* $k \to \infty$.

Remark: The Lindeberg condition also implies the validity of the uniform smallness condition.

Convergence to the Poisson distribution. *Let*

$$\begin{matrix} \xi_{1,1} \dots, \xi_{1,n_1} \\ \vdots \qquad \vdots \\ \xi_{k,1} \dots, \xi_{k,n_k} \\ \vdots \qquad \vdots \end{matrix}$$

be a triangular array which satisfies the following conditions:

1.) The random variables $\xi_{k,j}$ take non-negative integer values.

2.) $P(\xi_{k,j}=1)=\lambda_{k,j}$, $\lim\limits_{k\to\infty}\sum\limits_{j=1}^{n_k}\lambda_{k,j}=\lambda>0$.

3.) $\sup\limits_{1\le j\le n_k}\lambda_{k,j}\to 0$ if $k\to\infty$, and $\sum\limits_{j=1}^{n_k}P(\xi_{k,j}\ge 2)\to 0$ if $k\to\infty$.

Then the distributions of the random variables $S_k=\sum\limits_{j=1}^{n_k}\xi_{k,j}$ converge to the Poisson distribution with parameter λ as $k\to\infty$.

In the Appendix we shall prove this result.

2. The first question to be discussed.

Let us first consider the following question: Let $\xi_{k,1},\dots,\xi_{k,n_k}$ be such a triangular array whose elements in a row are not only independent but also identically distributed. Let us assume that the normalized versions of the sums $S_k=\sum\limits_{j=1}^{n_k}\xi_{k,j}$, the random variables S_k-A_k with some appropriate constants A_k converge in distribution to a distribution function F. What kind of limit distributions F may appear in such a case? The goal of the following heuristic argument is to justify the introduction of infinitely divisible distributions as the natural candidates for the limit distributions.

Let us split the sequence $\xi_{k,1},\dots,\xi_{k,n_k}$ to L blocks of the same length with some integer L. (That circumstance that the numbers n_k may be not divisible by the number L does not cause a hard problem. We can exploit for instance that because of the uniform smallness condition by leaving out finally many (less than L) terms from the rows we get a new triangular array for which the sums of the random variables in fixed rows satisfy the same limit theorem as the sums of the rows in the original triangular array. With the help of this fact we can restrict our attention to such triangular arrays whose rows have a length divisible by L.) Let $\eta_1^{(k)}$, ..., $\eta_L^{(k)}$ be the sum of the random variables in the blocks of the k-th row minus the number $\frac{A_k}{L}$. Then $\eta_1^{(k)}$, ..., $\eta_L^{(k)}$ are independent and identically distributed random variables, and

$$\eta_1^{(k)}+\cdots+\eta_L^{(k)}\Rightarrow S\quad\text{in distribution.}$$

Carrying out the limit procedure $k\to\infty$ we get that

$$\eta_1+\cdots+\eta_L\overset{\Delta}{=}S,$$

where $\overset{\Delta}{=}$ denotes identity in distribution, and $\eta_1,\dots,\eta_L$ are independent and identically distributed random variables, whose distribution agrees with the limit distribution of the random variables $\eta_1^{(k)}$ as $k\to\infty$. (Actually, this step would demand a more detailed explanation. The main problem is to justify that the random variables $\eta_1^{(k)}$ converge in distribution, or more precisely it is enough to show that this sequence of random variables have a convergent subsequence in distribution. This property could be proved, but we omit it from this heuristic argument.)

Definition of infinitely divisible distributions. *A distribution function F, (or an F distributed random variable S) is infinitely divisible if for all positive integers L there exist independent, and identically distributed random variables $\eta_1,\dots,\eta_L$ such that the sum $\eta_1+\cdots+\eta_L$ is F distributed.*

An equivalent definition of infinitely divisible distributions. *A distribution function F is an infinitely divisible distribution if and only if its characteristic function $\varphi(t)=\int e^{itx}F(\,dx)$ can be written for all positive integers L in the form $\omega(\cdot)^L=\varphi(\cdot)$, where the function $\omega(\cdot)$ is also a* characteristic function.

Questions to be investigated:

a.) Characterization of infinitely divisible distributions.

b.) To prove that only infinitely divisible distributions can appear as limit distribution in limit distributions for normalized sums of independent random variables.

Questions to be investigated later:

If we are interested in the limit distribution of the normalized partial sums of independent and identically distributed random variables $\xi_1,\xi_2,\dots$, then the characterization of an important subclass of the infinitely divisible random variables appears, the characterization of those distribution functions which appear as the limit in this particular case. This leads to the introduction of the so-called stable distributions.

Definition of stable distributions. *A distribution function F is stable, if for all positive integers L there exist such norming constants A_L and B_L for which the distribution functions $F_L(x)=F(B_Lx+A_L)$ satisfy th identity*

$$\underbrace{F_L*\cdots*F_L}_{L\text{-times convolution}}=F. \tag{*}$$

In an equivalent formulation: If $\eta_1,\eta_2,\dots$ are independent and identically distributed random variables with distribution function F, then $\eta_1\overset{\Delta}{=}\frac{(\eta_1-A_L)+\cdots+(\eta_L-A_L)}{B_L}$, or in a different formulation: $\varphi(t)=\left(e^{-tA_L/B_L}\varphi\left(\frac{t}{B_L}\right)\right)^L$, where $\varphi(t)=\int e^{itx}F(dx)$ is the characteristic function of the distribution function F. (In this definition $\overset{\Delta}{=}$ denotes again identity in distribution.) Actually we may impose the following restriction in this definition. We assume that the norming constant B_L in the definition of F_L is of the form $B_L=L^\alpha$ with some $\alpha>0$. Some deeper results show that only such norming constants can be chosen in the definition of the stable distributions.

Further questions to be investigated:

a.) The characterization of stable distributions and the norming constants A_L and B_L in their definition.

b.) Description of the domain of attraction of stable distributions, and the calculation of the right norming constants in limit theorems for normalized partial sums of independent and identically distributed random variables.

3. Examples of infinitely divisible and stable distributions.

a.) *Normal distribution.* This is an infinitely divisible and even stable distribution. Indeed, the distribution of a random variable η with expectation zero and variance σ^2 agrees with the distribution of the sum of L independent normally distributed random variables with expectation zero and variance $\frac{\sigma^2}{L}$. The distribution of the members in this sum agrees with the distribution of the random variable. $\frac{\eta}{\sqrt{L}}$. This means that the distribution of η is stable with constants $A_L = 0$ and $B_L = \sqrt{L}$.

b.) *Poisson distribution.* It is an infinitely divisible but not stable law. The sum of two independent Poisson distributed random variables with parameters λ and μ is a Poisson distributed random variable with parameter $\lambda + \mu$. Hence the distribution of a Poisson distributed random variable with parameter λ agrees with the distribution of L independent Poisson distributed random variables with parameter $\frac{\lambda}{L}$. This means that the Poisson distributed random variables are infinitely divisible. On the other hand, a Poisson distributed random variable with parameter $\frac{\lambda}{L}$ cannot be written as the linear transform of a Poisson distributed random variable with parameter λ, and the Poisson distribution is not stable. Let us remark that the characteristic function of a Poisson distributed random variable is nowhere zero, hence its logarithm can be defined on the whole real line. This implies that if the sum of independent and identically distributed random variables is Poisson distributed, then the parameter of this Poisson distribution determines the characteristic function, hence also the distribution of the summands.

The above two examples are the most important infinitely divisible distributions.

Further examples:

a.) The Cauchy distribution. The density function of this distribution is $f(x) = \frac{1}{\pi(1+x^2)}$, and its characteristic function is $\varphi(t) = e^{-|t|}$. As $e^{-|t|} = \left(e^{-|t|/L}\right)^L$, i.e. $\varphi(t) = \varphi\left(\frac{t}{L}\right)^L$, the Cauchy distribution is stable with the choice $A_L = 0$ and $B_L = L$.

b.) Γ-distributions. The density functions of these distribution functions are

$$f_{\alpha,\nu}(x) = \begin{cases} \frac{1}{\Gamma(\nu)} \alpha^\nu x^{\nu-1} e^{-\alpha x} & \text{if } x \ge 0 \\ 0 & \text{if } x < 0 \end{cases},$$

where $\alpha > 0$ and $\nu > 0$ are two parameters, and $\Gamma(t) = \int_0^\infty x^{t-1} e^{-x}\,dx$ is the Γ function. Some calculation yields that $f_{\alpha,\mu+\nu} = f_{\alpha,\mu} * f_{\alpha,\nu}$. Hence

$$f_{\alpha,\nu} = \underbrace{f_{\alpha,\frac{\nu}{L}} * \cdots * f_{\alpha,\frac{\nu}{L}}}_{L\text{-times convolution}},$$

and $f_{\alpha,\nu}$ is the density function of an infinitely divisible law. Let us also remark that the exponential distribution also belongs to this class with the choice $\nu = 1$. It may be also worth remarking that the characteristic function of a distribution with density function $f_{\alpha,\nu}$ can be given explicitly, and it equals $\varphi_{\alpha,\nu}(t) = (1 - i\frac{t}{\alpha})^{-\nu}$. The last identity written up for convolution follows from this formula.

The class of infinitely divisible distributions can be described explicitly. The Lévy–Hinchin formula supplies such a description. Before its formulation we show its probabilistic content. Namely, we show that with the help of the Gaussian and Poissonian distributions new infinitely divisible distributions can be constructed in a natural way. The main content of the Lévy–Hinchin formula is that this construction supplies all infinitely divisible laws.

4. Construction of random variables with infinitely divisible distribution.

Let us observe that if $\xi_1, \cdots, \xi_k$ are independent random variables with infinitely divisible distribution, then their linear combination $\alpha_1\xi_1 + \cdots + \alpha_k\xi_k + A$ is also a random variable with infinitely divisible distribution. Further, if F_n, $n = 1, 2, \dots$, is a sequence of infinitely distributed random variables, $F_n \Rightarrow F$, where $\Rightarrow$ denotes convergence in distribution, then the limit distribution F is also an infinitely distributed random variable. Indeed, by writing the representation $F_n = \underbrace{G_{n,L} * \cdot * G_{n,L}}_{L \text{ fold convolution}}$ for arbitrary positive integer L, then taking the limit procedure $n \to \infty$ we get the desired identity $F = \underbrace{G_L * \cdot * G_L}_{L \text{ fold convolution}}$. Actually, we should have some special argument to justify the possibility to carry out this limiting procedure. But we shall apply this procedure only in such special cases, where the possibility of such a limiting procedure can be simply justified.

We shall work with Poisson distributed random variables, and our procedures can be carried out more simply by means of Poisson fields. Hence we recall a result about the existence of Poisson fields. In the Appendix we shall present a simple construction of Poisson fields.

Theorem about the existence of Poisson fields. *Let $(X, \mathcal{A}, \mu)$ be a measurable space with a σ-finite measure μ. Then there exits a Poisson field with counting measure μ. More explicitly, a probability space $(\Omega, \mathcal{B}, P)$ can be constructed together with a finite or countably infinite set of random variables $\{x_1(\omega), x_2(\omega), \dots\}$, $\omega \in \Omega$, which take their values in the space X and satisfy the following properties:*

1.) *With probability 1 the Poisson field has only finitely many points in all measurable sets A with finite μ-measure, i.e. $P(\{\omega\colon \#\{n\colon x_n(\omega) \in A\} < \infty\}) = 1$, if $\mu(A) < \infty$.*

2.) *If $A_1 \in \mathcal{A}$, $A_2 \in \mathcal{A}, \dots, A_k \in \mathcal{A}$ are disjoint sets, and $\mu(A_j) < \infty$, $j = 1, \dots, k$, then the number of the points $x_n(\omega)$ which fall into the sets A_1, A_2 $\dots$, and A_k are independent Poisson distributed random variables with parameters $\mu(A_j)$, $j =$*

$1,\dots,k$, i.e. if we fix some non-negative integers $l_1,\dots,l_k$ and define the sets $B_j = \{\omega\colon \#\{n\colon x_n(\omega)\in A_j\}=l_j\}$, $1\le j\le k$, then $P\left(\bigcap\limits_{j=1}^k B_j\right)=\prod\limits_{j=1}^k \frac{\mu(A_j)^{l_j}}{l_j!}e^{-\mu(A_j)}$.

In the sequel we shall sometimes call a Poisson field a Poisson process if the measurable space X is the real line or a subset of it.

Let μ be a σ-finite measure on the set $\mathbf{R}$, ($\mathbf{R}$ denotes the real line in the sequel), and let $(x_1(\omega),x_2(\omega),\dots)$ be a Poisson field on the space $(\mathbf{R},\mathcal{A})$ with counting measure μ. (Here $\mathcal{A}$ denotes the Borel σ-algebra.) We should like to show that the sum $\xi(\omega)=\sum\limits_{n=1}^\infty x_n(\omega)$, i.e. the sum of the coordinates of the Poisson field is a random variable with infinitely divisible distribution. It is natural to expect this. Indeed, for all positive integers L let us consider L independent Poisson fields $(y_1^j(\omega),y_2^j(\omega),\dots)$, $j=1,2,\dots,L$ on the space $(\mathbf{R},\mathcal{A})$ with counting measure $\frac{\mu}{L}$, and define the random variables $\eta_j=\sum\limits_{n=1}^\infty y_n^j(\omega)$. We expect that the random variable $\xi(\omega)$ and the sum $\eta_1(\omega)+\dots+\eta_L(\omega)$ have the same distribution, since the random variable $\eta_1(\omega)+\dots+\eta_L(\omega)$ is the sum of all coordinates $y_n^j(\omega)$, $n=1,2,\dots$, $j=1,\dots,L$, of the Poisson fields $(y_1^j(\omega),y_2^j(\omega),\dots)$, $j=1,\dots,k$. But the union of these Poisson fields is a Poisson field with counting measure μ, because the number of points falling into disjoint sets A_s are independent random variables with parameters $\mu(A_s)$. Indeed the number of these points are the sum of L independent Poisson distributed random variables with parameters $\frac{\mu}{L}$.

There is one weak point in the above heuristic argument. Namely, the random sums defining the random variables $\xi(\omega)$ and $\eta_j(\omega)$ may be meaningless. On the other hand, we show that under appropriate not too restrictive conditions for the measure μ the above sums can be defined in a meaningful way with the help of a good regularization. With the help of this regularization we really get random sums with infinitely divisible distribution. Furthermore, as the later described Lévy–Hinchin formula states in such a way we get a sufficiently rich class of infinitely divisible distributions.

Let us assume that the measure μ satisfies the following condition:

$$\begin{matrix} \mu([a,\infty))<\infty & \mu((-\infty,-a])<\infty \\ \displaystyle\int_0^a x^2\mu(\,dx)<\infty, & \displaystyle\int_{-a}^0 x^2\mu(\,dx)<\infty \end{matrix} \qquad \text{for all numbers } a>0. \tag{$**$}$$

Let us define for all $N=0,1,\dots$ the sums $\xi_N(\omega)=\sum\limits_{n\colon\, |x_n(\omega)|>2^{-N}} x_n(\omega)$ containing randomly many terms These random variables are meaningful, since by the property

$$\mu((-\infty,-2^{-N})\cup(2^{-N},\infty))<\infty$$

the set $(-\infty,-2^{-N})\cup(2^{-N},\infty)$ contains only finitely many points of the Poisson field with probability 1, and the sum defining the random variable $\xi_N(\omega)$ contains only finitely many terms. We also claim that the limit $\xi(\omega)=\lim\limits_{N\to\infty}\xi_N(\omega)-(E\xi_N(\omega)-E\xi_1(\omega))$ exists with probability 1. As $\xi_N(\omega)-(E\xi_N(\omega)-E\xi_1(\omega))=\xi_0(\omega)+\sum\limits_{k=0}^N\zeta_k(\omega)$, where

$$\zeta_k(\omega)=\zeta_k'(\omega)-E\zeta_k'(\omega),\quad\text{and}\quad \zeta_k'(\omega)=\sum_{n\colon\, 2^{-k-1}<|\xi_n(\omega)|\le 2^{-k}}\xi_n(\omega),$$

it is enough to show that the sum $\sum\limits_{k=0}^\infty\zeta_k(\omega)$ is convergent with probability 1. On the other hand, the random variables ζ_k are independent. (Here we sum up the coordinates of the Poisson process lying in disjoint sets, and this implies the independence.) The convergence of the sum $\sum\limits_{k=0}^N\zeta_k(\omega)$ can be proved by means of a classical result of the probability theory, by means of the so-called three series theorem (actually we only need the simpler sufficiency part of this result.) Because of this result it is enough to show that $\sum\limits_{k=0}^\infty\operatorname{Var}\zeta_k<\infty$. (By the definition of the random variables ζ_k $E\zeta_k=0$.) This inequality follows from the following Lemma 1 and relation $(**)$.

Lemma 1. *Let μ be a finite measure on the σ-algebra of the Borel measurable sets of a finite interval $(a,b]$. Let $x_1(\omega),\dots,x_{k(\omega)}(\omega)$ (with a random index $k=k(\omega)$) be a Poisson field on the space $((a,b],\mathcal{A})$ with counting measure μ, and put $S(\omega)=\sum\limits_{j=1}^{k(\omega)}x_j(\omega)$. Then*

$$ES=\int_a^b x\mu(\,dx),\qquad \operatorname{Var}S=\int_a^b x^2\mu(\,dx).$$

Further, the logarithm of the characteristic function of the random variable S (which always exists) satisfies the identity

$$\log Ee^{itS}=\int_a^b\left(e^{itx}-1\right)\mu(\,dx)\quad\textit{for all } t\in\mathbf{R}.$$

Remark: In Lemma 1 we spoke about the logarithm of the characteristic function of a random variable. Let us explain its precise definition. The reason why this problem deserves some attention is that the logarithm of a complex number is a multi-valued function. If $\log z=z_1$, then we can take $\log z=z_1+2ki\pi$, $k=0,\pm1,\pm2,\dots$, with the same right. On the other hand, if a characteristic function $\varphi(\cdot)$ does not take the value zero in an interval $[a,t]$ such that $0\in[a,b]$, then the function $\log\varphi(t)$, $t\in[a,b]$ can be defined in a simple unique way. Namely, put $\log\varphi(0)=0$, and let us define the logarithm of the (continuous) function $\varphi(t)$ as a continuous function on the interval $[a,b]$. In such a way we can tell which branch of the logarithm of the function $\varphi(t)$ we choose.

By Lemma 1 and relation (∗∗)

$$\sum_{k=0}^{\infty} \operatorname{Var} \zeta_k = \sum_{k=0}^{\infty} \int_{2^{-k-1}<|x|\le 2^{-k}} x^2 \mu(\,dx) = \int_{0<|x|\le 1} x^2 \mu(\,dx) < \infty,$$

and this implies the desired convergence.

Proof of Lemma 1: If the measure μ is concentrated to finitely many points $u_1, \dots, u_n$, $\mu(u_j) = \mu_j$, $j = 1, \dots, n$, then $S = u_1 Z_1 + \cdots + u_n Z_n$, where $Z_1, \dots, Z_n$ are independent Poisson distributed random variables with parameters $\mu_1, \dots, \mu_n$. Hence in this case

$$ES = \sum u_j EZ_j = \sum u_j \mu_j = \int x\mu(\,dx)$$

$$\operatorname{Var} S = \sum u_j^2 \operatorname{Var} Z_j = \sum u_j^2 \mu_j = \int x^2 \mu(\,dx)$$

$$\log Ee^{itS} = \sum \log Ee^{itu_j Z_j} = \sum \mu_j \left(e^{itu_j} - 1\right) = \int \left(e^{itx} - 1\right) \mu(\,dx).$$

If μ is an arbitrary finite measure on the interval $(a, b]$, then let us fix an integer $T > 0$, and define the measures μ_T so that they are concentrated in the points $a + \frac{b-a}{T} t$, $t = 1, \dots, T$, and

$$\mu_T \left\{ a + \frac{b-a}{T} t \right\} = \mu \left\{ \left(a + \frac{b-a}{T}(t-1), a + \frac{b-a}{T} t \right] \right\}.$$

If $x_1(\omega), \dots, x_{k(\omega)}(\omega)$ is a Poisson field on the space $((a, b], \mathcal{A})$ with counting measure μ, then let us define the point process $x_{j,T}(\omega) = a + \frac{b-a}{T} t_j$ if the point $x_j(\omega)$ satisfies the inequality $a + \frac{b-a}{T}(t_j - 1) < x_j(\omega) \le a + \frac{b-a}{T} t_j$, $1 \le j \le k(\omega)$. Then $x_{1,T}(\omega), \dots, x_{k(\omega),T}$ is a Poisson field on the space $((a, b], \mathcal{A})$ with counting measure μ_T.

Let $S_T(\omega) = \sum\limits_{j=1}^{k(\omega)} x_{j,T}(\omega)$. Then $S_T(\omega) \to S(\omega)$ if $T \to \infty$. Hence

$$\lim_{T\to\infty} ES_T = ES, \quad \lim_{T\to\infty} \operatorname{Var} S_T = \operatorname{Var} S \quad \text{and} \quad \lim_{T\to\infty} \log Ee^{itES_T} = \log Ee^{itES},$$

and by taking the limit $T \to \infty$ we get the statements of Lemma 1 for arbitrary finite measure μ.

Remark: The formula expressing the logarithm of the characteristic function remains valid also for $a = -\infty$ and $b = \infty$ if $\mu([a, b]) < \infty$. Indeed, by applying this formula for such intervals $(a_n, b_n]$ for which $-\infty < a_n < b_n < \infty$, $a_n \to a$ and $b_n \to b$, then we obtain this identity with the help of the limit procedure $n \to \infty$ in the general case.

By means of Lemma 1 we can describe the logarithm of the characteristic function of the above defined random variable ξ. Namely,

$$\log \varphi(t) = \log Ee^{it\xi} = \int_{\mathbf{R}\setminus\{0\}} \left(e^{itx} - 1 - itA(x)\right) \mu(\,dx),$$

where μ is a measure satisfying condition $(**)$, and $A(x)=x$ if $|x|\le 1$, and $A(x)=0$ if $|x|>1$.

This formula follows from the following observation: As $\xi(\omega)=\xi_0(\omega)+\sum\limits_{k=0}^{\infty}\zeta_k(\omega)$, and the members of the sum are independent random variables, hence we get the logarithm of the characteristic function of the random variable $\xi(\omega)$ by summing up the logarithm of the characteristic functions of these terms. Furthermore, by means of Lemma 1

$$\log Ee^{it\zeta_k}=\log E^{it\zeta'_k}-itE\zeta'_k=\int_{2^{-k-1}<|x|\le 2^{-k}}\left(e^{itx}-1-itx\right)\mu(\,dx).$$

The definition of the above random variables $\xi(\omega)$ can be written in a slightly more general form. Let $B(N)\to 0$, $C(N)\to\infty$ if $N\to\infty$, $0<B(N)<C(N)\le\infty$ be arbitrary monotone deterministic sequences, and introduce the random variables $\xi_N(\omega)=\sum\limits_{n\colon\, B(N)<|x_n(\omega)|<C(N)} x_n(\omega)$,where $x_n(\omega)$, $n=1.2.\dots$, is a Poisson field on the space $(\mathbf{R},\mathcal{A})$ with counting measure μ which satisfies condition $(**)$. Then the following regularized sum exists with probability 1:

$$\begin{aligned}\xi(\omega)&=\lim_{N\to\infty}\operatorname{Reg}\xi_N(\omega)\\&=\lim_{N\to\infty}\left(\sum_{n\colon\, B(N)<|x_n(\omega)|<C(N)}x_n(\omega)-E\sum_{n\colon\, B(N)<|x_n(\omega)|<1}x_n(\omega)\right).\end{aligned}$$

Then for arbitrary positive integer L we can define similarly L independent copies of a random variable $\eta_{j,L}(\omega)$, $1\le j\le L$, with the help of a Poisson field on the space $(R,\mathcal{A})$ with counting measure $\frac{\mu}{L}$ as the regularized sum of the coordinates of this random field. Then $\eta_{1.L}+\cdots+\eta_{L,L}\stackrel{\Delta}{=}\xi$, where $\stackrel{\Delta}{=}$ denotes identity in distribution. Hence $\xi(\omega)$ is a random variable with infinitely divisible distribution.

We get a new random variable with infinitely divisible distribution if we consider instead of the above constructed random variable $\xi(\omega)$ a random variable of the form $\xi(\omega)+\eta(\omega)+D$, where η is independent of ξ, and it is a Gaussian random variable with expectation zero with some variance $\sigma^2\ge 0$. The logarithm of the characteristic function of this new random variable has the form

$$\log\bar\varphi(t)=\log\varphi(t)-\frac{\sigma^2t^2}{2}+itD=\int_{\mathbf{R}\setminus\{0\}}\left(e^{itx}-1-itA(x)\right)\mu(\,dx)-\frac{\sigma^2t^2}{2}+itD,\quad(1)$$

where $\varphi(t)$ is the characteristic function of the random variable ξ.

Now we formulate the Lévy-Hinchin formula. Roughly speaking, it states that the distribution of the above constructed infinitely divisible random variables give all possible infinitely divisible distributions. It is expressed in different equivalent form in different works, and there exists no “best version of the Lévy–Hinchin formula”. Here we formulate it in the form as it is done in the 17-th chapter of the book *An Introduction to the Probability Theory and Its Application II.* of William Feller. Then we shall show that this representation of infinitely divisible distribution is equivalent to that found by our construction. To formulate this result first we introduce the following definition.

Definition of canonical measures on the real line. *A measure M on the Borel σ-algebra of the real line is a canonical measure if for all finite intervals $[a,b]\subset\mathbf{R}$ the relation $M\{[a,b]\}<\infty$ holds, and for all numbers $a>0$*

$$\int_a^\infty \frac{1}{x^2}M(\,dx)<\infty,\quad\text{and}\quad \int_{-\infty}^{-a}\frac{1}{x^2}M(\,dx)<\infty.$$

Theorem. Lévy–Hinchin formula. *A probability distribution F is infinitely divisible if and only if its characteristic function $\varphi(t)=\int e^{itx}F(\,dx)$ has a logarithm, (i.e. $\varphi(t)\neq 0$ for all $t\in\mathbf{R}$, and this logarithm can be written in the form*

$$\log\varphi(t)=\int_{-\infty}^{\infty}\frac{e^{itx}-1-it\sin x}{x^2}M(\,dx)+itB, \tag{2}$$

where M is a canonical measure. The infinitely divisible distribution function F determines the canonical measure M and number B in the formula describing the logarithm of its characteristic function.

Remark: To explain completely the content of the above formula we have to explain the value of the integrand in the origin. We define the value of the integrand in the origin by extending this function to a continuous function on the whole real line. Hence we define

$$\left.\frac{e^{itx}-1-it\sin x}{x^2}\right|_{t=0}=-\frac{t^2}{2}.$$

After this remark we can compare the Lévy-Hinchin formula with formula (1) found by means of our construction.

Let us choose $M(0)=\sigma^2$ and introduce the measure $\mu(dx)=\frac{M(dx)}{x^2}$ if $x\neq 0$ with the help of the quantities in formula (2). Then some consideration shows the equivalence of formulas (1) and (2). Indeed, the measure $\mu(dx)=\frac{M(dx)}{x^2}$, $x\in\mathbf{R}\setminus\{0\}$, satisfies the condition $(**)$ if and only if $M(\cdot)$ is a canonical measure, and the term $-\frac{\sigma^2t^2}{2}$ in formula (1) equals to the contribution of the origin to the integral (2). By rewriting the integral in formula (1) as an integral with respect to the M instead of the measure μ we get that the difference of the expressions in formulas (1) and (2) equals $\int_{-\infty}^\infty it\frac{A(x)-\sin x}{x^2}M(\,dx)+i(B-D)t$. The integral in this formula is finite, since

$$\sup_{x\neq 0}|A(x)-\sin(x)|<\infty,\quad\text{and}\quad \lim_{x\to 0}\frac{A(x)-\sin x}{x^2}=0.$$

Hence we can make the difference of these to expressions zero by an appropriate choice or the constant B or D. The above argument also showed that the contribution of the origin to the integral in formula (2) gave the Gaussian part of the infinitely divisible distributions.

The difference between different representations of infinitely distributed random variables, beside the application of different measures μ and M, is caused by the fact that in the definition of the characteristic functions we have to guarantee that the integrals appearing in these formulas are finite. To achieve this we have to make some kind of regularization. This can be done in different ways, and different regularizations were applied in formulas (1) and (2). Let us remark that there exists no "most natural regularization".

Let us finally remark that there exists a third equivalent and frequently used description of the characteristic function of infinitely divisible distributions. We shall also give this representation. Actually, in the second and third part of this work we shall use this representation of infinitely divisible distributions, because we can better work with it. To describe this representation we fix a positive number $a > 0$ and define the function

$$\tau(x) = \tau_a(x) = \begin{cases} x & \text{if } |x| \le a \\ a & \text{if } x \ge a \\ -a & \text{if } x \le -a. \end{cases} \tag{3}$$

Then the logarithm of the characteristic functions of the infinitely divisible distributions can be written with the help of a canonical measure M and real number$B$ in the following form:

$$\log \varphi(t) = \int_{-\infty}^{\infty} \frac{e^{itx} - 1 - it\tau(x)}{x^2} M(\,dx) + itB. \tag{4}$$

The representation (4) is very similar to the representation (1). The main difference is that here we work with the function $\tau(\cdot) = \tau_a(\cdot)$ instead of the function $A(\cdot)$. Let us emphasize that the function $\tau(\cdot)$, unlike the function $A(\cdot)$, is a continuous (and bounded) function. This property will simplify the limiting procedures in later proofs.

5. Construction of infinitely divisible process.

Let us consider an F infinitely divisible distribution which has no Gaussian component. Then the logarithm of its characteristic function $\varphi(t) = \int e^{itx} F(\,dx)$ can be written by the Lévy–Hinchin formula as

$$\log \varphi(t) = \int_{x \neq 0} (e^{itx} - 1 - it\tau(x))\mu(\,dx) + iDt \tag{5}$$

with the function $\tau(x)$ defined in formula (3) and a measure μ which satisfies the condition $(**)$. (Here we applied the Lévy–Hinchin formula in the form (4), and replaced the canonical measure M by the measure $\mu(\,dx) = \frac{M(\,dx)}{x^2}$.)

Now we show that the ideas of Section 4 enables one to construct not only random variables with infinitely divisible distribution, but also so-called infinitely divisible stochastic processes with nice, so-called càdlàg trajectories. Such processes are closely related to infinitely divisible distributions. Infinitely divisible processes play a role similar to the role of infinitely distributed random variables if we want to prove not only limit theorems for sums of independent random variables but also limit theorems for random broken lines made from partial sums of independent random variables in a natural way. Such results are called functional limit theorems, and they will be the subject of the third part of this text. Before formulating the result about the existence of infinitely divisible processes we introduce the following definition.

Definition of càdlàg functions. *We call a real valued function $x(\cdot)$ on the interval $[0,1]$ a càdàg (continue à droite, limite à gauche) function if it is continuous from the right and has a left-side limit in all points $t \in [0,1]$.*

The reason for introducing the notion of càdlàg function is that we want to take such a version of the stochastic processes we shall work with which has nice trajectories. In very nice cases we can have a version with continuous trajectories. General infinitely divisible do not have such a nice version, but as we shall show they have a version whose trajectories are càdlàg functions.

We shall prove that there exists a stochastic process on a probability space $(\Omega, \mathcal{A}, P)$ $\xi(t) = \xi(t,\omega)$, $\omega \in \Omega$, $0 \le t < \infty$, which satisfies the following properties:

1. The random variable $\xi(1,\omega)$ has a prescribed infinitely divisible distribution F (without Gaussian component) whose characteristic function is defined in formula (5).
2. The stochastic process $\xi(t,\omega)$ has independent and stationary increments, i.e. for all numbers $0 \le t_1 < t_2 < \cdots < t_k \le 1$, $\xi(0) \equiv 0$, the random variables $\xi(t_1)$, $\xi(t_2) - \xi(t_1)$, … , $\xi(t_k) - \xi(t_{k-1})$ are independent, and for all pairs of non-negative numbers s and t such that $s + t \le 1$ the distribution of the random variable $\xi(t+s) - \xi(t)$ does not depend on the parameter t.
3. For almost all $\omega \in \Omega$ the trajectory $\xi(\cdot,\omega)$ is a càdlàg function on the interval $[0,1]$.

Remark: There exists a Gaussian process $W(t,\omega)$, $0 \le t \le 1$, with independent and stationary increments whose trajectories are continuous functions $W(0,\omega) \equiv 0$, and

$W(1,\omega)$ is a random variable with standard normal distribution. Such stochastic processes are called Wiener processes in the literature. The existence of Wiener processes enables us to embed a general infinitely distribution function F to an infinitely divisible process $X(t)$, $0\le t\le 1$, with properties 2.) and 3.) such that $X(1)$ has distribution F. Indeed, by the previously formulated statement a process of the form $X(t)=\xi(t)+\sigma W(t)$ with an appropriate infinitely divisible process $\xi(t)$ and number σ satisfies the desired properties.

We can construct the infinitely divisible stochastic process with the desired properties in the following way. Introduce the measure $\bar\mu=\mu\times\lambda$ on the set $\mathbf{R}\times[0,1]$, the product of the measure μ appearing in formula (5) and the Lebesgue measure on the unit interval. Let us then consider a Poisson field

$$(x_1(\omega),x_2(\omega),\dots)=((x_1^{(1)}(\omega),x_1^{(2)}(\omega)),(x_2^{(1)}(\omega),x_2^{(2)}(\omega),\dots)$$

on the space $(\mathbf{R}\times[0,1],\mathcal{A})$ with counting measure $\bar\mu$, where $\mathcal{A}$ denotes the Borel σ-algebra on $\mathbf{R}\times[0,1]$.

We apply a construction similar to the regularized sum of a Poisson process carried out in the previous Section by which we constructed a random variable with infinitely divisible distribution. We also make some small changes in the construction. Thus, now we only consider an appropriate monotone decreasing $B(N)$, $\lim\limits_{N\to\infty}B(N)=0$, and put $C(N)=\infty$ for all N. Besides, we work with the function $\tau(\cdot)$ instead of the function $A(\cdot)$ in the regularization in order to get a representation of the characteristic function in the form (5). We shall define the random variables $\xi(t)$ simultaneously for all $0\le t\le 1$ by taking the (regularized) sum of the first coordinate of the Poisson field with counting measure $\bar\mu$. But for a fixed $0\le t\le 1$ we take only those points of the Poisson field in the definition of the random variable $\xi(t)$ whose second coordinates are less than or equal to t. More explicitly,

$$\begin{aligned}\xi(t,\omega)&=\lim_{N\to\infty}\operatorname{Reg}\xi_N(t,\omega)\\&=\lim_{N\to\infty}\left(\sum_{\substack{n:\ B(N)<|x_n^{(1)}(\omega)|\\x_n^{(2)}(\omega)\le t}}x_n^{(1)}(\omega)-E\sum_{\substack{n:\ B(N)<|x_n^{(1)}(\omega)|\\x_n^{(2)}(\omega)\le t}}\tau(x_n^{(1)}(\omega))\right).\end{aligned}$$

The argument of the previous Section yields that the limit $\lim\limits_{N\to\infty}\operatorname{Reg}\xi_N(t,\omega)$ exists with probability 1 for all fixed $0\le t\le 1$. Moreover, this argument also shows that $\xi_1(\omega)$ satisfies formula (5) with $D=0$. To satisfy formula (5) in the case of a general constant D we have to consider the process $\xi(t)+Dt$, and if the process $\xi(\cdot)$ satisfies Properties 2.) and 3.), then the new process satisfies all Properties 1.)—3.).

Hence we have to check the validity of Properties 2.) and 3.). Let us first consider the approximating stochastic processes $\operatorname{Reg}\xi_N(t,\omega)$, $0\le t\le 1$. They satisfy Property 2.) because of the independence property of the Poisson field and the invariance property of its distribution with respect to the shift of the second coordinate. They also satisfy Property 3.), and besides the definition of the sum $\operatorname{Reg}\xi_N(t,\omega)$ is meaningful. To see this observe that the Poisson process has only finitely many points in the domain $((-\infty,-B(N))\cup(B(N),\infty))\times[0,1]$. Hence the regularized sums are meaningful, because they consist of finitely many terms, and the trajectories $\operatorname{Reg}\xi_N(t,\omega)$, $0\le t\le 1$, have only finitely many jumps, where they are continuous from the right.

A simple limiting procedure $N\to\infty$ shows that the stochastic process $\xi(t,\omega)$ also satisfies Property 2.). To prove Property 3.), that is to show that the trajectories $\xi(\cdot,\omega)$ are càdlàg functions, we have to make a more careful limiting procedure. Let us exploit our freedom in the choice of the sequence $B(N)$, and choose this sequence in such a way that the inequality $\int_{\{x\colon\, 0<|x|<B(N)\}} x^2\mu(\,dx)\le 4^{-N}$ holds.

Let us observe that the processes $\operatorname{Reg}\xi_N(t,\omega)-\operatorname{Reg}\xi_{N+1}(t,\omega)$, $0\le t\le 1$, are stochastic processes with independent increments and zero expectation which satisfies the inequality

$$E\left(\operatorname{Reg}\xi_{N+1}(1,\omega)-\operatorname{Reg}\xi_N(1,\omega)\right)^2=\int_{\{x\colon\, B(N+1)<|x|<B(N)\}} x^2\mu(\,dx)\le 4^{-N}.$$

for all $N\ge 1$ because of Lemma 1. Hence, by the Kolmogorov inequality

$$\begin{aligned}P&\left(\sup_{0\le t\le 1}|\operatorname{Reg}\xi_{N+1}(t,\omega)-\operatorname{Reg}\xi_N(t,\omega)|\ge 2^{-N/2}\right)\\ &\qquad\le 2^N\int_{\{x\colon\, 0<|x|<B(N)\}} x^2\mu(\,dx)\le 2^{-N}.\end{aligned}$$

Since the sum we get by summing up the expressions at the right-hand side of the last formula for all N is convergent, and

$$\xi(t,\omega)=\operatorname{Reg}\xi_1(t,\omega)+\sum_{N=1}^\infty\left[\operatorname{Reg}\xi_{N+1}(\omega)-\operatorname{Reg}\xi_N(t,\omega)\right],$$

hence the Borel–Cantelli lemma implies that

$$\sup_{0\le t\le 1}|\xi(t,\omega)-\operatorname{Reg}\xi_N(t,\omega)|\le\frac{2^{-N/2}}{\sqrt2-1}\quad\text{if } N\ge N_0(\omega)$$

for almost all $\omega\in\Omega$.

The last relation means that for almost all $\omega\in\Omega$ the trajectories $\operatorname{Reg}\xi_N(t,\omega)$ converge to the trajectory $\xi(\cdot,\omega)$ in the interval $[0,1]$ supremum norm. This implies that not only the functions $\operatorname{Reg}\xi_N(t,\omega)$ but also the functions $\xi(\cdot,\omega)$ are càdlàg functions, i.e. Property 3.) holds.

Actually we need a more general result in Part III of this work. In that part more general processes appear as the limit process in the functional limit theorem. Those processes $X(t)$, $0\le t\le 1$, are also stochastic processes with independent increments and cádlàg trajectories and such that $X(t)$ is an infinitely divisible random variable for all $0\le t\le 1$. But these processes may have not stationary increments. Such processes appear if we consider the limit of such random broken lines which are determined by partial sums of independent random variables with not necessarily identical distributions. We formulate below the statement about the existence of the stochastic processes we shall need in Part III of this work. medskip**Lemma 2.** *Let μ be a measure on the strip $\mathbf{R}\times[0,1]$ such that its projection $\mu^{(1)}$ to the first coordinate, defined by the formula $\mu^{(1)}(B)=\mu(B\times[0,1])$ for all measurable sets $B\in\mathbf{R}$ satisfies the relation $(**)$, and $\mu(\mathbf{R}^1\times\{0\})=0$. Then there exists a stochastic process $\xi(t,\omega)$ which satisfies the following properties:*

1. *All differences $\xi(t_2,\omega)-\xi(t_1,\omega)$ are random variables with infinitely divisible distribution functions, and their characteristic functions $\varphi_{t_1,t_2}(u)=Ee^{iu(\xi(t_2,\omega)-\xi(t_1,\omega))}$ satisfy the identity*

$$\log\varphi_{t_1,t_2}(u)=\int_{\{(y,s)\colon\ -\infty<y<\infty,\ t_1<s<\le t_2\}}\left(e^{iuy}-1-iu\tau(y)\right)\mu(\,dy,\,ds),\quad u\in\mathbf{R}.$$

   for all $0\le t_1<t_2\le 1$. (This integral depends on the argument s through the determination of the domain of integration.)
2. *The stochastic process $\xi(t,\omega)$ has independent increments, i.e. for all numbers $0\le t_1<t_2<\cdots<t_k\le 1$, $\xi(0)\equiv 0$, the random variables $\xi(t_1)$, $\xi(t_2)-\xi(t_1)$, … , $\xi(t_k)-\xi(t_{k-1})$ are independent.*
3. *For almost all $\omega\in\Omega$ the trajectory $\xi(\cdot,\omega)$ is a càdlàg function on the interval $[0,1]$.*

The proof of Lemma 2 together with the construction of an appropriate stochastic process $\xi(t,\omega)$ is a natural adaptation of the argument of this section. Let us consider a Poisson field $(x_1(\omega),x_2(\omega),\dots)=((x_1^{(1)}(\omega),x_1^{(2)}(\omega)),(x_2^{(1)}(\omega),x_2^{(2)}(\omega),\dots)$ on the space $(\mathbf{R}\times[0,1],\mathcal{A})$ with counting measure μ, where $\mathcal{A}$ denotes the Borel σ-algebra on the strip $\mathbf{R}\times[0,1]$. Then we define the process $\xi(t,\omega)$ with the help of this Poisson field and the formula

$$\begin{aligned}\xi(t,\omega)&=\lim_{N\to\infty}\operatorname{Reg}\xi_N(t,\omega)\\&=\lim_{N\to\infty}\left(\sum_{\substack{n\colon\ B(N)<|x_n^{(1)}(\omega)|\\x_n^{(2)}(\omega)\le t}}x_n^{(1)}(\omega)-E\sum_{\substack{n\colon\ B(N)<|x_n^{(1)}(\omega)|\\x_n^{(2)}(\omega)\le t}}\tau(x_n^{(1)}(\omega))\right),\end{aligned}$$

with the above Poisson field and a monotone decreasing function $B(N)$ such that $\lim\limits_{N\to\infty}B(N)=0$, and $\int_{\{(x,t)\colon\ 0<|x|<B(N),\ 0\le t\le 1\}}x^2\mu(\,dx)\le 4^{-N}$. Then the argument described in this Section supplies with slight modifications the proof of Lemma 2.

Appendix

A.) Stable distributions:

The stable distributions and their embedding to such infinitely divisible processes whose one-dimensional distributions are stable distributions can be simply constructed by means of the results described in this text. They are infinitely divisible distributions which are determined in the Lévy–Hinchin formula by such measures μ which have appropriate homogeneity properties. Namely, define the characteristic function of the stable distributions by formula (5) with a measure μ whose density function is

$$\frac{d\mu}{dx}(x)=\begin{cases} C_1x^{-\alpha} & \text{if } x>0\\ C_2|x|^{-\alpha} & \text{if } x<0\end{cases}$$

where $C_1\ge 0$, $C_2\ge 0$, $C_1+C_2>0$, $-3<\alpha<-1$. The last condition satisfies that the measure μ satisfies property $(**)$.

A detailed investigation shows that the above formula describes all stable distributions. Moreover, the domain of attraction of stable distributions, together with the appropriate normalization can be given explicitly in a relatively simple way. Let us also mention that the integral (5) defining the characteristic functions can be calculated explicitly, and it is a homogeneous function. Nevertheless, in most investigations it is simpler to work with the original representation of the characteristic function and not with its shorter, integrated form. Here we shall not discuss the details. Let us remark that a complete description of stable distributions and their domain of attraction is based on the results described in the second part of this work. Besides, the investigation has still another important ingredient. It is the investigation of the so-called slowly varying functions, that is of such functions $L(\cdot)$ in the interval $[1,\infty]$ which satisfy the relation $\lim\limits_{t\to\infty}\frac{L(st)}{L(t)}=s^\alpha$ for all $0<s<\infty$ with some $-\infty<\alpha<\infty$.

B.) A simple construction of Poisson processes.

One of the basic properties of the Poisson distribution is that the sum of two independent Poisson distributed random variables with parameters λ and μ is Poisson distributed with parameter $\lambda+\mu$. The following Lemma B is a reverse statement to this result. It helps to construct Poisson fields.

Lemma B. *Let k urns be given, and let us throw a random number of balls into them. Let us denote the number of balls thrown into these urns by ξ, and let us assume that ξ is a Poisson distributed random variable with parameter $\lambda>0$. Let us throw all balls independently of each other and the random variable ξ, and let each ball fall into the j-th urn with probability $p_j\ge 0$, $j=1,\dots,k$, $\sum\limits_{j=1}^k p_j=1$. Let η_j denote the number of balls thrown into the j-th urn, $1\le j\le k$. Then the random variables η_j, $j=1,\dots,k$, are independent, and η_j is Poisson distributed with parameter $p_j\lambda$, $j=1,\dots,k$.*

Proof of Lemma B:

$$P(\eta_1 = l_1, \dots, \eta_k = l_k) = P(\xi = l_1 + \cdots + l_k) \frac{(l_1 + \cdots + l_k)!}{l_1! \cdots l_k!} p_1^{l_1} \cdots p_k^{l_k}$$
$$= \frac{\lambda^{(l_1 + \cdots + l_k)}}{l_1! \cdots l_k!} p_1^{l_1} \cdots p_k^{l_k} e^{-\lambda} = \prod_{j=1}^{k} \frac{(\lambda p_j)^{l_j}}{l_j!} e^{-\lambda p_j}$$

for arbitrary integers $l_1 \ge 0, \dots, l_k \ge 0$. This identity implies Lemma B.

We formulate the following Corollary of Lemma B.

Corollary of Lemma B. *Let a measurable space $(X, \mathcal{A})$ be given together with a probability measure μ on it. Let ξ be a Poisson distributed random variable with parameter $\lambda > 0$. Let us choose ξ points $x_1, \dots, x_\xi$ on the space X independently of each other, and the random variable ξ in such a way that the distribution of the random points x_l satisfies the identity $P(x_l \in \mathbf{A}) = \mu(\mathbf{A})$ for all sets $\mathbf{A} \in \mathcal{A}$ and $l = 1, \dots, \xi$. Then for all disjoint sets $\mathbf{A}_1 \in \mathcal{A}, \dots, \mathbf{A}_k \in \mathcal{A}$ the number of the points x_l, $1 \le l \le \xi$ contained in the sets $\mathbf{A}_j$, $j = 1, \dots, k$, are independent Poisson distributed random variables with parameters $\lambda\mu(\mathbf{A}_j)$.*

Let a measurable space $(X, \mathcal{B})$ be given together with a σ-finite measure μ on it. With the help of the above construction such a set of points $x_1, x_2, \dots$ can be chosen in the space X which satisfies the following properties: For all sets A with finite μ measure the number of points from the set $\{x_1, x_2, \dots\}$ which are contained in the set A is a Poisson distributed random variable with parameter $\mu(A)$. Beside this, the number of the points from the set $\{x_1, x_2, \dots\}$ contained in disjoint sets of finite μ measure are independent random variables.

Proof of the Corollary. Le us adjust to the sets $A_1, \dots, A_k$ the set $\mathbf{A}_{k+1} = X \setminus \bigcup_{j=1}^{k} \mathbf{A}_j$, and put $p_j = \mu_j(\mathbf{A}_j)$, $j = 1, \dots, k+1$. Then by Lemma B the number of points falling into the sets $\mathbf{A}_j$ are independent, Poisson distributed random variables with parameter $\lambda\mu(\mathbf{A}_j)$, and this is the statement of the first paragraph in the Corollary.

To prove the second statement of the Corollary let us consider a partition of the space X such that $X = \bigcup_{j=1}^{\infty} X_j$, the sets X_j, $j = 1, 2, \dots$, are disjoint, and $\mu(X_j) = \lambda_j < \infty$. Let us construct with the help of the already proven part of the Corollary (with the choice $\lambda = \lambda_j = \mu(X_j)$ and probability measure $\bar{\mu}$, $\bar{\mu}(A) = \frac{1}{\lambda_j}\mu(A)$ on the measurable sets $A \subset X_j$) a set of points $\{x_{j,1}, x_{j,2}, \dots\} \subset X_j$ on each set X_j in such a way that the number of the points from the set $\{x_{j,1}, x_{j,2}, \dots\}$ contained in a set $\mathbf{A}_j \subset X_j$ is Poisson distributed with parameter $\mu(\mathbf{A}_j)$, and the number of the points contained in disjoint subsets of the set X_j are independent random variables. Let us choose these sets $\{x_{j,1}, x_{j,2}, \dots\}$ independently from each other for different indices j. Let us take the union $\bigcup_{j=1}^{\infty} \{x_{j,1}, x_{j,2}, \dots\}$ of these sets. We claim that this set of points satisfies the

Conditions of the Corollary. Really, the number of the points of this set falling into a set A equals the sum of the numbers of points falling into the sets $A\cap X_j$, $j=1,2,\dots$, which are independent Poisson distributed random variables with parameters $\mu(A\cap X_j)$. Hence the number of the points of the above constructed set falling into a set A is a Poisson distributed random variable with parameter $\sum\limits_{j=1}^{\infty}\mu(A\cap X_j)=\mu(A)$. The needed independence property can be checked similarly.

The above Corollary contains actually a construction of a Poisson field. Its first statements describes this construction in the case when $\mu(X)<\infty$, i.e. if the measure of the space is finite. The second statement of the Corollary reduces the general case where we only know that the space is σ-finite to this former case by splitting the space to countable many disjoint sets with finite measure.

C.) Proof of the Poissonian limit theorem for sums of independent and integer valued random variables.

Here we prove the Poissonian limit theorem formulated in pages 3 and 4. We shall give two different proofs. It may be instructive to consider both of them, because they show a simple example of the two different methods applied in this paper. The first method applied mainly in Part II is based on the characteristic function technique to prove limit theorems. The second method applied in Part III exploits the fact that a small perturbation of a sequence of random variables does not change the limit behaviour of this sequence. This fact combined with a good coupling makes possible to reduce the problem we are interested in to a much simpler problem.

First proof: We show that the characteristic functions of the random variables S_k converge to the characteristic function of a Poisson distributed random variable with parameter λ. The characteristic function of a Poisson distributed random variable η with parameter λ equals $Ee^{it\eta}=\sum\limits_{k=0}^{\infty}\frac{\lambda^k}{k!}e^{-\lambda+ikt}=\exp\{-\lambda+\lambda e^{it}\}$. Let $\varphi_{k,j}(t)$ denote the characteristic function of the random variable $\xi_{k,j}$. Then by the Condition 1. of the Theorem

$$\varphi_{k,j}(t)=P(\xi_{k,j}=0)+P(\xi_{k,j}=1)e^{it}+\varepsilon(k,j,t)=1+\lambda_{k,j}(e^{it}-1)+\bar\varepsilon(k,j,t),$$

where $|\varepsilon(k,j,t)|\le P(\xi_{k,j}\ge2)$, and $|\bar\varepsilon(k,j,t)|\le 2P(\xi_{k,j}\ge2)$. Hence

$$\begin{aligned}Ee^{itS_k}&=\prod_{j=1}^{n_k}\varphi_{k,j}(t)=\prod_{j=1}^{n_k}\left(1+\lambda_{k,j}(e^{it}-1)+\bar\varepsilon(k,j,t)\right)\\&=\prod_{j=1}^{n_k}\exp\left\{\lambda_{k,j}(e^{it}-1)+O(\lambda_{k,j}^2+\bar\varepsilon(k,j,t))\right\}\\&=\exp\left\{(e^{it}-1)\left(\sum_{j=1}^{n_k}\lambda_{k,j}\right)+O\left(\sum_{j=1}^{n_k}\left(\lambda_{k,j}^2+\bar\varepsilon(k,j,t)\right)\right)\right\}\to\exp\{\lambda(e^{it}-1)\},\end{aligned}$$

if $k \to \infty$, because $\lim\limits_{k\to\infty} \sum\limits_{j=1}^{n_k} \lambda_{k,j} = \lambda$, $\sum\limits_{j=1}^{n_k} \lambda_{k,j}^2 \le \sup\limits_{1\le j\le n_k} \lambda_{k,j} \cdot \sum\limits_{j=1}^{n_k} \lambda_{k,j} \to 0$ if $k \to \infty$ by conditions 2 and 3 of the Theorem. Furthermore, $\sum\limits_{j=1}^{n_k} \bar\varepsilon(k,j,t) \le 2 \sum\limits_{j=1}^{n_k} P(\xi_{k,j} \ge 2) \to 0$ if $k \to \infty$ by condition 3. Since $\exp\{\lambda(e^{it}-1)\}$ is the characteristic function of a Poisson distributed random variable with parameter λ, these relations imply the Theorem.

The second proof is based on a Lemma C formulated bellow. We shall prove Lemma C in Part III of this work.

Lemma C. *Let S_k and $\bar S_k$, $k = 1, 2, \dots$, be two sequences of random variables such that the differences $S_k - \bar S_k$ converge stochastically to zero as $k \to \infty$. If the sequence of random variables $\bar S_k$ converges in distribution to a distribution F, then the sequence of random variables S_k converges to the same distribution function F.*

Second proof: We shall prove that for all indices k a sequence of independent and Poison distributed random variables $\bar\xi_{k,j}$, $1 \le j \le n_k$, can be constructed with an appropriate parameter $\bar\lambda_{k,j}$ for which the differences of the random sums $\bar S_k = \sum\limits_{j=1}^{n_k} \bar\xi_{k,j}$ and $S_k = \sum\limits_{j=1}^{n_k} \xi_{k,j}$, the expression $\bar S_k - S_k$ converges stochastically to zero if $k \to \infty$, and $\lim\limits_{k\to\infty} \sum\limits_{j=1}^{n_k} \bar\lambda_{k,j} = \lambda$. This implies the Statement of the Theorem. Indeed, $\bar S_k$ is a Poisson distributed random variable with parameter $\sum\limits_{j=1}^{n_k} \bar\lambda_{k,j}$. Hence the distributions of the random variables $\bar S_k$ converge to the Poisson distribution with parameter λ, and by Lemma C the same statement holds for the distributions of the random variables S_k.

Let $\bar\lambda_{k,j}$, the parameter of the Poisson distributed random variable $\bar\xi_{k,j}$, be the solution of the equation $\lambda_{k,j} = xe^{-x}$ in the interval $[0,1]$. If $\lambda_{k,j} \le e^{-1}$, then such a solution exists, and $|\lambda_{k,j} - \bar\lambda_{k,j}| = \bar\lambda_{k,j}|1 - e^{-\bar\lambda_{k,j}}| \le \text{const.}\, \bar\lambda_{k,j}^2 \le \text{const.}\, \lambda_{k,j}^2$. Hence the relations $\lim\limits_{k\to\infty} \sum\limits_{j=1}^{n_k} \lambda_{k,j} = \lambda$ and $\lim\limits_{k\to\infty} \sup\limits_{1\le j\le n_k} \lambda_{k,j} = 0$ imply that $\lim\limits_{k\to\infty} \sum\limits_{j=1}^{n_k} \bar\lambda_{k,j} = \lambda$, and $\lim\limits_{k\to\infty} \sup\limits_{1\le j\le n_k} \bar\lambda_{k,j} = 0$. Let us define the random variable $\bar\xi_{k,j}$ in such a way that the events $\bar\xi_{k,j} = 1$ and $\xi_{k,j} = 1$ agree. Let us remark that this is possible, since we defined the number $\bar\lambda_{k,j}$ in such a way that a Poisson distributed random variable $\bar\xi_{k,j}$ with parameter $\bar\lambda_{k,j}$ satisfies the identity $P(\bar\xi_{k,j} = 1) = \bar\lambda_{k,j} e^{-\bar\lambda_{k,j}} = P(\xi_{k,j} = 1)$. We shall define the events $\bar\xi_{k,j} = l$, $l \ne 1$ in such a way that $P(\bar\xi_{k,j} = l) = \frac{\bar\lambda_{k,j}^l}{l!} e^{-\bar\lambda_{k,j}}$, and the random variables $\bar\xi_{k,j}$, $1 \le j \le n_k$, are independent for a fixed index k.

In a sufficiently rich probability space such a construction is possible. A possible construction is the following one: Let $\eta_1, \dots, \eta_{n_k}$ be a sequence of independent random variables with uniform distribution on the interval $[0,1]$ which random variables are also independent of the random variables $\xi_{k,j}$, $1 \le j \le n_k$. Let us consider for all numbers $1 \le j \le n_k$ a partition $A_{0,j} = [0, a_{1,j}]$, $A_{l,j} = [a_{l-1,j}, a_{l,j}]$, $l = 2, 3, \dots$, of the interval

$[0,1]$ (depending on the parameters k and j) in such a way that the length of the interval $A_{0,j}$ is $a_{0,j}=\frac{e^{-\bar\lambda}}{1-\bar\lambda e^{-\bar\lambda}}$ and the length of the interval $A_{l,j}$ is $a_{l,j}-a_{l,j-1}=\frac{\bar\lambda^l e^{-\bar\lambda}}{l!(1-\bar\lambda e^{-\bar\lambda})}$, $l=2,3\dots$. In the case $l\neq 1$ let the set $\{\omega\colon \bar\xi_{j,k}(\omega)=l\}$ agree with the set $\{\omega\colon \eta_j(\omega)\in A_{l,j}\}$. The random variables $\xi_{k,j}$ constructed in such a way are independent for a fixed k, and they have the prescribed distributions, since the conditional probabilities $P(\bar\xi_{k,j}=l|\bar\xi_{k,j}\neq 1)$ have the right values.

The assumption that the probability space where we made this construction is sufficiently rich does not mean an unpleasant restriction, because the validity of the statement to be proved does not depend on the properties of the probability space where we are working. We show that with the above constructed random variables $\bar\xi_{k,j}$ the differences of the sums $\bar S_k$ and S_k, the expressions $\bar S_k-S_k$, converge stochastically to zero.

To prove this statement let us observe that for an arbitrary number $\varepsilon>0$

$$P(|S_k-\bar S_k|>\varepsilon)\le \sum_{j=1}^{n_k}P(\xi_{k,j}\ge 2)+\sum_{j=1}^{n_k}P(\bar\xi_{k,j}\ge 2),$$

since the relation $S_k-\bar S_k\neq 0$ can only hold if either $\xi_{k,j}\ge 2$ or $\bar\xi_{k,j}\ge 2$ for some index $1\le j\le n_k$. On the other hand, $\lim\limits_{k\to\infty}\sum\limits_{j=1}^{n_k}P(\xi_{k,j}\ge 2)=0$ because of the conditions of the Theorem. Furthermore, $\lim\limits_{k\to\infty}\sum\limits_{j=1}^{n_k}P(\bar\xi_{k,j}\ge 2)=0$, since $\sum\limits_{j=1}^{n_k}P(\bar\xi_{k,j}\ge 2)\le \text{const.}\sum\limits_{j=2}^{\infty}\bar\lambda_{k,j}^2$, and also the relations $\lim\limits_{k\to\infty}\sum\limits_{j=1}^{n_k}\bar\lambda_{k,j}=\lambda$ and $\lim\limits_{k\to\infty}\sup\limits_{1\le j\le n_k}\bar\lambda_{k,j}=0$ hold. Hence the desired inequality and also the statement of the Theorem holds.

Example for a non-stable distribution that satisfies a weakened form of the stability condition.

We construct such a random variable S whose distribution F does not satisfy the condition $(*)$ of stability, but it satisfies a weakened version of it.

Let us fix a number $\alpha>\frac12$. We construct such a non-stable distribution F that satisfies the identity $F(2^\alpha(x-A))*F(2^\alpha(x-A))=F(x)$ with an appropriate number $A=A(\alpha)$, or in a different formulation, we show that if S is an F distributed random variable, and S' and S'' are two independent random variables with the same distribution as S, then $\frac{(S'+A)+(S''+A)}{2^\alpha}\stackrel{\Delta}{=}S$. Here $\stackrel{\Delta}{=}$ means that on the two sides of the formula we have two random variables with the same distribution.

We construct the random variable S with the desired distribution F in the following way. Let $\eta(2^n)$, $n=0,\pm1,\pm2,\dots$, be independent Poisson distributed random variables with parameter $\mu=2^n$, and put

$$S=\sum_{n=-\infty}^{0}2^{-n\alpha}\eta(2^n)+\sum_{n=1}^{\infty}2^{-n\alpha}[\eta(2^n)-E(\eta(2^n)]=S_1+S_2.$$

I claim that in the case $\alpha > 1/2$ the above definition of the random variable S is meaningful, i.e. the random sum defining it is convegent (with probability 1), and if S' and S'' are two independent random variables with the same distribution as S, then $\frac{(S'+A)+(S''+A)}{2^\alpha} \stackrel{\Delta}{=} S$.

To prove the above convergence let us observe that $E2^{-n\alpha}[\eta(2^n)-E\eta(2^n)]=0$, and $\operatorname{Var}(2^{-n\alpha}[\eta(2^n)-E\eta(2^n)])=2^{-2n\alpha+n}$, which implies that $\sum\limits_{n=1}^{\infty}\operatorname{Var}(2^{-n\alpha_n}[\eta(2^n)-E\eta(2^n)])<\infty$, if $\alpha>\frac12$. Hence the partial sum $\sum\limits_{n=1}^{\infty}2^{-n\alpha}[\eta(2^n)-E(\eta(2^n)]$ defining the expression S_2 is convergent. On the other hand, in the case $n\le 0$ the estimate $P(\eta(2^n)\neq 0)=1-e^{-2^n}\le \text{const.}\,2^n$ holds. Hence $\sum\limits_{n=-\infty}^{0}P(\eta(2^n)\neq 0)\le \text{const.}\sum\limits_{n=0}^{\infty}2^{-n}<\infty$, and the Borel–Cantelli lemma implies that the sum defining the expression S_1 contains only finitely many term, and as a consequence it is also convergent.

Let $\eta'(2^n)$ and $\eta''(2^n)$, $n=0,\pm1,\pm2,\dots$, be similarly to $\eta(2^n)$ independent Poisson distributed random variables with parameter $\mu=2^n$, and define the random variables S' and S'' similarly to S, only let us reply the random variables $\eta(2^n)$ in the sum defining them by the random variables $\eta'(2^n)$ and $\eta''(2^n)$ respectively. I claim that $\frac{(S'+A)+(S''+A)}{2^\alpha}\stackrel{\Delta}{=}S$. To prove this let us observe that

$$\frac{2^{-\alpha(n-1)}\eta'(2^{n-1})+2^{-\alpha(n-1)}\eta''(2^{n-1})}{2^\alpha}\stackrel{\Delta}{=}2^{-\alpha n}\eta(2^n),$$

and

$$\frac{2^{-\alpha(n-1)}(\eta'(2^{n-1})-E(\eta'(2^{n-1}))+2^{-\alpha(n-1)}(\eta''(2^{n-1})-E\eta''(2^n))}{2^\alpha}$$
$$\stackrel{\Delta}{=}2^{-\alpha n}(\eta(2^n)-E\eta(2^n))$$

for all integers n, and the expression appearing in these expressions are independent random variables for different indices n. By summing up these identities for all indices n (by summing up the first identity for indices $n\le 0$ and the second identity for $n\ge 1$) we get the identity we wanted to prove. The constant A with an appropriate choice appears in this identity, because in the sum defining the random variable S we took the term $\eta(2^0)=\eta(1)$ for the index $n=0$ and the term $\eta(2)-E\eta(2)$ for the index $n=1$.

We prove with the help of a result proved in Part II. that the above defined random variable S is not stable distributed. To do this let us observe that the distribution function of S is infinitely distributed, and the logarithm of its characteristic function is defined by such a measure μ which is concentrated in the pointe $y_n=2^{n\alpha}$, $n=0,\pm1,\pm2,\dots$, and $\mu(2^{n\alpha})=2^{-n}$. In Part II. of this work we shall prove that an infinitely divisible distribution uniquely determines the measure μ which appears in the Lévy–Hinchin formula representing its characteristic function. On the other hand, observe that if S_1, S_2 are S_3 independent random variables with the same distribution as S, then the measure $\bar{\mu}$ determing the characteristic function of the random variable $\frac{S_1+S_2+S_3-B}{3^\alpha}$ with the help of the Lévy–Hinchin formula is different from the measure μ. This measure $\bar{\mu}$ is defined by the identity $\bar{\mu}(A) = 3\mu(\frac{A}{3^\alpha})$ for all measurable sets A. It is even concentrated in a different set as the measure μ.

There is the following idea behind the above construction. An infiniely divisible distribution is stable if and only if the measure μ (or the canonical measure M) defining its characteristic function satisfies certain homogeneity properties. If only a weakened, discretized version of this homogeneity property holds, if the desired identitities hold only for special parameters $t = k^n$ with a fixed integer k and $n = 0, \pm 1, \pm 2, \dots$, then a weakened version of the stability property expressed in $(*)$ may hold, but the infinitely divisible distribution under consideration is not stable.