

A useful lemma of Arcones

by Péter Major

Mathematical Institute of the Hungarian Academy of Sciences

In Lemma 1 of Arcones' paper Limit theorems for non-linear functionals of a stationary Gaussian sequence of vectors (The Annals of Probability 1994 Vol. 22 No. 4, 2242–2274) the following result is proved (with a slightly different notation).

Lemma 1 in Arcones' paper. *Let $X = (X_1, \dots, X_d)$ and $Y = (Y_1, \dots, Y_d)$ be two d -dimensional (jointly Gaussian) standard normal random vectors with covariances $EX_j Y_k = r(j, k)$, $1 \leq j, k \leq d$, let $f(x_1, \dots, x_d) \in R^d$ be such a function for which $Ef^2(X_1, \dots, X_d) < \infty$, and it has rank τ with some positive integer τ , i.e.*

$$Ef(X_1, \dots, X_d)P_k(X_1, \dots, X_d) = 0$$

for any polynomial $P_k(x_1, \dots, x_d)$ of order $0 \leq k < \tau$. Suppose that

$$\psi = \left(\sup_{1 \leq j \leq d} \sum_{k=1}^d |r(j, k)| \right) \vee \left(\sup_{1 \leq k \leq d} \sum_{j=1}^d |r(j, k)| \right) \leq 1. \quad (1)$$

Then

$$|Ef(X_1, \dots, X_d)f(Y_1, \dots, Y_d)| \leq \psi^\tau Ef^2(X_1, \dots, X_d). \quad (2)$$

Here I present a more detailed proof of this result together with some explanation about its role in the study of some problems.

Remark: First I explain the role of the above lemma in the study of limit theorems (central or non-central limit theorems) for non-linear functionals of vector-valued Gaussian stationary random fields.

In the scalar valued case where limit theorems are proved for the partial sums of random variables of the form $f(X_p)$, where $f(\cdot)$ is an appropriate real valued function on the real line, and X_p , $p \in R^\nu$, is a stationary random field consisting of standard normal random variables the following argument helps to simplify the problem.

Let us expand the function f with respect to the normalized Hermite polynomials as $f(x) = \sum_{t=1}^{\infty} a_t \frac{H_t(x)}{\sqrt{t!}}$ with $\sum_{t=1}^{\infty} a_t^2 < \infty$ if $Ef(X_p) = 0$ and $Ef^2(X_p) < \infty$ for the random variables X_p of the random field. Let $r(p, q) = EX_p X_q$, $p, q \in R^\nu$, denote the covariance function of the random field. Then $EH_t(X_p)H_{t'}(X_q) = 0$ for all indices p and q if $t \neq t'$, and $Ea_t \frac{H_t(X_p)}{\sqrt{t!}} a_t \frac{H_t(X_q)}{\sqrt{t!}} = a_t^2 r(p, q)^t$, hence $\left| Ea_t \frac{H_t(X_p)}{\sqrt{t!}} a_t \frac{H_t(X_q)}{\sqrt{t!}} \right| \leq a_t^2 |r(p, q)|^t$.

Since $Ef(X_p)f(X_q) = \sum_{t=1}^{\infty} Ea_t \frac{H_t(X_p)}{\sqrt{t!}} a_t \frac{H_t(X_q)}{\sqrt{t!}}$ the last inequality implies that the high-term elements in the expansion of the function $f(\cdot)$ with respect to the Hermite polynomials have a negligible contribution to the value of the expectation $Ef(X_p)f(X_q)$ if the covariance function $r(\cdot, \cdot)$ satisfies certain conditions, (if $r(p, q) \rightarrow 0$ sufficiently

fast, as $|p - q| \rightarrow \infty$). This has the consequence that in such cases they have a negligible role in the limit theorems we are interested in.

The above lemma of Arcones is a natural generalization of the above estimate of $Ef(X_p)f(X_q)$ to the vector-valued case. It provides a similar reduction in the study of limit theorems for non-linear functionals of vector-valued stationary random fields. In such cases we consider a vector-valued Gaussian stationary random field whose elements $X_p = (X_{p,1}, \dots, X_{p,d})$, $p \in R^\nu$, are d -dimensional random vectors with standard normal distribution together with a function $f(x_1, \dots, x_d)$ defined in the d -dimensional Euclidean space.

Let us fix two elements $X_p = (X_{p,1}, \dots, X_{p,d})$ and $X_q = (X_{q,1}, \dots, X_{q,d})$ of this random field, apply the notation $X_p = (X_{p,1}, \dots, X_{p,d}) = X = (X_1, \dots, X_d)$, $X_q = (X_{q,1}, \dots, X_{q,d}) = Y = (Y_1, \dots, Y_d)$ and $EX_{p,j}X_{q,k} = EX_jY_k = r(j, k)$, $1 \leq j, k \leq d$, for the elements of the covariance matrix of these two random vectors. Arcones' lemma gives an estimate on $|Ef(X_{p,1}, \dots, X_{p,d})f(X_{q,1}, \dots, X_{q,d})|$ with such a notation.

Let us remark that this approach to this problem is very similar in the scalar and vector valued cases. Formulas (4) and (5) in this note are the natural multivariate versions of the expansion of the function $f(\cdot)$ with respect to normalized Hermite polynomials. The estimate (6) corresponds to the estimate $\left| Ea_t \frac{H_t(X_p)}{\sqrt{t!}} a_t \frac{H_t(X_q)}{\sqrt{t!}} \right| \leq a_t^2 |r(p, q)|^t$ in the scalar valued case. Arcones' estimate can be applied similarly to the corresponding result in the scalar-valued case, only the absolute value $|r(p, q)|$ of the covariance function has to be replaced with the function $\psi_{p,q} = \psi$ defined in formula (1) with the arguments $r(j, k) = r_{p,q}(j, k) = EX_{p,j}X_{q,k}$, $1 \leq j, k \leq d$.

The proof of Arcones' lemma. In this proof the so-called diagram formula plays an important role. This formula deals with the following problem. Let us have a Gaussian random vector $X = (X_1, \dots, X_s)$ whose coordinates are standard normal random variables, and let us calculate the expectation of the product $H_{k_1}(X_1) \cdots H_{k_s}(X_s)$ with the help of the elements of the covariance matrix of the random vector X , where $H_k(x)$ denotes the k -th Hermite polynomial. The diagram formula yields a formula for this expected value with the help of some diagrams. More explicitly, the diagram formula contains the following result.

Diagram formula. *Let $(X_1, \dots, X_s)$ be a Gaussian random vector with $EX_1 = 0$, $EX_i^2 = 1$ for all $1 \leq i \leq s$. Let $r(i, j) = EX_iX_j$ for all $1 \leq i, j \leq s$, and let $H_k(x)$ denote the k -th Hermite polynomial (with leading coefficient 1). Let us take some non-negative integers $k_1, \dots, k_s$ and consider the expected value $EH_{k_1}(X_1) \cdots H_{k_s}(X_s)$. The diagram formula expresses this expectation with the help of some diagrams defined in the following way.*

Let $\Gamma = \Gamma(k_1, \dots, k_s)$ denote the set of diagrams $\gamma \in \Gamma$, defined in the following way. Each diagram $\gamma \in \Gamma$ consists of s rows, and the i -th row, (which corresponds to the random variable X_i), contains k_i vertices, denoted by $(i, 1), (i, 2), \dots, (i, k_i)$. A diagram $\gamma \in \Gamma$ has vertices defined in the above way, and it contains edges with the following properties. From each vertex of γ starts exactly one edge. An edge can connect only vertices from different rows. The set $\Gamma = \Gamma(k_1, \dots, k_s)$ contains the diagrams γ with the above properties. Given a diagram $\gamma \in \Gamma(k_1, \dots, k_s)$ let $E(\gamma)$ denote the set of edges

contained in γ .

With the above notations the following identity holds.

$$EH_{k_1}(X_1) \cdots H_{k_s}(X_s) = \sum_{\gamma \in \Gamma(k_1, \dots, k_s)} \prod_{((i, p_i), (j, p_j)) \in E(\gamma)} r(i, j). \quad (3)$$

In words: Given a diagram $\gamma \in \Gamma$ we attach a (possibly negative) weight to it by giving the weight $r(i, j)$ to an edge connecting a vertex from the i -th row with a vertex from the j -th row, and then the weight of a diagram γ is defined as the product of the weights of its edges. The expected value we are interested in equals the sum of the weights of the diagrams in the set $\Gamma(k_1, \dots, k_s)$.

In the proof of Arcones' lemma we exploit that the products $\frac{H_{k_1}(X_1)}{\sqrt{k_1!}} \cdots \frac{H_{k_d}(X_d)}{\sqrt{k_d!}}$ for all possible indices $(k_1, \dots, k_d)$ yield an orthonormal basis in the Hilbert space generated by the random variables $X_1, \dots, X_d$, and the products $\frac{H_{k_1}(Y_1)}{\sqrt{k_1!}} \cdots \frac{H_{k_d}(Y_d)}{\sqrt{k_d!}}$ yield an orthonormal basis in the Hilbert space generated by the random variables $Y_1, \dots, Y_d$. It is useful to consider the expansion of the functions $f(X_1, \dots, X_d)$ and $f(Y_1, \dots, Y_d)$ with respect to these orthonormal systems together with their following natural decomposition.

$$\begin{aligned} f(X_1, \dots, X_d) &= \sum_{t=\tau}^{\infty} f_t(X_1, \dots, X_d) \\ f(Y_1, \dots, Y_d) &= \sum_{t=\tau}^{\infty} f_t(Y_1, \dots, Y_d) \end{aligned} \quad (4)$$

with

$$f_t(x_1, \dots, x_d) = \sum_{(k_1, \dots, k_d): k_1 + \dots + k_d = t} a_{k_1, \dots, k_d}(t) \frac{H_{k_1}(x_1)}{\sqrt{k_1!}} \cdots \frac{H_{k_d}(x_d)}{\sqrt{k_d!}} \quad (5)$$

with some appropriate coefficients $a_{k_1, \dots, k_d}(t)$. Then $\sum_{(k_1, \dots, k_d): k_1 + \dots + k_d = t} a_{k_1, \dots, k_d}^2(t) = Ef_t^2(X_1, \dots, X_d)$.

Summation in (4) is taken for $\tau \leq t < \infty$, because the function f has rank τ . The function $f_t(\cdot)$ defined in (5) has rank t . $EH_{k_1}(X_1) \cdots H_{k_d}(X_d) H_{k'_1}(X_1) \cdots H_{k'_d}(X_d) = 0$ if $(k_1, \dots, k_d) \neq (k'_1, \dots, k'_d)$, hence the random variables $f_t(X_1, \dots, X_d)$ are orthogonal for different indices t , and $Ef^2(X_1, \dots, X_d) = \sum_{t=\tau}^{\infty} Ef_t^2(X_1, \dots, X_d)$.

Let us also observe that $Ef_t(X_1, \dots, X_d) f_{t'}(Y_1, \dots, Y_d) = 0$ if $t \neq t'$. This follows from the fact that $EH_{k_1}(X_1) \cdots H_{k_d}(X_d) H_{k'_1}(Y_1) \cdots H_{k'_d}(Y_d) = 0$ if $t \neq t'$ for $t = k_1 + \dots + k_d$ and $t' = k'_1 + \dots + k'_d$. The last identity follows from the diagram formula, because both the random variables $X_1, \dots, X_d$ and the random variables $Y_1, \dots, Y_d$ are orthogonal. These orthogonalities imply that only such diagrams yield a non-zero contribution to the expectation of the above product of Hermite polynomials of standard normal random variables in the application of the diagram formula in which each edge connects a vertex from a row corresponding to a random variable X_i with a vertex from a row corresponding to a random variable Y_j . But there are no such diagrams if $t \neq t'$.

Remark. Given a set of Gaussian random variables X_t , $t \in T$, and the Hilbert space $\mathcal{H}$ generated by these random variables, there is a so-called Fock–space representation of this Hilbert space $\mathcal{H}$. In the above discussion we applied this Fock–space representation in a special case, and showed some of its basic properties.

I claim that under the conditions of Lemma 1 in Arcones’ paper the following inequality holds for all $t \geq \tau$:

$$|Ef_t(X_1, \dots, X_d)f_t(Y_1, \dots, Y_d)| \leq \psi^t Ef_t^2(X_1, \dots, X_d). \quad (6)$$

It is not difficult to see that relation (6) implies Arcones’ lemma. Indeed, it yields (together with the condition $\psi \leq 1$) that

$$\begin{aligned} |Ef(X_1, \dots, X_d)f(Y_1, \dots, Y_d)| &\leq \sum_{t=\tau}^{\infty} |Ef_t(X_1, \dots, X_d)f_t(Y_1, \dots, Y_d)| \\ &\leq \sum_{t=\tau}^{\infty} \psi^t Ef_t^2(X_1, \dots, X_d) \leq \psi^\tau \sum_{t=\tau}^{\infty} Ef_t^2(X_1, \dots, X_d) = \psi^\tau Ef^2(X_1, \dots, X_d). \end{aligned}$$

Formula (6) will be proved with the help of the following lemma.

Lemma. *Let two random vectors $(X_1, \dots, X_d)$, $(Y_1, \dots, Y_d)$ and a function f of d variables satisfy the conditions of Lemma 1 in Arcones’ paper. Let us fix an integer $t \geq \tau$ and a vector $(l_1^{(0)}, \dots, l_d^{(0)})$ or $(m_1^{(0)}, \dots, m_d^{(0)})$ with non-negative integer coordinates such that $l_1^{(0)} + \dots + l_d^{(0)} = t$, and $m_1^{(0)} + \dots + m_d^{(0)} = t$. Then the following inequalities hold:*

$$\begin{aligned} &\sum_{(m_1, \dots, m_d): m_1 + \dots + m_d = t} \left| EH_{l_1^{(0)}}(X_1) \cdots H_{l_d^{(0)}}(X_d) \frac{H_{m_1}(Y_1) \cdots H_{m_d}(Y_d)}{m_1! \cdots m_d!} \right| \\ &\leq \prod_{i=1}^d (|r(i, 1)| + \dots + |r(i, d)|)^{l_i^{(0)}} \leq \psi^t \\ &\sum_{(l_1, \dots, l_d): l_1 + \dots + l_d = t} \left| EH_{m_1^{(0)}}(Y_1) \cdots H_{m_d^{(0)}}(Y_d) \frac{H_{l_1}(X_1) \cdots H_{l_d}(X_d)}{l_1! \cdots l_d!} \right| \\ &\leq \prod_{j=1}^d (|r(1, j)| + \dots + |r(d, j)|)^{m_j^{(0)}} \leq \psi^t. \end{aligned} \quad (7)$$

First I prove formula (6) with the help of the Lemma, and then I prove the Lemma. To prove formula (6) write the inequality

$$\begin{aligned} &|Ef_t(X_1, \dots, X_d)f_t(Y_1, \dots, Y_d)| \\ &\leq \sum_{(l_1, \dots, l_d): l_1 + \dots + l_d = t} \sum_{(m_1, \dots, m_d): m_1 + \dots + m_d = t} \left| \frac{a_{l_1, \dots, l_d}(t)}{\sqrt{l_1! \cdots l_d!}} \cdot \frac{a_{m_1, \dots, m_d}(t)}{\sqrt{m_1! \cdots m_d!}} \right| \\ &\quad |EH_{l_1}(X_1) \cdots H_{l_d}(X_d)H_{m_1}(Y_1) \cdots H_{m_d}(Y_d)|. \end{aligned}$$

By applying this inequality together with the relation

$$\left| \frac{a_{l_1, \dots, l_d}(t)}{\sqrt{l_1! \dots l_d!}} \cdot \frac{a_{m_1, \dots, m_d}(t)}{\sqrt{m_1! \dots m_d!}} \right| \leq \frac{1}{2} \left(\frac{a_{l_1, \dots, l_d}^2(t)}{m_1! \dots m_d!} + \frac{a_{m_1, \dots, m_d}^2(t)}{l_1! \dots l_d!} \right)$$

we get with the help of the Lemma that

$$\begin{aligned} |Ef_t(X_1, \dots, X_d)f_t(Y_1, \dots, Y_d)| &\leq \sum_{(l_1, \dots, l_d): l_1 + \dots + l_d = t} \frac{a_{l_1, \dots, l_d}^2(t)}{2} \\ &\quad \sum_{(m_1, \dots, m_d): m_1 + \dots + m_d = t} \left| EH_{l_1}(X_1) \dots H_{l_d}(X_d) \frac{H_{m_1}(Y_1) \dots H_{m_d}(Y_d)}{m_1! \dots m_d!} \right| \\ &+ \sum_{(m_1, \dots, m_d): m_1 + \dots + m_d = t} \frac{a_{m_1, \dots, m_d}^2(t)}{2} \\ &\quad \sum_{(l_1, \dots, l_d): l_1 + \dots + l_d = t} \left| EH_{m_1}(Y_1) \dots H_{m_d}(Y_d) \frac{H_{l_1}(X_1) \dots H_{l_d}(X_d)}{l_1! \dots l_d!} \right| \\ &\leq \psi^t \sum_{(l_1, \dots, l_d): l_1 + \dots + l_d = t} a_{l_1, \dots, l_d}^2(t) = \psi^t Ef_t^2(X_1, \dots, X_d), \end{aligned}$$

i.e. relation (6) holds true.

Next I prove an identity which is actually not applied in the proof of the Lemma. But relation (7) can be proved by a small modification of this proof which may explain the idea behind the Lemma. To formulate this identity I fix a vector $(l_1^{(0)}, \dots, l_d^{(0)})$ with non-negative integer coordinates such that $l_1^{(0)} + \dots + l_d^{(0)} = t$. I formulate the following identity with its help.

$$\begin{aligned} &\sum_{(m_1, \dots, m_d): m_1 + \dots + m_d = t} EH_{l_1^{(0)}}(X_1) \dots H_{l_d^{(0)}}(X_d) \frac{H_{m_1}(Y_1) \dots H_{m_d}(Y_d)}{m_1! \dots m_d!} \\ &= \prod_{i=1}^d (r(i, 1) + \dots + r(i, d))^{l_i^{(0)}}. \end{aligned} \tag{8}$$

In the proof of (8) all terms in the sum at the left-hand side of this formula will be calculated with the help of the diagram formula. In this proof the numbers $l_i^{(0)}$, m_j , $u(\cdot, \cdot)$ will always denote non-negative integers.

Let us fix some vector $(m_1, \dots, m_d)$ such that $m_1 + \dots + m_d = t$, and let us compute the expected value

$$EH_{l_1^{(0)}}(X_1) \dots H_{l_d^{(0)}}(X_d) \frac{H_{m_1}(Y_1) \dots H_{m_d}(Y_d)}{m_1! \dots m_d!} \tag{9}$$

by means of the diagram formula.

In this calculation only such diagrams $\gamma \in \Gamma(l_1^{(0)}, \dots, l_d^{(0)}, m_1, \dots, m_d)$ have to be considered in which all edges connect a vertex from a row indexed by i with some $1 \leq i \leq d$ with a vertex from a row indexed by $d+j$ with some $1 \leq j \leq d$, i.e. a vertex from a row corresponding to some X_i with a vertex in a row corresponding to some Y_j . The omitted diagrams yield zero contribution in the diagram formula.

Let us define a partition of these diagrams in the following way. Define the set $\mathcal{U}(l_1^{(0)}, \dots, l_d^{(0)}, m_1, \dots, m_d)$ consisting of some vectors by the formula

$$\mathcal{U}(l_1^{(0)}, \dots, l_d^{(0)}, m_1, \dots, m_d) = \left\{ \{u(i, j), 1 \leq i, j \leq d\} : \right. \\ \left. \sum_{j=1}^d u(i, j) = l_i^{(0)} \text{ for all } 1 \leq i \leq d, \quad \sum_{i=1}^d u(i, j) = m_j \text{ for all } 1 \leq j \leq d \right\},$$

and introduce the set of diagrams

$$\Gamma(\{u(i, j), 1 \leq i, j \leq d\}) = \{\gamma : \gamma \in \Gamma(l_1^{(0)}, \dots, l_d^{(0)}, m_1, \dots, m_d), \\ \text{there are } u(i, j) \text{ vertices between the } i\text{-th and } d+j\text{-th rows of } \gamma, 1 \leq i, j \leq d\}$$

for all $\{u(i, j), 1 \leq i, j \leq d\} \in \mathcal{U}(l_1^{(0)}, \dots, l_d^{(0)}, m_1, \dots, m_d)$.

Let us consider the sets $\Gamma(\{u(i, j), 1 \leq i, j \leq d\})$ with all possible arguments $\{u(i, j), 1 \leq i, j \leq d\} \in \mathcal{U}(l_1^{(0)}, \dots, l_d^{(0)}, m_1, \dots, m_d)$. They give a partition of those diagrams of $\Gamma(l_1^{(0)}, \dots, l_d^{(0)}, m_1, \dots, m_d)$ that have to be taken into account when the expression in (9) is computed by means of the diagram formula. For each argument $\{u(i, j), 1 \leq i, j \leq d\}$ the set $\Gamma(\{u(i, j), 1 \leq i, j \leq d\})$ contains

$$\frac{\prod_{i=1}^d l_i^{(0)}!}{\prod_{1 \leq i, j \leq d} u(i, j)!} \cdot \frac{\prod_{j=1}^d m_j!}{\prod_{1 \leq i, j \leq d} u(i, j)!} \prod_{1 \leq i, j \leq d} u(i, j)! = \frac{\prod_{i=1}^d l_i^{(0)}! \prod_{j=1}^d m_j!}{\prod_{1 \leq i, j \leq d} u(i, j)!}$$

diagrams, and all these diagrams have weight $\prod_{1 \leq i, j \leq d} r(i, j)^{u(i, j)}$. Hence in the calculation of the expression in (9) by means of the diagram formula the sum of the contributions of the weights of the diagrams $\gamma \in \Gamma(\{u(i, j), 1 \leq i, j \leq d\})$ equals

$$\frac{\prod_{i=1}^d l_i^{(0)}! \prod_{1 \leq i, j \leq d} r(i, j)^{u(i, j)}}{\prod_{1 \leq i, j \leq d} u(i, j)!} = \prod_{i=1}^d \left(l_i^{(0)}! \left(\prod_{j=1}^d \frac{r(i, j)^{u(i, j)}}{u(i, j)!} \right) \right).$$

The sum of these expressions for all $\{u(i, j), 1 \leq i, j \leq d\} \in \mathcal{U}(l_1^{(0)}, \dots, l_d^{(0)}, m_1, \dots, m_d)$

equals the expression in formula (9). In such a way we get the following identity:

$$\begin{aligned}
& EH_{l_1^{(0)}}(X_1) \cdots H_{l_d^{(0)}}(X_d) \frac{H_{m_1}(Y_1) \cdots H_{m_d}(Y_d)}{m_1! \cdots m_d!} \\
&= \sum_{\substack{\{(u(i,j), 1 \leq i,j \leq d): \\ \sum_{j=1}^d u(i,j)=l_i^{(0)}, 1 \leq i \leq d, \\ \sum_{j=1}^d u(i,j)=m_j, 1 \leq j \leq d\}}} \prod_{i=1}^d \left(l_i^{(0)}! \left(\prod_{j=1}^d \frac{r(i,j)^{u(i,j)}}{u(i,j)!} \right) \right). \tag{10}
\end{aligned}$$

By summing up these identities for all vectors $(m_1, \dots, m_d)$ such that $m_1 + \dots + m_d = t$ we get that

$$\begin{aligned}
& \sum_{(m_1, \dots, m_d): m_1 + \dots + m_d = t} EH_{l_1^{(0)}}(X_1) \cdots H_{l_d^{(0)}}(X_d) \frac{H_{m_1}(Y_1) \cdots H_{m_d}(Y_d)}{m_1! \cdots m_d!} \\
&= \sum_{\substack{\{(u(i,j), 1 \leq i,j \leq d): \\ \sum_{j=1}^d u(i,j)=l_i^{(0)}, 1 \leq i \leq d\}}} \prod_{i=1}^d \left(l_i^{(0)}! \left(\prod_{j=1}^d \frac{r(i,j)^{u(i,j)}}{u(i,j)!} \right) \right). \tag{11}
\end{aligned}$$

To prove identity (8) I present a new identity in the next formula (12). In the second line of this formula the expression at the right-hand side of formula (11) is rewritten as the product of certain expressions. Then this product is rewritten in a simpler form by means of the multinomial theorem. Here is formula (12).

$$\begin{aligned}
& \sum_{\substack{\{(u(i,j), 1 \leq i,j \leq d): \\ \sum_{j=1}^d u(i,j)=l_i^{(0)}, 1 \leq i \leq d\}}} \prod_{i=1}^d \left(l_i^{(0)}! \left(\prod_{j=1}^d \frac{r(i,j)^{u(i,j)}}{u(i,j)!} \right) \right) \\
&= \prod_{i=1}^d \left(\sum_{\substack{\{(u(i,j), 1 \leq j \leq d): \\ \sum_{j=1}^d u(i,j)=l_i^{(0)}\}}} l_i^{(0)}! \left(\frac{r(i,1)^{u(i,1)}}{u(i,1)!} \cdots \frac{r(i,d)^{u(i,d)}}{u(i,d)!} \right) \right) \\
&= \prod_{i=1}^d (r(i,1) + \cdots + r(i,d))^{l_i^{(0)}}. \tag{12}
\end{aligned}$$

Formulas (11) and (12) together imply identity (8).

To prove the first identity in formula (12) let us first observe that if we carry out the term by term multiplications in the product of sums at the right-hand side of this identity, then both sides of this identity are sums of terms which have the following form:

$$\prod_{i=1}^d l_i^{(0)}! \left(\frac{r(i,1)^{u(i,1)}}{u(i,1)!} \cdots \frac{r(i,d)^{u(i,d)}}{u(i,d)!} \right).$$

We have to understand that the same terms appear in the sums of the two sides of this identity. But in both cases those terms of the above form appear in these sums which are defined by those vectors $(u(i, 1), \dots, u(i, d))$, $1 \leq i \leq d$, which satisfy the identity $u(i, 1) + \dots + u(i, d) = l_i^{(0)}$ for all $1 \leq i \leq d$. Hence the first identity in (12) holds.

The second identity in formula (12) is a consequence of the multinomial theorem.

The proof of the first inequality in the first statement of formula (7) can be obtained with some small, natural modifications in the proof of relation (8).

Let us first observe that the absolute value of the expression at the left-hand side of (10) is smaller than the expression we get by replacing the terms $r(i, j)$ by $|r(i, j)|$ at the right-hand side of this identity. Then summing up these inequalities we get such a version of relation (11) where in the sum at the left-hand side the summands are replaced by their absolute value, at the right-hand side the variables $r(i, j)$ are replaced by their absolute value $|r(i, j)|$, and in this new version of formula (11) the expression at the left-hand side is less than or equal to the expression at the right-hand side.

Formula (12) remains valid if we replace $r(i, j)$ by $|r(i, j)|$ in it. These new versions of formulas (11) and (12) imply the first inequality in the first statement of (7).

The proof of the second inequality in the first statement of (7) is much simpler.

Since $|r(i, 1)| + \dots + |r(i, d)| \leq \psi$ for all $1 \leq i \leq d$, and $l_1^{(0)} + \dots + l_d^{(0)} = t$

$$\prod_{i=1}^d (|r(i, 1)| + \dots + |r(i, d)|)^{l_i^{(0)}} \leq \prod_{i=1}^d \psi^{l_i^{(0)}} = \psi^t.$$

The second statement of (7) is equivalent to the first one. Only the variables $l_i^{(0)}$ and m_j must be replaced by the variables $m_j^{(0)}$ and l_i , $1 \leq i, j \leq d$, and the role of the random vectors $(X_1, \dots, X_d)$ and $(Y_1, \dots, Y_d)$ have to be exchanged. Lemma 1 in Arcones' paper is proved.