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**Probability Theory and Related Fields**



# On the Tail Behaviour of the Distribution Function of Multiple Stochastic Integrals

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**Summary.** Let $F_n(u)$ denote the empirical distribution function of a sample of i.i.d. random variables with uniform distribution on $[0,1]$. Define $\bar{\mu}_n^*(u)=\sqrt{n}[F_n(u)-u]$, and consider the integrals $I(t)=\int_0^t\int_0^1\dots\int_0^1 f(u_1,\dots,u_s)$ $\cdot\,\bar{\mu}_n^*(du_1)\dots\mu_n^*(du_s)$, where $f$ is a bounded measurable function. We give a good upper bound on the probability $P\left(\sup\limits_{0\leqq t\leqq 1}|I(t)|\geqq x\right)$. An analogous estimate is given for multiple integrals with respect to a Poisson process.

## Introduction

The main results of this paper are the following two theorems: Let $\xi_1,\xi_2,\dots$ be a sequence of i.i.d. random variables with uniform distribution on the interval $[0,1]$, define the empirical distribution function $F_n(u)=\frac{1}{n}\sum\limits_{i=1}^n I(\xi_i<u)$ and its standardization $\bar{\mu}_n^*(u)=\sqrt{n}(F_n(u)-u)$. We shall prove the following.

**Theorem 1.** *If $f(u_1,\dots,u_s)$ is a bounded measurable function, $|f(u_1,\dots,u_s)|\leqq K$ then there exist some universal constants $C_s>0$ and $\alpha_s>0$ depending only on the dimension $s$ such that*

$$P\left(\sup_{0\leqq t\leqq 1}\left|\int_0^t\int_0^1\dots\int_0^1 f(u_1,\dots,u_s)\,\bar{\mu}_n^*(du_1)\dots\bar{\mu}_n^*(du_s)\right|>x\right)$$
$$\leqq C_s\exp\left(-\alpha_s\frac{x^{2/s}}{K^{2/s}}\right) \tag{1.1}$$

*for all $x>0$.*

Let $P_n(u)$ denote a Poisson process on $[0,1]$ with parameter $n$, i.e. let $P_n(u)$, $0\leqq u\leqq 1$, be a process with independent increments such that $P_n(0)=0$ and

$P_n(v)-P_n(u)$, $0\leqq u<v\leqq 1$, is Poisson distributed with parameter $n(v-u)$. Set $\mu_n^*(u)=\frac{1}{\sqrt{n}}(P_n(u)-nu)$. We shall prove the following.

**Theorem 2.** *Let $f(u_1,\ldots,u_s)$ be a bounded measurable function $|f(u_1,\ldots,u_s)|\leqq K$. There exist some universal constants $C_s>0$ and $\alpha_s>0$ depending only on the dimension $s$ such that*

$$P\left(\sup_{0\leqq t\leqq 1}\left|\int_0^t\int_0^1\cdots\int_0^1 f(u_1,\ldots,u_s)\,\mu_n^*(du_1)\ldots\mu_n^*(du_s)\right|>x\right)$$
$$\leqq C_s\exp\left(-\alpha_s\frac{x^{2/s}}{K^{2/s}}\right) \tag{1.2}$$

*for all $0\leqq x<Kn^{s/2}$.*

The investigations leading to the proof of Theorem 1 were motivated by paper [1]. Here an estimate of this type was needed to bound an error term. We could prove Theorem 1 only by first proving Theorem 2 and then applying a Poisson approximation.

We wrote $\sup\limits_{0\leqq t\leqq 1}$ in formulas (1.1) and (1.2) because we needed such an estimate in [1]. It is natural to expect that the estimates would not improve considerably if the sup were dropped in formulas (1.1) and (1.2). This is really so, but we had to overcome several technical difficulties when we proved that the sup can be inserted into formulas (1.1) and (1.2). Actually the greatest part of the paper deals with this problem.

The need for a $\sup\limits_{0\leqq t\leqq 1}$ in formulas (1.1) and (1.2) arose in [1] in a natural way. We had to show that a process defined by a multiple stochastic integral of the same type as in (1.1), and multiplied by a small number can be considered as a small error term. The sup in Theorems 1 and 2 guarantees that also this process is small, and not only its values at a fixed time $t$. We expect that similar problems arise in other investigations. It seems very likely that the sup could be taken in a more general way. But since this would make the paper more complicated we do not investigate this question.

The following simple example gives some information about the sharpness of Theorems 1 and 2. Choose $f(u_1,\ldots,u_s)=\prod\limits_{j=1}^{s} I(u_j<\frac{1}{2})$. In this case

$$P\left(\int_0^1\cdots\int_0^1 f(u_1,\ldots,u_s)\,\bar\mu_n^*(du_1)\ldots\bar\mu_n^*(du_s)>x\right)=P(\bar\mu_n^*([0,\tfrac{1}{2}])>x^{1/s}), \tag{1.3}$$

and the expression at the right hand side of (1.3) is of order $\exp\left(-\frac{x^{2/s}}{4}\right)$ by the central limit theorem. This means that in this case Theorem 1 gives a good estimate. The same can be told about Theorem 2. Moreover, one can understand with the help of this example that the condition $x<Kn^{s/2}$ in Theorem 2 is essential. Indeed, since $P(\mu_n^*([0,\frac{1}{2}])>x\sqrt{n})>\exp(-cnx\log x)\gg\exp(-cnx^2)$ if $x\gg 1$, the same choice of the function $f$ as in (1.3) and a simple calculation show that for $x=x_n\gg n^{s/2}$ relation (1.2) does not hold any longer.

It is natural to ask whether in (1.1) and (1.2) the constant $K=\sup|f(u_1,\dots,u_s)|$ can be replaced by $K_p=\left[\int_0^1\dots\int_0^1 |f(u_1,\dots,u_s)|^p du_1\dots du_s\right]^{1/p}$ with some $\infty>p>0$. The norm $K_2$ would be a natural candidate. The following simple example shows that the answer to this question is in the negative. Set $s=1$ and $f(u)=I(u\leqq\varepsilon_n)$. $2x\sqrt{n}$ with $\varepsilon_n=n^{-\beta}$, where $\beta>1$ is appropriately chosen. Then

$$P\left(\int_0^1 f(u)\bar{\mu}_n^*(du)>x\right)=P\text{ (one of the sample points }\xi_i,\ 1\leqq i\leqq n,$$

$$\text{belongs to the interval }[0,\varepsilon_n])>\varepsilon_n=n^{-\beta}. \tag{1.4}$$

On the other hand $K_p=2x\sqrt{n}\,\varepsilon_n^{1/p}$ hence $c\exp\left(\frac{\alpha x^2}{K_p^2}\right)=c\exp\left(-\frac{\alpha}{4}n^{1-\frac{2\beta}{p}}\right)$, and this expression does not bound (1.4) if $\beta>p$.

We shall prove Theorems 1 and 2 in a slightly different formulation. Let us introduce the measures

$$\bar{\mu}_n(u)=n(F_n(u)-u) \tag{1.5}$$

and

$$\mu_n(u)=P_n(u)-nu. \tag{1.5$'$}$$

We formulate the following

**Theorem 1′.** *Let $|f(u_1,\dots,u_s)|\leqq 1$. There exist some universal constants $c_1>0$, $c_2>0$, $C>0$ and $\alpha>0$ depending only on the dimension s such that*

$$P\left(\sup_{0\leqq t\leqq 1}\left|\int_0^t\int_0^1\dots\int_0^1 f(u_1,\dots,u_s)\,\bar{\mu}_n(du_1)\dots\bar{\mu}_n(du_s)\right|>x\right)\leqq C\exp\left(-\alpha\frac{x^{2/s}}{n}\right) \tag{1.6}$$

*for $c_1n^{s/2}<x<c_2n^s$.*

*and*

**Theorem 2′.** *Let $|f(u_1,\dots,u_s)|\leqq 1$. There exist some universal constants $c_1>0$, $c_2>0$, $C>0$ and $\alpha>0$ depending only on the dimension s such that*

$$P\left(\sup_{0\leqq t\leqq 1}\left|\int_0^t\int_0^1\dots\int_0^1 f(u_1,\dots,u_s)\,\mu_n(du_1)\dots\mu_n(du_s)\right|>x\right)\leqq C\exp\left(-\alpha\frac{x^{2/s}}{n}\right) \tag{1.7}$$

*for $c_1n^{s/2}<x<c_2n^s$.*

Theorems 1′ and 2′ imply Theorems 1 and 2. This can be seen with the help of a natural rescaling and the following two observations: 1) The condition $x>c_1n^{s/2}$ can be dropped if $C$ is chosen sufficiently large. Indeed, if $C$ is chosen sufficiently large in Theorems 1′ and 2′ then the right-hand side of the inequalities will be larger than 1 for $x\leqq c_1n^{s/2}$, and the inequalities hold in this case trivially. 2) The condition $x<c_2n^s$ can be dropped from Theorem 1′ if $\alpha>0$ is chosen sufficiently small. Indeed, since $\bar{\mu}_n$ is the difference of two measures with total volume $n$, hence

$$\left|\int_0^t\int_0^1\dots\int_0^1 f(u_1,\dots,u_s)\,\bar{\mu}_n(du_1)\dots\bar{\mu}_n(du_s)\right|\leqq 2^sn^s$$

with probability 1, and Theorem 1′ holds for $x > 2^s n^s$. Since it holds for $x = c_2 n^s$, it also holds for $c_2 n^s < x < 2^s n^s$ if $\alpha$ is chosen in (1.6) sufficiently small.

The paper consists of four sections. In Sect. 2 we prove an inequality which is important in the proof of Theorem 2′. This inequality implies immediately a weakened version of Theorem 2′, where the sup is omitted from (1.7). Section 3 contains the proof of Theorem 2′ and Sect. 4 the proof of Theorem 1′.

*Remark.* We define the integrals with respect to $\mu_n$, $\bar{\mu}_n$, etc. as usual Lebesgue integrals with respect to the product measure induced by the processes $\mu_n(t)$ and $\bar{\mu}_n(t)$. There is no problem with this definition, since $\mu_n(t)$ and $\bar{\mu}_n(t)$ have finite total variation with probability 1. In probability literature one often defines integrals with respect to Poisson processes differently, namely the diagonals $u_i = u_j$ are deleted from the domain of integration. Our results easily follow for such integrals if we consider such functions which vanish at the diagonals.

## 2. Some Estimates

Let us introduce the random variables

$$\eta_f(t) = \int_0^t \int_0^1 \dots \int_0^1 f(u_1, \dots, u_s)\, \mu_n(du_1) \dots \mu_n(du_s), \tag{2.1}$$

where $\mu_n$ is the same as in (1.5)′.

Our main task in this section is to give a good upper bound on the probabilities $P(|\eta_f(t+v) - \eta_f(v)| > z)$. Obviously

$$P(|\eta_f(t+v) - \eta_f(t)| > z) \leqq \frac{E([\eta_f(t+v) - \eta_f(v)]^{2k})}{z^{2k}} \tag{2.2}$$

for arbitrary $z > 0$ and positive integer $k$. Inequality (2.2) gives us a good estimate, if we can make a good estimate on the moments of $\eta_f(t+v) - \eta_f(v)$, and choose the number $k$ in (2.2) appropriately. For this sake first we prove the following two lemmas. In the sequel we use the letters $C$, $C_1$, $\alpha$, etc. for some appropriate constant. The same letter may denote different constants in different formulas.

**Lemma 1.** *Let $\xi$ be a Poisson distributed random variable with parameter $\lambda$. Then*

*a) $E(\xi - E\xi)^k \geqq 0$ for all $k = 0, 1, 2, \dots$*
*b) There exists some $C > 0$ such that $E(\xi - E\xi)^k \leqq C^k (k\lambda)^{k/2}$ for all $k \leqq \lambda$.*

**Lemma 2.** *Let $f(u_1, \dots, u_s)$ and $g(u_1, \dots, u_s)$ be two bounded measurable functions such that $|g(u_1, \dots, u_s)| \leqq f(u_1, \dots, u_s)$ for all $u_1, \dots, u_s$. Then*

$$E\left[\int_0^1 \dots \int_0^1 g(u_1, \dots, u_s)\, \mu_n(du_1) \dots \mu_n(du_s)\right]^k$$
$$\leqq E\left[\int_0^1 \dots \int_0^1 f(u_1, \dots, u_s)\, \mu_n(du_1) \dots \mu_n(du_s)\right]^k \tag{2.3}$$

*for all $k = 0, 1, 2, \dots$.*

*Remark 2.* Lemma 2 enables us to estimate the moments of $\eta_f(t+u)-\eta_f(u)$. Actually it is Lemma 2 which makes the proof of Theorem 2′ simpler than that of Theorem 1′. If the measure $\mu_n$ is replaced by $\bar{\mu}_n$ in (2.3) then Lemma 2 does not hold any longer. This can be seen with the help of the following observation: For $f(u_1,\ldots,u_s)\equiv 1$

$$\int_0^1 \ldots \int_0^1 f(u_1,\ldots,u_s)\,\bar{\mu}_n(du_1)\ldots\bar{\mu}_n(du_s)=0$$

with probability 1.

*Proof of Lemma 1.* The moment generating function of $\xi-E\xi$ is

$$f(t)=E\exp\left[t(\xi-E\xi)\right]=\exp\{\lambda(e^t-t-1)\}$$

and
$$E(\xi-E\xi)^k=\frac{d^k}{dt^k}f(t)|_{t=0}. \tag{2.4}$$

Since $\dfrac{d^k}{dt^k}(\lambda(e^t-t-1))\geqq 0$ for all $k=0,1,2,\ldots$, it is not difficult to see that $\dfrac{d^k}{dt^k}f(t)|_{t=0}\geqq 0$. This implies part a) of Lemma 1.

Let us consider $f(t)$ as a complex valued analytic function. Then we get by Cauchy's formula that

$$\left|\frac{d^k}{dt^k}f(t)|_{t=0}\right| = \left|\frac{k!}{2\pi i}\int_{|\zeta|=r}\frac{f(\zeta)}{\zeta^{k+1}}d\zeta\right|\leqq k!\cdot\sup_{|\zeta|=r}|f(\zeta)|\,\frac{1}{r^k} \tag{2.5}$$

for all $r>0$.

We have
$$\sup_{|\zeta|=r}|f(\zeta)|=\exp(\lambda(e^r-r-1))\leqq\exp\left(\lambda\frac{r^2}{2}e^r\right) \tag{2.6}$$

choose $r=\sqrt{\dfrac{k}{\lambda}}\leqq 1$. Then (2.4), (2.5) and (2.6) imply that

$$E(\xi-E\xi)^k\leqq k!\left(\frac{k}{\lambda}\right)^{-\frac{k}{2}}\exp\left(\frac{\lambda}{2}\frac{k}{\lambda}e^r\right)\leqq C^k k^k\left(\frac{k}{\lambda}\right)^{-\frac{k}{2}}\exp\left(\frac{e}{2}k\right)\leqq\bar{C}^k\cdot(k\lambda)^{\frac{k}{2}}.$$

Lemma 1 is proved.

*Remark 3.* Part b) of Lemma 1 can be interpreted in the following way. The $k$-th moment of $\xi-E\xi$ is smaller than the $k$-th absolute moment of a normal random variable with expectation zero and variance const. $\lambda$. Some calculation would show that for $k\gg\lambda$ this relation does not hold, i.e. the restriction $k\leqq\lambda$ in part b) of Lemma 1 is essential.

*Proof of Lemma 2.* First we restrict ourselves to the special case when $f$ and $g$ are step functions. More precisely we assume that there are some constants $0=a(1)<a(2)<\ldots<a(p)=1$ such that both $f$ and $g$ are constants on all rectangles $\prod_{j=1}^{s}[a(i_j),a(i_j+1)]$. Then

$$\int f(u_1,\ldots,u_s)\,\mu_n(du_1)\ldots\mu_n(du_s)=\sum A_i^f\prod_{j=1}^{s}\mu_n([a(i_j),a(i_j+1)])$$

with

$$A_i^f = f(u_{i_1}, \dots, u_{i_s}),\ u_{i_j} \in [a(i_j), a(i_j+1)],\ j = 1, \dots, s,$$

and

$$\begin{aligned} & E[\int f(u_1, \dots, u_s)\, \mu_n(du_1) \dots \mu_n(du_s)]^k \\ &= \sum_{i_1, \dots, i_k} A_{i_1}^f \dots A_{i_k}^f E \prod_{j=1}^{s} \mu_n([a(i_{1_j}), a(i_{1_j}+1)]) \dots \mu_n([a(i_{k_j}), a(i_{k_j}+1)]). \end{aligned} \tag{2.7}$$

A similar formula can be written for the $k$-th moment of the integral of $g$, only $A_i^f$ must be replaced by $A_i^g$. Observe that the terms $E(\ )$ on the right-hand side of (2.7) are non-negative because of part a) of Lemma 1 and the independence of the random variables $\mu_n([a(i), a(i+1)])$, $i = 1, \dots, p-1$. Since $|A_i^g| \leqq A_i^f$ by the conditions of Lemma 2, the above fact together with formula (2.7) and its version for the integral of the function $g$ imply Lemma 2 in this special case. Then a simple limiting procedure supplies the proof in the general case.

The main result of this section is the following

**Proposition 3.** *Let $f$ be a measurable function such that $|f(u_1, \dots, u_s)| \leqq 1$, and define $\eta_f(t)$ by formula (2.1). If $0 \leqq v < t + v \leqq 1$, $d_1 t^{1/2} n^{s/2} < z < d_2 t^{\frac{s+1}{2}} n^s$ with appropriate $d_1 > 0$, $d_2 > 0$ then there exist some $\alpha > 0$ and $\alpha_1 > 0$ such that*

*a)* $P(|\eta_f(t+v) - \eta_f(v)| > z) \leqq \exp\left(-\alpha \frac{z^{2/s}}{n t^{1/s}}\right)$

*b)* $P(|\eta_f(t+v) - \eta_f(v)| > Bz) \leqq \exp\left(-(\alpha + \alpha_1 \log B) \frac{z^{2/s}}{n t^{1/s}}\right)$

*for arbitrary $B > 1$.*

*The constants $d_1$, $d_2$, $\alpha_1$ and $\alpha$ depend only on the dimension $s$.*

*Proof of Proposition 3.* We prove part a) with the help of (2.2). We want to estimate $E[\eta_f(t+v) - \eta_f(v)]^{2k}$. To this end let us introduce the function $h(u_1, \dots, u_s) = h_{t,v}(u_1, \dots, u_s)$

$$h_{t,v}(u_1, \dots, u_s) = \begin{cases} 1 & \text{if } v < u_1 < t+v \\ 0 & \text{otherwise} \end{cases}$$

Then because of Lemma 2 and the Schwarz inequality

$$\begin{aligned} E[\eta_f(t+v) - \eta_f(v)]^{2k} &\leqq E\left[\int_0^1 \dots \int_0^1 h(u_1, \dots, u_s)\, \mu_n(du_1) \dots \mu_n(du_s)\right]^{2k} \\ &= E\{\mu_n([0,1])^{2k(s-1)}\, \mu_n([v, t+v])^{2k}\} \\ &\leqq E(\mu_n([0,1])^{4k(s-1)})^{1/2}\, E(\mu_n([v, t+v])^{4k})^{1/2}. \end{aligned} \tag{2.8}$$

Choose $k = \left[\beta \frac{z^{2/s}}{n t^{1/s}}\right]$, where $\beta > 0$ is a sufficiently small fixed number, and [ ] denotes integer part. If the constants $d_1 > 0$, $d_2 > 0$ and $\beta > 0$ are appropriately chosen then $1 \leqq k \leqq \frac{nt}{4s}$ for sufficiently large $n$.

On the other hand $\mu_n([v,t+v]) = P_n([v,v+t]) - EP_n([v,t+v])$, and $P_n([v,t+v])$ is Poisson distributed with parameter $nt$. A similar statement holds for $\mu_n([0,1])$. Hence we can apply part b) of Lemma 1 to estimate the right-hand side of (2.8). In this way we get that

$$E[\eta_f(t+v) - \eta_f(v)]^{2k} \leqq C^k t^k (kn)^{ks} \tag{2.9}$$

with our choice of $k$. Since

$$\frac{Ctk^s n^s}{z^2} \leqq \frac{1}{2} \frac{Ctz^2\beta^s n^s}{z^2 n^s t} = \frac{1}{2} C\beta^s,$$

formula (2.2) and (2.9) imply that if $\beta > 0$ is chosen so small that $\frac{1}{2} C\beta^s < 1$ (first we choose $\beta$ then $d_1$ and $d_2$ during the proof) then

$$P(|\eta(t+v) - \eta(v)| > z) \leqq \left(\frac{1}{2} C\beta^s\right)^k \leqq \exp\left(-\alpha \frac{z^{2/s}}{nt^{1/s}}\right).$$

Part a) is proved.

The proof of part b) is the same. In this case we get with the same choice of $k$ that

$$P(|\eta(t+v) - \eta(v)| > Bz) \leqq \frac{E(\eta(t+v) - \eta(v))^{2k}}{B^{2k} z^{2k}}$$

$$\leqq \exp\left(-\alpha \frac{z^{2/s}}{nt^{1/s}} - 2k \log B\right) = \exp\left(-(\alpha + \alpha_1 \log B) \frac{z^{2/s}}{nt^{1/s}}\right).$$

*Remark 4.* Part a) of Proposition 3 with $t=1$ and $v=0$ implies that $P(|\eta_f(1)| > z) \leqq \exp\left(-\alpha \frac{z^{2/s}}{nt^{1/s}}\right)$ if $d_1 n^{s/2} < z < d_2 n^s$, i.e. Theorem 2′ hold if the sup in (1.7) is dropped. Theorem 1′ without the sup in (1.6) can also be deduced from this result. It is enough to apply the same Poisson approximation of an empirical process as in Sect. 4, only the argument of the proof becomes much simpler.

## 3. The Proof of Theorem 2′

First we formulate the following

**Proposition 4.** *Let $|f(u_1, \ldots, u_s)| \leqq 1$ be a measurable function, and define $\eta_f(t)$ by formula (2.1). Let $c_1 n^{s/2} < x < c_2 n^s$ with some appropriate $c_1 > 0$, $c_2 > 0$ which depend only on the dimension d, and define the number $t_0 = t_0(x,n)$ by the relations $t_0 = \frac{1}{2^j}$, j is integer, and $x^{\frac{2}{s+1}} n^{-\frac{2s}{s+1}} \leqq t_0 < 2x^{\frac{2}{s+1}} n^{-\frac{2s}{s+1}}$. The following inequalities hold true:*

a) $$P\left(\max_{0 \leqq l < \frac{1}{t_0}} |\eta_f(lt_0)| > \frac{x}{2}\right) \leqq \exp\left(-\alpha \frac{x^{2/s}}{n}\right)$$

b) $$P\left(\max_{0 \leqq l < \frac{1}{t_0}} \sup_{lt_0 \leqq t < (l+1)t_0} \left| n \int_{lt_0}^{t} \int_0^1 \ldots \int_0^1 f(u_1, \ldots, u_s)\, du_1\, \mu_n(du_2) \ldots \mu_n(du_s) \right| > \frac{x}{4}\right) \leqq \exp\left(-\alpha \frac{x^{2/s}}{n}\right)$$

c) $$P\left(\max_{0\le l<\frac{1}{t_0}} \sup_{lt_0<t<(l+t)t_0} \left| \int_{lt_0}^{t} \int_0^1 \dots \int_0^1 f(u_1,\dots,u_s)\, P_n(du_1)\, \mu_n(du_2)\dots\mu_n(du_s)\right| > \frac{x}{4}\right) \le \exp\left(-\alpha \frac{x^{2/s}}{n}\right)$$

*The constants $c_1$ and $c_2$ are the same as in Theorem 2 (we shall choose $c_2 < 1$, so that $t_0 \le 1$).*

Let us observe that Theorem 2′ follows from Proposition 4. Indeed, part a) reduces the proof of Theorem 2′ to the estimation of

$$P\left(\sup_{0<l<\frac{1}{t_0}} \sup_{lt_0<t<(l+l)t_0} |\eta_f(t)-\eta_f(lt_0)| > \frac{x}{2}\right),$$

and this is done in parts b) and c) with the decomposition $\mu_n(du_1) = P_n(du_1) - n\,du_1$. Part a) of Proposition 4 will be proved with the help of part a) of Proposition 3 with the same halving procedure, as it is done e.g. is the proof of Kolmogorov's continuity theorem.

But in this way we can give a good bound on the tail behaviour of $\sup_{l=1,2,\dots} |\eta_f(lt_0)|$ only if $t_0$ is not to small. This is the reason why part a) had to be treated separately. On the other hand, when we want to bound $|\eta_f(t)-\eta_f(t')|$, $|t-t'|<t_0$, then we can make the decomposition $\mu_n(du_1) = P_n(du_1) - n\,du_1$, as it is done in the formulation of parts b) and c), i.e. we do not have to exploit the cancellation which is caused by the fact that $\mu_n$ is a signed measure.

*Proof of Proposition 4*

*Part a).*

$$P\left(\sup_{0\le l<2^j} \left|\eta_f\left(\frac{l}{2^j}\right)\right| > \frac{x}{2}\right) \le \sum_{p=1}^{j} \sum_{l=1}^{2^p} P\left(\left|\eta_f\left(\frac{l}{2^p}\right)-\eta_f\left(\frac{l-1}{2^p}\right)\right| > \frac{x(A-1)}{2A^{p+1}}\right) \tag{3.1}$$

where $A>1$ is arbitrary. To see why relation (3.1) holds one has to observe that if $\left|\eta_f\left(\frac{l}{2^p}\right)-\eta_f\left(\frac{l-1}{2^p}\right)\right| \le \frac{x(A-1)}{2A^{p+1}}$ for all pairs $(l,p)$ such that $p\le j$, $l\le 2^p$ then $\left|\eta_f\left(\frac{k}{2^j}\right)\right| \le \frac{x}{2}$ for all $k=0,1,\dots,2^j$. Indeed, in this case we can write

$$\left|\eta_f\left(\frac{k}{2^j}\right)\right| \le \sum_{p=0}^{2^j} \sup_{l\le 2^p} \left|\eta_f\left(\frac{l}{2^p}\right)-\eta_f\left(\frac{l-1}{2^p}\right)\right| \le \frac{x(A-1)}{2A} \sum_{p=0}^{2^j} \frac{1}{A^p} \le \frac{x}{2}.$$

We shall estimate the terms at the right-hand side of (3.1) with the help of part a) of Proposition 3. First we have to check that the conditions of Proposition 3 are satisfied with $t=\frac{1}{2^p}$ and $z=\frac{(A-1)x}{2A^{p+1}}$, i.e. we have to show that

$$d_1 2^{-\frac{p}{2}} n^{\frac{s}{2}} < \frac{(A-1)x}{2A^{p+1}} < d_2 2^{-p\frac{s+1}{2}} n^s. \tag{3.2}$$

The left hand side of (3.2) holds if we choose $A<\sqrt{2}$ and $c_1>\frac{4d_1}{A-1}$. On the other hand, since $2^{-p}\geqq 2^{-j}\geqq x^{\frac{2}{s+1}}n^{-\frac{2s}{s+1}}$ the right hand side of (3.2) holds if $A$ is chosen so that $A-1<d_2$. Hence relations (3.1), (3.2) and Proposition 3 imply that

$$P\left(\sup_{0\leqq l\leqq 2^j}\left|\eta_f\left(\frac{l}{2^j}\right)\right|\geqq\frac{x}{2}\right)\leqq\sum_{p=1}^{j}2^p\exp\left(-\alpha\frac{(A-1)^{2/s}x^{2/s}2^{p/s}}{2^{2/s}A^{\frac{2(p+1)}{s}}n}\right)$$

$$=\sum_{p=1}^{j}2^p\exp\left(-\frac{\bar\alpha\lambda^p x^{2/s}}{n}\right)\leqq\exp\left(-\frac{\bar\alpha}{2}\frac{x^{2/s}}{n}\right)\tag{3.3}$$

with $\bar\alpha=\left(\frac{A-1}{2A}\right)^{2/s}$ and $\lambda=2^{1/s}A^{-\frac{2}{s}}>1$ (because of $A<\sqrt{2}$) if $x>c_1n^{s/2}$, and $c_1>0$ is chosen sufficiently large. The last inequality in (3.3) holds since $2^p\exp\left(-\frac{\bar\alpha}{2}\frac{\lambda^p x^{2/s}}{n}\right)<2^{-p}$ if $c_1$ is sufficiently large. Part a) is proved.

*Part b).* We apply a halving procedure similarly to the proof of part a). But this method works only when $\mu_n(du_1)$ is replaced by $n\,du_1$, as it is done in part b). Similarly to the proof of (3.1) one gets that

$$I=P\left(\sup_{0\leqq l<2^j}\ \sup_{t_0\leqq t<(l+1)t_0} n\left|\int_{lt_0}^{t}\int_0^1\dots\int_0^1 f(u_1,\dots,u_s)\,du_1\,\mu_n(du_2)\dots\mu_n(du_s)\right|>\frac{x}{4}\right)$$

$$\leqq\sum_{l=0}^{2^j-1}\sum_{r=0}^{\infty}\sum_{p=0}^{2^r-1}P\left(n\left|\int_{lt_0+pt_0 2^{-r}}^{lt_0+(p+1)t_0\cdot 2^{-r}}\int_0^1\dots\int_0^1 f(u_1,\dots,u_s)\right.\right.$$

$$\left.\left.\cdot\,du_1\,\mu_n(du_2)\dots\mu_n(du_s)\right|>\frac{x}{8}\frac{A-1}{A^{r+1}}\right)\tag{3.4}$$

with arbitrary $A>1$. Let us introduce the function $G(u_2,\dots,u_s)=G_{l,r,p}(u_2,\dots,u_s)$

$$G(u_2,\dots,u_s)=\frac{2^r}{t_0}\int_{lt_0+pt_0\cdot 2^{-r}}^{lt_0+(p+1)t_0\cdot 2^{-r}}f(u_1,\dots,u_s)\,du_1\,.$$

Then we have $|G(u_2,\dots,u_s)|\leqq 1$, and a general term in (3.4) can be written in the form

$$II=P\left(\left|\int_0^1\dots\int_0^1 G(u_2,\dots,u_s)\,\mu_n(du_2)\dots\mu_n(du_s)\right|>\frac{x}{8}\frac{A-1}{A^{r+1}}\frac{2^r}{nt_0}\right).$$

We estimate the expression in (3.4) with the help of part b) of Proposition 3 with $z=\frac{(A-1)x}{8Ant_0}$ and $B=\left(\frac{2}{A}\right)^r$, $1<A<2$. First we have to check that the conditions of Proposition 3 are satisfied. A simple calculation shows that

$$d_1n^{\frac{s-1}{2}}\leqq\frac{(A-1)x}{8Ant_0}\leqq d_2n^{s-1}$$

if $c_1>0$ is sufficiently large, and $c_2>0$ is sufficiently small in the condition $c_1 n^{s/2}<x<c_2 n^s$. Since $1<A<2$, hence $B=\left(\frac{2}{A}\right)^r>1$. Thus we get that

$$II \leqq \exp\left(-(\alpha+\alpha_1 r)\frac{1}{n}\left(\frac{x}{n t_0}\right)^{\frac{2}{s-1}}\right) \leqq \exp\left(-(\alpha+\alpha_1 r)\, x^{\frac{2}{s+1}} n^{-\frac{s-1}{s+1}}\right),$$

and the expression in (3.4) can be bounded as

$$I \leqq \sum_{r=0}^{\infty} 2^{j+r} \exp\left(-(\alpha+\alpha_1 r)\, x^{\frac{2}{s+1}} n^{-\frac{s-1}{s+1}}\right). \tag{3.5}$$

We estimate (3.5) with the help of the following inequalities:

$$\sum_{r=0}^{\infty} 2^r \exp\left(-\alpha_1 r x^{\frac{2}{s+1}} n^{-\frac{s-1}{s+1}}\right) \leqq \sum_{r=0}^{\infty} 2^r \exp\left(-\alpha r c_1^{\frac{2}{s+1}} n^{\frac{1}{s+1}}\right) \leqq B_1,$$

$$2^j \exp\left(-\frac{\alpha}{2} x^{\frac{2}{s+1}} n^{-\frac{s-1}{s+1}}\right) \leqq C n^{\frac{s}{s+1}} \exp\left(-\frac{\alpha}{2} C_1^{\frac{2}{s+1}} n^{\frac{1}{s+1}}\right) \leqq B_2.$$

These inequalities together with (3.5) imply that

$$I \leqq B_1 B_2 \exp\left(-\frac{\alpha}{2} x^{\frac{2}{s+1}} n^{-\frac{s-1}{s+1}}\right) \leqq \exp\left(-\bar{\alpha}\frac{x^{2/s}}{n}\right)$$

if $c_1 n^{s/2}<x<c_2 n^s$. Part b) of Proposition 4 is proved.

*Part c)*. Let us first observe that if $\xi_\lambda$ is a Poisson distributed random variable with parameter $\lambda$, $\lambda \geqq 1$, then

$$P(\xi_\lambda>2\lambda) \leqq \exp(-\alpha\lambda)$$

with some $\alpha>0$. Since $P_n((l+1)t_0)-P_n(l t_0)$ is Poisson distributed with parameter $n t_0$, $n t_0>1$, the above inequality implies that

$$P(P_n((l+1)t_0)-P_n(l t_0)>2n t_0) \leqq \exp(-\alpha n t_0) \leqq \frac{t_0}{2}\exp\left(-\bar{\alpha}\frac{x^{2/s}}{n}\right).$$

The last inequality enables us to reduce the proof of part c) to the verification of the inequality

$$P\left(\sup_{l t_0 \leqq t (l+1) t_0}\left|\int_{l t_0}^{t}\int_0^1 \dots \int_0^1 f(u_1,\dots,u_s)\, P_n(du_1)\right.\right. \tag{3.6}$$

$$\left.\left.\mu_n(du_2)\dots\mu_n(du_s)\right| > \frac{x}{4}\; P_n((l+1)t_0)-P_n(l t_0)<2n t_0\right) \leqq \frac{t_0}{2}\exp\left(-\alpha\frac{x^{2/s}}{n}\right)$$

for arbitrary $l=0,1,\dots,\frac{1}{t_0}-1$.

The main point in (3.6) is that we can assume that the $P_n$ measure of the interval $[l t_0\,(l+1)\,t_0]$ is less than const. $n t_0$ when investigating the integral in (3.6). Hence we can expect that the probability in (3.6) has the same order as in the case when $P_n(du_1)$ is replaced by $n\,du_1$, and this was investigated in part b). The main difficulty in the proof of (3.6) is that $P_n(du)$ is a random measure. We shall overcome this difficulty by exploiting that the random measure $P_n(du)$ in $[lt_0,(l+1)t_0]$ is independent of this measure outside this interval. We exploit this independence with the help of some conditioning. We also make a decomposition of the measure $\mu_n(du)$, which appears naturally when carrying out this conditioning.

Let us introduce the event $A_m(y_1,\dots,y_m)=A_{n,l,m,x}(y_1,\dots,y_m)\ A_m(y_1,\dots,y_m)$ $=\{$the Poisson process $P_n$ has exactly $m$ jumps in the interval $[lt_0\,(l+1)\,t_0]$ and they are in the points $y_1<y_2<\dots<y_m\}$.

We claim that

$$P\left(\sup_{lt_0\leqq t\leqq(l+1)t_0}\left|\int_{lt_0}^{t}\int_0^1\dots\int_0^1 f(u_1,\dots,u_s)\,P_n(du_1)\,\mu_n(du_2)\dots\mu_n(du_s)\right|\right.$$
$$\left.>\frac{x}{4}\,\middle|\,A_m(y_1,\dots,y_m)\right)\leqq\frac{t_0}{2}\exp\left(-\alpha\frac{x^{2/s}}{n}\right)\tag{3.7}$$

for all $m<2nt_0$ and $t_0<y_1<\dots<y_m<(l+1)\,t_0$. Relation (3.7) implies (3.6).

Let us write

$$\mu_n(du)=\mu_n^{(1)}(du)+\bar{P}_n(du)-n\bar{\lambda}(du),\tag{3.8}$$

where $\mu_n^{(1)}$ denotes the restriction of $\mu_n$ to the complementary set of the interval $[lt_0,(l+1)\,t_0]$, $\bar{P}_n$ and $\bar{\lambda}$ the restriction of the Poisson measure $P_n$ resp. the Lebesgue measure to the interval $[lt_0,(l+1)\,t_0]$. Let us decompose the measure $\mu_n(du_i)$ in the integral (3.7) for all $2\leqq i\leqq s$ by formula (3.8). In this way we decompose the integral in (3.7) as the sum of $3^{s-1}$ integrals. To prove (3.7) we are going to estimate for each such integral the conditional probability that it is larger than $\frac{x}{4}3^{1-s}$ under the condition $A_m(y_1,\dots,y_m)$. Let us observe that the measure $\mu_n^{(1)}$ is independent of $A_m(\cdot,\dots,\cdot)$, and $\bar{P}_n$ is measurable with respect to it. This enables us after the above decomposition of the integral in (3.7) to rewrite the conditional probability that a term is larger than $\frac{x}{4}3^{1-s}$ in the form of an unconditional distribution. After rewriting these conditional probabilities we have to prove inequalities of the following type to prove (3.7).

$$P\left(\sup_{lt_0<t<(l+1)t_0}\left|\int_{lt_0}^{t}\int_0^1\dots\int_0^1 f(u_1,\dots,u_s)\,\bar{\bar{P}}_n(du_1)\dots\bar{\bar{P}}_n(du_{k_1})\right.\right.$$
$$\left.\left.\cdot\, n^{k_2}\bar{\lambda}(du_{k_1+1})\dots\bar{\lambda}(du_{k_1+k_2})\,\mu_n^{(1)}(du_{k_1+k_2+1})\dots\mu_n^{(1)}(du_s)\right|>\frac{x}{4}3^{1-s}\right)$$
$$\leqq\frac{t_0}{2\cdot 3^s}\exp\left(-\frac{\alpha x^{2/s}}{n}\right)\tag{3.9}$$

for all $m\leqq 2nt_0$, $lt_0\,y_1<\dots<y_m<(l+1)\,t_0$, $k_1\geqq 1$, $k_2\geqq 0$, $k=k_1+k_2\leqq s$, where $\bar{\bar{P}}_n(du)=\sum_{r=1}^{m}\delta(y_r)$, and $\delta(x)$ denotes the point mass concentrated in the point $x$. A general term which we have to bound is similar. We have to bound the distribution function of the integral of the function $f$ with respect to a (random) product measure, where the first component of this product is $\bar{\bar{P}}_n(du)$, and the other components can be either $\bar{\bar{P}}_n$ or $\mu_n^{(1)}$ or $\bar{\lambda}$. All such terms can be estimated in the same way as we shall estimate the left hand side of (3.9), and the same upper bound can be obtained. Only the notation would become more complicated in the general case.

In the integral in (3.9) let us first integrate with respect to the coordinates $u_1, \dots, u_{k_1}$ (in this case this means summation). We get that (3.9) would follow from the following inequality:

$$\sum_{p_1=1}^{m} \dots \sum_{p_{k_1}=1}^{m} P\Bigg(\bigg|\int_0^1 \dots \int_0^1 f(y_{p_1}, \dots, y_{p_{k_1}}, u_{k_1+1}, \dots, u_s) \cdot \bar{\lambda}(du_{k_1+1}) \dots \bar{\lambda}(du_k)\, \mu_n^{(1)}(du_{k+1}) \dots \mu_n^{(1)}(du_s)\bigg| > \frac{x}{4 \cdot 3^{s-1} m^{k_1} n^{k_2}}\Bigg) \leqq \frac{t_0}{2 \cdot 3^{s-1}} \exp\left(-\alpha \frac{x^{2/s}}{n}\right) \tag{3.10}$$

under the same conditions as in (3.9). We shall bound each term at the left-hand side of (3.10) by first integrating with respect to the $k_1+1$-th, $k_1+2$-th ... and $k$-th coordinates, and then estimating the integral arising in this way with the help of Proposition 3. Introduce the functions

$$F(u_1, \dots, u_{s-k}) = F(u_1, \dots, u_{s-k}, y_{p_1}, \dots, y_{p_{k_1}}, l, t_0)$$

$$F(u_1, \dots, u_{s-k}) = \int_{lt_0}^{(l+1)t_0} \dots \int_{lt_0}^{(l+1)t_0} f(y_{p_1}, \dots, y_{p_{k_1}}, v_{k_1+1}, \dots, v_k, u_1, \dots, u_{s-k})\, dv_{k_1+1} \dots dv_k$$

and

$$\bar{F}(u_1, \dots, u_{s-k}) = t_0^{-k_2} F(u_1, \dots, u_{s-k}) \prod_{i=1}^{s-k} I(u_i \leqq lt_0 \text{ or } u_i > (l+1)t_0),$$

where $I(A)$ denotes the indicator function of the set $A$.

Then we have

$$|\bar{F}(u_1, \dots, u_{s-k})| \leqq 1,$$

and a general term at the left hand side of (3.10) can be rewritten in the form

$$P\left(\left|\int_0^1 \dots \int_0^1 \bar{F}(u_1, \dots, u_{s-k})\, \mu_n(du_1) \dots \mu_n(du_{s-k})\right| > \frac{x}{4 \cdot 3^{s-1} m^{k_1} (nt_0)^{k_2}}\right). \tag{3.11}$$

In the case $s=k$ the integral in (3.11) is defined as the number 1. In this case the probability in (3.11) equals zero if $c_2$ in the conditions of Proposition 4 is sufficiently small, since the relation $m < 2nt_0$ in this case implies that

$$\bar{x} = \frac{x}{4 \cdot 3^{s-1} m^{k_1} (nt_0)^{k_2}} > Cx(nt_0)^{-s} > C' \left(\frac{n^s}{x}\right)^{\frac{s-1}{s+1}} > 1.$$

To bound (3.11) in the case $1 \leqq k \leqq s-1$ let us introduce the number $z = z(k, x, n)$, $z = x^{\frac{s-k}{s}} \left(\frac{n^s}{x}\right)^{\frac{s-k}{2s(s+1)}}$. Some calculation shows that $d_1 n^{\frac{s-k}{2}} < z < d_2 n^{s-k}$, and since $m < 2nt_0$, $\bar{x} > z$ if $c_1 n^{s/2} < x < c_2 n^s$, and $c_1 > 0$ is sufficiently large, $c_2 > 0$ is sufficiently small. This implies that we increase the probability in (3.11) by replacing $\bar{x}$ with $z$ in it, and the latter probability can be estimated with the help of part a) of Proposition 3. This estimation gives the upper bound $\exp\left(-\alpha \frac{z^{\frac{2}{s-k}}}{n}\right) = \exp\left(-\alpha \frac{x^{2/s}}{n} \cdot \left(\frac{n^s}{x}\right)^{\frac{1}{s(s+1)}}\right)$ for the expression in (3.11). Since the left-hand side of (3.10) consists of $m^{k_1} \leqq (2nt_0)^s$ such terms we get the upper bound $C(nt_0)^s \exp\left(-\alpha \frac{x^{2/n}}{n} \left(\frac{n^s}{x}\right)^{\frac{1}{s(s+1)}}\right)$ for it. Observe that $\exp\left(-\frac{\alpha}{2} \frac{x^{2/s}}{n} \cdot \left(\frac{n^s}{x}\right)^{\frac{1}{s(s+1)}}\right) < n^K$ for arbitrary $K > 0$ if $c_1 n^{s/2} < x < c_2 n^s$, and $n > n_0(K)$, since $-\frac{\alpha}{2}\left(\frac{n^s}{x}\right)^{\frac{1}{s(s+1)}} \cdot \frac{x^{2/s}}{n} < -\varepsilon \left[\frac{x^{2/s}}{n} + \left(\frac{n^s}{x}\right)^{\frac{1}{s(s+1)}}\right] < -K \log n$. Thus we get that the left hand side of (3.10) can be bounded by

$$\frac{1}{n} \exp\left(-\frac{\alpha}{n} \frac{x^{2/s}}{n} \left(\frac{n^s}{x}\right)^{\frac{1}{s(s+1)}}\right) \leqq \frac{t_0}{2 \cdot 3^s} \exp\left(-\frac{\alpha}{2} \frac{x^{2/s}}{n}\right).$$

Thus we verified (3.9), which implies (3.7) and hence also (3.6). Part c) is proved.

## 4. The Proof of Theorem 1′

Our aim in this section is to construct a Poisson process and an empirical process simultaneously which are close to each other. More precisely we want to construct the measures $\mu_n$ and $\bar{\mu}_n$ in such a way that the relation

$$P\Bigg(\sup_{0 \leqq t \leqq 1} \Bigg| \int_0^t \int_0^1 \dots \int_0^1 f(u_1, \dots, u_s)\, \mu_n(du_1) \dots \mu_n(du_s) \tag{4.1}$$
$$- \int_0^t \int_0^1 \dots \int_0^1 f(u_1, \dots, u_s)\, \bar{\mu}_n(du_1) \dots \bar{\mu}_n(du_s) \Bigg| > x\Bigg) \leqq C \exp\left(-\alpha \frac{x^{2/s}}{n}\right)$$

hold for all $|f(u_1, \dots, u_s)| \leqq 1$, $c_1 n^{s/2} < x < c_2 n^s$, where $C > 0$, $c_1 > 0$, $c_2 > 0$ and $\alpha > 0$ depend only on the dimension $s$. We make the following construction: Let $\gamma_1, \gamma_2, \dots$ be a sequence of independent uniformly distributed random variables on the interval $[0, 1]$, and let $\eta_n$ be a Poisson distributed random variable with parameter $n$, which is independent of the sequence $\{\gamma\}_{i=1}^{\infty}$.

Set

$$\bar{\mu}_n = \sum_{i=1}^{n} \delta(\gamma_i) - n\lambda$$

and

$$\mu_n = \sum_{i=1}^{\eta_n} \delta(\gamma_i) - n\lambda,$$

where $\lambda$ denotes the Lebesgue measure on [0, 1] and $\delta(x)$ the point mass measure concentrated in the point $x$. Then $\bar{\mu}_n$ and $\mu_n$ have the prescibed distributions. Moreover, we claim that they satisfy relation (4.1). It is clear that relation (4.1) and Theorem 2′ together imply Theorem 1′. Let us first observe that there exists some $A > 0$ such that

$$P\left(|\eta_n - n| > \frac{1}{3} x^{1/s}\right) \leqq 2 \exp\left(-A \frac{x^{2/s}}{n}\right). \tag{4.2}$$

Because of (4.2) it is enough to show that

$$P\left(B_n \cap \{|\eta_n - n| < \frac{1}{3} x^{1/s}\}\right) \leqq C \exp\left(-\alpha \frac{x^{2/s}}{n}\right) \tag{4.3}$$

with

$$\begin{aligned} B_n = \Bigg\{ \sup_{0 \leqq t \leqq 1} \Bigg| & \int_0^t \int_0^1 \dots \int_0^1 f(u_1, \dots, u_s)\, \mu_n(du_1) \dots \mu_n(du_s) \\ & - \int_0^t \int_0^1 \dots \int_0^1 f(u_1, \dots, u_s)\, \bar{\mu}_n(du_1) \dots \bar{\mu}_n(du_s) \Bigg| > x \Bigg\} \end{aligned} \tag{4.4}$$

in order to prove (4.1).

For $s = 1$ the condition $|\eta_n - n| \leqq \frac{1}{3} x$ implies that the difference of the integrals in (4.4) is less than $|\eta_n - n| \leqq \frac{1}{3} x$ hence the left hand side of (4.3) equals zero. For $s \geqq 2$ we shall prove (4.3) by induction. Our inductive hypothesis is that for $s' < s$ Theorem 1′ holds. Moreover, as the argument at the end of Sect. 1 shows, the condition $c_1 n^{s/2} < x < c_2 n^s$ can be dropped from Theorem 1′. Then if we prove (4.3) with the help of our inductive hypothesis then we we also prove (4.1) and hence Theorem 1′ for $s$. We shall prove (4.3) by applying a conditioning argument. Namely, we are going to prove that

$$P(B_n | \eta_n = n + l,\ \gamma_{n+1} = y_1, \dots, \gamma_{n+l} = y_l) \leqq C \exp\left(-\alpha \frac{x^{2/s}}{n}\right), \tag{4.5}$$

and

$$P(B_n | \eta_n = n - l,\ \gamma_{n-l+1} = y_1, \dots, \gamma_n = y_l) \leqq C \exp\left(-\alpha \frac{x^{2/s}}{n}\right) \tag{4.5$'$}$$

for all $0 \leqq l \leqq \frac{1}{3} x^{1/s}$ and $0 \leqq y_j \leqq 1$, $1 \leqq j \leqq l$. Relations (4.5) and (4.5)′ imply (4.3).

First we prove (4.5). Let us introduce the measure $\nu_n = \nu_n(l)$

$$\nu_n = \sum_{j=1}^{l} \delta(y_j), \tag{4.6}$$

where $y_1, \dots, y_l$ are the same as in (4.5). Then under the conditions appearing at the left-hand side of (4.5) $\mu_n = \bar{\mu}_n + \nu_n$, and the conditional distribution of $\bar{\mu}_n$ under this condition agrees with its unconditional distribution. Hence, by applying the decomposition $\mu_n = \bar{\mu}_n + v_n$ we can write

$$\begin{aligned}
&P(B_n | \eta_n = n+l,\ \gamma_{n+1} = y_1, \ldots, \gamma_{n+l} = y_l) \\
&\leqq \Sigma' P\left( \sup_{0 \leqq t \leqq 1} \left| \int_0^t \int_0^1 \ldots \int_0^1 f(u_1, \ldots, u_s)\, v_n(du_1)\, m_{\varepsilon(2)}(du_2) \ldots m_{\varepsilon(s)}(du_s) \right| > \frac{x}{2^s} \right) \\
&\quad + \Sigma'' P\left( \sup_{0 \leqq t \leqq 1} \left| \int_0^t \int_0^1 \ldots \int_0^1 f(u_1, \ldots, u_s)\, \bar{\mu}_n(du_1)\, m_{\varepsilon(2)}(du_2) \ldots m_{\varepsilon(s)}(du_s) \right| > \frac{x}{2^s} \right) \\
&= \Sigma_1 + \Sigma_2, \qquad (4.7)
\end{aligned}$$

where $\varepsilon(j) = 0$ or $1$, $j = 2, \ldots, s$, $m_0(du) = v_n(du)$, $m_1(du) = \bar{\mu}_n(du)$ in $\Sigma'$, the summation is taken for all possible sequences $\varepsilon(j) = 0$ or $1, j = 2, \ldots, s$, and in $\Sigma''$ again it is taken for all such sequences with the exception of the term where $\varepsilon(j) = 1$ for all $j = 2, \ldots, s$. The terms in $\Sigma_1$ and $\Sigma_2$ were separated, because their estimations require a slightly different argument.

$\Sigma_1$ contains the term where $f$ is integrated with respect to the measure $\prod_{i=1}^{s} v_n(du_1)$. This integral is less than $l^s \leqq 3^{-s} x$, hence this term equals zero. To estimate the other terms in $\Sigma_1$ we define a partition $I_{j,k} = [t_{j,k},\ t_{j,k+1})$, $k = 0, 1, \ldots, \bar{k}(j) = \left[\frac{l-1}{2^j}\right]$, of the interval $[0, 1]$ for all $j \leqq \log_2 l$ in the following way: $t_{j,0} = 0$, $y_{k \cdot 2^j} < t_{j,k} < y_{k \cdot 2^j + 1}$ for $1 \leqq k \leqq \bar{k}(j)$, $y_{\bar{k}(j)+1} = 1$. We made this partition in such a way that for fixed $j$ all intervals $I_{j,k}$, with the possible exception of the last one, contains exactly $2^j$ points $y$ (the last interval may contain less points). Then, by using a halving procedure similarly to the start of the proof part a) of Proposition 4 we get the following estimate for a general term of $\Sigma_1$:

$$\begin{aligned}
I_k &= P\left( \sup_{0 \leqq t \leqq 1} \left| \int_0^t \int_0^1 \ldots \int_0^1 f(u_1, \ldots, u_s)\, v_n(du_1) \ldots v_n(du_k)\, \bar{\mu}_n(du_{k+1}) \ldots \bar{\mu}_n(du_s) \right| > \frac{x}{2^s} \right) \\
&\leqq \sum_{j=1}^{\log_2 l} \sum_{m=1}^{[\frac{l-1}{2^j}]+1} P\left( \left| \int_{I_{j,m}} \int_0^1 \ldots \int_0^1 f(u_1, \ldots, u_s)\, v_n(du_1) \ldots v_n(du_k)\, \bar{\mu}_n(du_{k+1}) \ldots \bar{\mu}_n(du_s) \right| \right. \\
&\quad \left. > \frac{x}{2^s} \frac{\sqrt{2}-1}{\sqrt{2}} \left(\frac{2^j}{l}\right)^{1/2} \right). \qquad (4.8)
\end{aligned}$$

We get, by integrating with respect to the first $k$ coordinates that

$$\left| \int_{I_{j,m}} \int_0^1 \ldots \int_0^1 f(u_1, \ldots, u_s)\, v_n(du_1) \ldots v_n(du_k) \right| \leqq 2^j l^{k-1} \leqq \frac{2^j}{l} \cdot x^{k/s}.$$

Now we can estimate (4.8) with the help of Theorem 1′ for $s' = s - k$ in the following way:

$$\begin{aligned}
I_k &\leqq 2 \sum_{j=1}^{\log_2 l} \frac{l}{2^j} \exp\left( -\alpha \frac{x^{2/s}}{n} \left(\frac{l}{2^j}\right)^{\frac{1}{s-k}} \right) \\
&\leqq C \sum_{j=1}^{\log_2 l} \frac{2^j}{l} \exp\left( -\frac{\alpha}{2} \frac{x^{2/s}}{n} \left(\frac{l}{2^j}\right)^{\frac{1}{s-k}} \right) \leqq 2C \exp\left( -\frac{\alpha}{n} x^{2/s} \right),
\end{aligned}$$

if $x > c_1 n^{s/2}$ with a sufficiently large $c_1 > 0$ and $1 \leqq k < s$, since in this case $\exp\left(-\frac{\alpha}{n}\frac{x^{2/s}}{n}\left(\frac{l}{2^j}\right)^{\frac{1}{s-k}}\right) \leqq \left(\frac{2^j}{l}\right)^2$. All other terms of $\Sigma_1$ can be estimated in the same way, and one gets that

$$\Sigma_1 \leqq C \exp\left(-\frac{\alpha}{n} x^{2/s}\right). \tag{4.9}$$

Let us estimate a term of the following type in $\Sigma_2$:

$$I'_k = P\left(\sup_{0 \leqq t \leqq 1} \left| \int_0^t \int_0^1 \dots \int_0^1 f(u_1, \dots, u_s)\, \bar{\mu}_n(du_1)\, \nu_n(du_2) \dots \nu_n(du_{k+1})\, \bar{\mu}_n(du_{k+2}) \cdots \bar{\mu}_n(du_s) \right| > \frac{x}{2^s}\right).$$

By integrating first with respect to the coordinates $u_2, \dots, u_{k+1}$ we get with the function

$$F(u_1, u_{k+2}, \dots, u_s) = x^{-\frac{k}{s}} \int_0^1 \dots \int_0^1 f(u_1, \dots, u_s)\, \nu_n(du_2) \dots \nu_n(du_{k+1})$$

that

$|F(u_1, u_{k+2}, \dots, u_s)| \leqq l^k x^{-\frac{k}{s}} \leqq 1$, and

$$I'_k = P\left(\sup_{0 \leqq t \leqq 1} \left| \int_0^t \int_0^1 \dots \int_0^1 F(u_1, u_{k+2}, \dots, u_s)\, \bar{\mu}_n(du_1)\, \bar{\mu}_n(du_{k+2}) \dots \bar{\mu}_n(du_s) \right| > \frac{x^{1-\frac{s}{k}}}{2^s}\right).$$

The term $I'_k$ can be estimated with the help of Theorem 1′ for $s' = s - k$. One gets

$$I'_k \leqq C \exp\left(-\alpha \frac{x^{2/s}}{n}\right).$$

The general terms in $\Sigma'_2$ can be estimated in the same way, and one gets that

$$\Sigma_2 \leqq C \exp\left(-\alpha \frac{x^{2/s}}{n}\right). \tag{4.9$'$}$$

Relations (4.9) and (4.9)′ imply (4.5). The proof of (4.5)′ is similar. In this case we define $\nu_n$ again by (4.6), only $y_1, \dots, y_l$ are the $y_i - s$ appearing in (4.5)′. Then we get that under the conditions in formula (4.5)′ $\bar{\mu}_n = \mu_n + \nu_n$, and the conditional distribution of $\mu_n$ agrees with the unconditional distribution of $\bar{\mu}'_{n-l}$,

$$\bar{\mu}'_{n-l} = \bar{\mu}_{n-l} - l \cdot \lambda, \tag{4.10}$$

where $\bar{\mu}_{n-l}$ is defined by (1.5), only $n$ is substituted by $n-l$, and $\lambda$ is the Lebesgue measure. Then (4.5)′ can be proved in the same way as (4.5), only one has to prove that Theorem 1′ remains valid for $s' < s$ if $\bar{\mu}_n$ is replaced by $\bar{\mu}'_{n-l}$. Since $n - l > \frac{2}{3}n$, we prove this statement if we show that for all $|f(u_1, \dots, u_s)| \leqq 1$

$$P\left(\sup_{0 \leqq t \leqq 1} \left| \int_0^t \int_0^1 \dots \int_0^1 f(u_1, \dots, u_s)\, \bar{\mu}'_{n-l}(du_1) \dots \bar{\mu}'_{n-l}(du_{s'}) - \int_0^t \int_0^1 \dots \int_0^1 f(u_1, \dots, u_s)\, \mu_{n-l}(du_1) \dots \bar{\mu}_{n-l}(du_{s'}) \right| > \frac{x}{2}\right) \leqq C \exp\left(-\frac{\alpha x^{2/s}}{n}\right) \tag{4.11}$$

for all $x > 0$, $l \leqq \frac{1}{3} x^{1/s}$, $s' \leqq s-1$, where $C$ and $\alpha$ are universal constants depending only on $s$.

By applying the decomposition (4.10) of $\bar{\mu}_{n-l}$ we get that (4.11) follows from the following type of inequalities:

$$P\left(\sup_{0 \leqq t \leqq 1} l^{s'-k} \left| \int_0^t \int_0^1 \dots \int_0^1 f(u_1, \dots, u_{s'})\, \bar{\mu}_{n-l}(du_1) \dots \bar{\mu}_{n-l}(du_k)\, du_{k+1} \dots du_{s'} \right| > \frac{x}{2^{s'}}\right)$$
$$\leqq C \exp\left(-\alpha \frac{x^{2/s'}}{n}\right), \tag{4.12}$$

and

$$P\left(\sup_{0 \leqq t \leqq 1} l^{s'-k} \left| \int_0^t \int_0^1 \dots \int_0^1 f(u_1, \dots, u_{s'})\, du_1\, \bar{\mu}_{n-l}(du_2) \dots \bar{\mu}_{n-l}(du_{k+1}) \right.\right.$$
$$\left.\left. \cdot\, du_{k+2} \dots du_{s'} \right| > \frac{x}{2^{s'}}\right) \leqq C \exp\left(-\alpha \frac{x^{2/s'}}{n}\right). \tag{4.12$'$}$$

By integrating first with respect to the coordinates $u_{k+1}, \dots, u_{s'}$ and exploiting that $l < \frac{1}{3} x^{1/s}$, we get (4.12) from Theorem 1′ for the dimension $s' - k$. The proof of (4.12)′ is similar, first we integrate with respect to the coordinates where Lebesgue measure stands, and then apply Theorem 1′. But here we must apply, because of the sup in (4.12)′ a halving procedure in the first coordinate in the same way as it is done at the beginning of the proof of part b) in Proposition 4. We have to estimate the probability of the event that the integral in (4.12)′ is larger than $\frac{x}{2^{s'} l^{s'-k}} \cdot \frac{\sqrt{2}-1}{\sqrt{2}} \cdot 2^{-j/2}$ if we integrate with respect to the first coordinate only in the interval $[k 2^{-j}, (k+1) 2^{-j})$. Then the same argument which is done in formula (3.4) leads to the proof of (4.12)′. We get in this way that formula (4.11) and hence also (4.5)′ holds true. Theorem 1′ is proved.

## References

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