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Limit Theorems for Non-Linear Functionals of Gaussian Sequences

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Summary. We prove limit theorems for sums of non-linear functionals of Gaussian sequences. In certain cases we obtain a non-Gaussian limit with a norming factor n^α, $0<\alpha<1/2$. The class of functionals we are investigating is a natural enlargement of the class investigated by M. Rosenblatt in [7]. We prove our results by refining the method of the paper [3].

1. Introduction

In recent time, several papers dealt with limit theorems for sums of dependent random variables. Examples when the central limit theorem ceases to hold are of special interest. The first example of this type was given by M. Rosenblatt in [6] (see also [8]). He considered quadratic functionals of a stationary Gaussian sequence with a correlation function which slowly tends to zero. This result was generalized by Dobrushin and Major in [3] and by Taqqu in [10] independently of each other. The problem is more naturally put in the following modified form:

Let ξ_n, $n=\dots-1,0,1,\dots$ be a strictly stationary sequence, $E\xi_n=0$, $E\xi_n^2=1$. For every positive integer N, $N=1,2,\dots$ define a new sequence:

$$Z_n^N=A_N^{-1}\sum_{m=(n-1)N+1}^{nN}\xi_m, \qquad n=\dots-1,0,1,\dots,N=1,2,\dots, \tag{1.1}$$

where A_N is an appropriate norming constant. Question: Under what conditions does the sequence Z_n^N tend to a sequence Z_n^*, $n=\dots-1,0,1,\dots$ in distribution? In the most general case this question seems to be very difficult, but in the case when the ξ are functionals of a Gaussian sequence some interesting results have already been proved.

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The first problem to be solved when answering the above question is the characterization of the possible limiting sequences Z_n^*. For every $N=1,2,\dots$ formula (1.1) induces a transformation $T_N(A_N)$ on the distribution of sequences of random variables ξ_n, $n=\dots-1,0,1,\dots$. It can be seen (cf. [2] for a proof) that the distribution of a limiting sequence Z_n^* must be a fixed point of the transformations $T_N(A_N)$ for all N. More precisely, Z_n^* can be the limit of some sequences Z_n^N defined by (1.1) only if its distribution remains invariant under all transformations $T_N(N^\alpha)$ with some fixed positive constant $\alpha\geqq 0$. In this case, A_N must be chosen in (1.1) as $A_N=N^\alpha L(N)$, where $L(\cdot)$ is a slowly varying function. Sequences whose distributions are invariant under the transformations $T_N(N^\alpha)$ will be called self-similar processes with self-similarity parameter $1-\alpha$. (There is no standard definition of the self-similarity parameter in the literature. We follow the terminology of Dobrushin in [1].)

A large class of non-Gaussian self-similar processes was constructed by Dobrushin in [1]. Essentially the same class was found also by Taqqu in [9], but he gave a different representation. In the study of limit theorems for sums of dependent random variables in [3] the following self-similar processes played an important role.

$$\bar{Z}_n=\bar{Z}_n(k,\alpha)=\int \exp[in(x_1+\dots+x_k)] \cdot K_0(x_1+\dots+x_k)|x_1|^{\frac{\alpha-1}{2}}\dots|x_k|^{\frac{\alpha-1}{2}} W(dx_1)\dots W(dx_k)$$
$$n=\dots=-1,0,1,\dots \tag{1.2}$$

where

$$K_0(x)=\frac{\exp(ix)-1}{ix}, \tag{1.3}$$

i.e. K_0 is the characteristic function of the uniform distribution on $[0,1]$, $W(\cdot)$ denotes the white noise process, and the integral is meant as multiple Wiener-Itô integral. More precisely, we consider here and also in the sequel a modified version of the Wiener-Itô integral which is defined in [1]. The process $\bar{Z}_n(k,\alpha)$ is well-defined if $0<k\alpha<1$, and its self-similarity parameter is $k\frac{\alpha}{2}<\frac{1}{2}$. Therefore, they can appear as the limit of sequences Z_n^N defined by (1.1) only if $A_N=N^{1-\frac{k\alpha}{2}}L(N)$, where $1-\frac{k\alpha}{2}>\frac{1}{2}$, and $L(\cdot)$ is a slowly varying function. Quite recently, M. Rosenblatt [7] found a class of stationary sequences whose partial sums satisfy a non-central limit theorem with a norming factor N^c, $c<1/2$. Our aim is to prove limit theorems which contain Rosenblatt's result as a very special case.

Let us introduce some notation:

Let $\dots Y_{-1}, Y_0, Y_1, \dots$ $\mathbf{E}Y_0=0$, $\mathbf{E}Y_0^2=1$, be a stationary Gaussian sequence with correlation function

$$r(n)=\mathbf{E}Y_0Y_n=n^{-\alpha}L(n), \qquad 0<\alpha<1, \tag{1.4}$$

where $L(\cdot)$ is a slowly varying function. Given a real function $H(x)$ and a sequence of real numbers a_n, $n=\dots-1,0,1,\dots$ we define the sequences of random variables

$$X_n=H(Y_n), \quad n=\dots-1,0,1,\dots \tag{1.5}$$

and

$$U_m=U_m(H)=\sum_{n=-\infty}^{\infty} a_n X_{m+n}, \quad m=\dots-1,0,1,\dots. \tag{1.6}$$

In formula (1.6) convergence is meant in L_2 sense. If the sum on the right-hand side of (1.6) does not converge in L_2 sense, then we consider formula (1.6) to be meaningless. Now we define the sequences

$$Z_n^N=Z_n^N(H)=A_N^{-1}\sum_{m=nN}^{(n+1)N-1} U_m, \quad n=\dots,-1,0,1,\dots,N=1,2,\dots \tag{1.7}$$

where A_N is an appropriate norming constant. Our aim is to prove limit theorems for the distribution of the sequences Z_n^N, $n=\dots-1,0,1,\dots$ when $N\to\infty$. We shall consider some processes Z_n^* which belong to the class of self-similar processes constructed in [1]. They will appear as the limit of some sequences Z_n^N defined in (1.7), when $N\to\infty$.

$$Z_n^*=Z_n^*(\alpha,\beta,k,b,c)=\int \exp[in(x_1+\dots+x_k)] \cdot \bar{K}_0(x_1+\dots+x_k)|x_1|^{\frac{\alpha-1}{2}}\dots|x_k|^{\frac{\alpha-1}{2}} W(dx_1)\dots W(dx_k)$$
$$n=\dots-1,0,1,\dots \tag{1.8}$$

where

$$\bar{K}_0(x)=K_0(x)M_0(x) \tag{1.9}$$

$$M_0(x)=bM_0^{(1)}(x)+cM_0^{(2)}(x) \tag{1.10}$$

$$M_0^{(1)}(x)=|x|^\beta, \quad M_0^{(2)}(x)=i|x|^\beta \operatorname{sign} x \tag{1.11}$$

and $K_0(x)$ is defined in (1.3).

Here, $W(\cdot)$ denotes again the white noise process, and we consider multiple Wiener-Itô integrals with respect to it. The process Z_n^* is well-defined if and only if

$$\int |\bar{K}_0(x_1+\dots+x_k)|^2|x_1|^{\alpha-1}\dots|x_k|^{\alpha-1}\,dx_1\dots dx_k<\infty. \tag{1.12}$$

This integral is convergent if and only if

$$0<k\alpha<1, \quad \text{and} \quad 0<1-\beta-\frac{k}{2}\alpha<1. \tag{1.13}$$

The first inequality in (1.13) ensures that

$$\int_{1<|x_1+\dots+x_k|<2} |\bar{K}_0(x_1+\dots+x_k)|^2|x_1|^{\alpha-1}\dots|x_k|^{\alpha-1}\,dx_1\dots dx_k<\infty,$$

and the second one that the singularities of the integrand in the region $x_1+\dots+x_k=0$ and $x_1+\dots+x_k\to\infty$ do not make the integral (1.12) divergent. The finiteness of the integral (1.12) under the condition (1.13) will follow also from the results of this paper.

The self-similarity parameter of the process $Z_n^*(\alpha,\beta,k,b,c)$ is $\beta+\frac{k}{2}\alpha$. In case $\beta>0$ it may occur that $\gamma=1-\beta-\frac{k}{2}\alpha<\frac{1}{2}$, and in this case limit theorems with norming factor N^γ, $\gamma<\frac{1}{2}$ can be expected.

Let H_k denote the k-th Hermite polynomial with first coefficient 1. The convergence problem for $Z_n^N(H)$ with a general function H can be reduced to the convergence problem for $Z_n^N(H_k)$, $k=1,2,\dots$. The main result of this paper is the following one: If $a_n=C_1n^{-\beta-1}+o(n^{-\beta-1})$ for $n>0$, $a_n=C_2|n|^{-\beta-1}+o(|n|^{-\beta-1})$ for $n<0$, $Z_n^N(H_k)$ is defined by (1.4), (1.5), (1.6) and (1.7), α and β satisfy (1.13), then $Z_n^N(H_k)$ converges with an appropriate norming to a field $Z_n^*(\alpha,\beta,k,b,c)$, where the constants b and c are appropriately chosen. But in the case $\beta>0$ this statement holds only under the additional condition $\sum_{n=-\infty}^{\infty}a_n=0$.

We formulate this result in more detail. Let a sequence a_n satisfy the relations

$$\begin{aligned} a_n&=a_n(\beta)=C(1)\,n^{-\beta-1}+o(n^{-\beta-1}) && \text{if } n\geqq 0\\ a_n&=a_n(\beta)=C(2)\,|n|^{-\beta-1}+o(|n|^{-\beta-1}) && \text{if } n<0. \end{aligned} \tag{1.14}$$

Define the new sequences

$$b_n=\tfrac{1}{2}(a_n+a_{-n}),\qquad c_n=\tfrac{1}{2}(a_n-a_{-n}),\qquad n=\dots-1,0,1,\dots. \tag{1.15}$$

Obviously,

$$\begin{aligned} b_n&=\tfrac{1}{2}(C(1)+C(2))\,|n|^{-\beta-1}+o(|n|^{-\beta-1}), && b_n=b_{-n},\ b_0=a_0.\\ c_n&=\tfrac{1}{2}(C(1)-C(2))\,|n|^{-\beta-1}\operatorname{sign} n+o(|n|^{-\beta-1}), && c_n=-c_{-n},\ c_0=0, \end{aligned}$$

and $a_n=b_n+c_n$.

We shall prove the following

Theorem. *Given a stationary Gaussian sequence Y_n, $n=\dots-1,0,1,\dots$ $\mathbf{E}Y_0=0$, $\mathbf{E}Y_0^2=1$ satisfying* (1.4) *and a sequence of real numbers a_n satisfying* (1.15) *we define the stationary sequences X_n, U_n, Z_n^N by means of formulae* (1.5), (1.6) *and* (1.7) *with the choice $H(x)=H_k(x)$, the k-th Hermite polynomial with first coefficient* 1. *If condition* (1.13) *is satisfied, then the processes X_n, U_n, Z_n are well-defined. Further assume one of the following conditions to be satisfied*

(i) $0<\beta<1$ *and* $\sum_{n=-\infty}^{\infty}a_n=0$,

(ii) $-1<\beta<0$,

(iii) $\beta=0$, $C(1)=-C(2)$ *and* $\sum|b_n|<\infty$.

Then the finite dimensional distributions of the sequences Z_n^N tend, with the choice $A_N=N^{1-\beta-k/2\alpha}L(N)^{k/2}$, to the finite dimensional distributions of the se-

quence $D^{-k} Z_n^* = D^{-k} Z_n^*(\alpha, \beta, k, b, c)$ *defined in* (1.8), *as* $N \to \infty$. D *is chosen as* $D = 2\Gamma(\alpha)\cos\left(\frac{\alpha}{2}\pi\right)$ *in all three cases. In cases* (i) *and* (ii) $b = 2[C(1) + C(2)]$ $\Gamma(-\beta)\sin\left(\frac{\beta+1}{2}\pi\right)$, $c = 2[C(1) - C(2)]\ \Gamma(-\beta)\cos\left(\frac{\beta+1}{2}\pi\right)$.

In case (iii), $b = \sum_{n=-\infty}^{\infty} b_n$, $c = C(1)$.

Let us now consider a general function $H(x)$ with the following properties

$$\int H(x)\exp\left(-\frac{x^2}{2}\right)dx = 0, \qquad \int [H(x)]^2 \exp\left(-\frac{x^2}{2}\right)dx < \infty. \tag{1.16}$$

The function $H(x)$ can be expanded by means of Hermite polynomials in the form

$$H(x) = \sum_{k=1}^{\infty} d_k H_k(x), \qquad \sum d_k^2 k! < \infty. \tag{1.17}$$

We shall prove the following

Corollary. *Let the function* $H(x)$ *satisfy* (1.16), *and let* k *be the smallest index in its expansion* (1.17) *such that* $d_k \neq 0$. *Let us define the sequences* $Z_n^N = Z_n^N(H)$ *in the same way as in the Theorem, only substituting* $H_k(x)$ *by* $H(x)$. *If the conditions of the Theorem hold with the* k *defined at the beginning of the Corollary, then the sequences* Z_n^N *are well-defined and their finite-dimensional distributions tend to that of the sequence* $d_k D^{-k} Z_n^*(\alpha, \beta, k, b, c)$ *as* $N \to \infty$. *The constants* D, b, c *are determined in the same way as in the Theorem.*

It is worthwhile comparing the Theorem with the following Theorem A, which is a consequence of Theorem 3 in [3].

Theorem A. *Given a stationary Gaussian sequence* Y_n, $n = \dots -1, 0, 1, \dots$ $\mathbf{E}Y_0 = 0$, $\mathbf{E}Y_0^2 = 1$ *satisfying* (1.4) *and a sequence of real numbers* a_n, $\sum_{n=-\infty}^{\infty} |a_n| < \infty$, $\sum_{n=-\infty}^{\infty} a_n \neq 0$, *we define the sequences* X_n, U_n, Z_n^N *by aid of formulae* (1.5), (1.6) *and* (1.7) *with the choice of* $H(x) = H_k(x)$. *These sequences are well-defined, and in the case* $0 < k\alpha < 1$ *the finite dimensional distributions of the sequences* Z_n^N *tend with the choice* $A_N = N^{1-k\alpha/2} L^{k/2}(N)$, *to the finite dimensional distributions of the sequence* $D^{-k/2}\left(\sum_{n=-\infty}^{\infty} a_n\right) \cdot \bar{Z}_n(k, \alpha)$ *as* $N \to \infty$, *where* $\bar{Z}_n$ *is defined in* (1.2) *and* D *is the same constant as in the Theorem.*

Let us observe that the exponent of the norming factor in case (i) of the Theorem is smaller than in Theorem A. This shows that the condition $\sum a_n = 0$ in case (i) is essential.

The self-similar sequences $\bar{Z}_n$ belong to the class of self-similar fields Z_n^* defined in (1.8). Nevertheless, a comparison of the Theorem and Theorem A indicates that the sequence $\bar{Z}_n$ plays a special role. The sequences Z_n^* appear as the limit of sequences Z_n^N only if the coefficients a_n are very specially chosen, namely only if the sequence a_n behaves asymptotically like a power of n. On the other hand, in Theorem A a rather general sequence a_n can be chosen. As the proof of the Theorem will show, this particular behaviour of the sequence Z_n^* is closely connected with the following fact in Fourier analysis. If $f(x)$ is a sufficiently smooth function on the real line with support in $[-\frac{\pi}{2}, \frac{\pi}{2})$, then for every β, $-1<\beta<1$,

$$\int_{-\pi}^{\pi} \exp(inx)|x|^\beta \operatorname{sign} x \cdot f(x)\,dx = C(\beta)|n|^{-\beta-1} \operatorname{sign} n(1+o(1)), \qquad C(\beta)\neq 0.$$

A similar relation holds if $|x|^\beta \operatorname{sign} x$ is substituted by $|x|^\beta$. The only exceptional case is when $\beta=0$, and the term $\operatorname{sign} x$ is absent. In this case, the Fourier transform tends fast to zero at infinity, since the function $g(x)\equiv 1$ has no singularity. We shall see that this exceptional behaviour of the constant function has some implication about the behaviour of the sequence Z_n^*.

The present paper consists of four sections. In Sect. 2 we prove the Theorem with the help of a lemma and some relations in Fourier analysis. In Sect. 3, this lemma and the Corollary are proved together with the facts from Fourier analysis needed in Sect. 2. In Sect. 4, some comments are made and some possible generalizations are discussed.

2. Proof of the Theorem

The bulk of the proof consists of checking that certain sums and integrals are finite. To explain the idea of the proof better, we first give a brief outline of it.

We shall represent the sequences Z_n^N by means of multiple Wiener-Itô integrals. After an appropriate substitution Z_n^N can be written in the form

$$Z_n^N=\int \exp[in(x_1+\ldots+x_k)]\,\bar{K}_N(x_1+\ldots+x_k)\,Z_{G_N}(dx_1)\ldots Z_{G_N}(dx_k).$$

with such functions $\bar{K}_N$ and G_N that $\bar{K}_N(x)\to\bar{K}_0(x)$ and $G_N(x)\to G_0(x)$. These relations would suggest a formal limiting procedure which yields

$$Z_n^N\to\int \exp[in(x_1+\ldots+x_k)]\,\bar{K}_0(x_1+\ldots+x_k)\,Z_{G_0}(dx_1)\ldots Z_{G_0}(dx_k).$$

To justify this limiting procedure, we shall prove that the sequence of measures μ_N,

$$\mu_N(A)=\int_A |\bar{K}_N(x_1+\ldots+x_k)|^2\,G_N(dx_1)\ldots G_N(dx_k), \qquad A\subset R^k$$

tends weakly to the measure μ_0, where

$$\mu_0(A)=\int_A |K_0(x_1+\ldots+x_k)|^2\,G_0(dx_1)\ldots G_0(dx_k), \qquad A\subset R^k.$$

We prove the last statement by means of Fourier analysis. Now if we want to follow the argument of [3] in the proof of the Theorem then we meet with some problems. They arise because the convergence $\bar{K}_N \to \bar{K}_0$ may be non-uniform even in bounded regions. To overcome this difficulty we decompose Z_n^N into the sum $Z_n^N = (Z_n^N)_1 + (Z_n^N)_2$ in an appropriate way. We make this decomposition in such a way that the limiting procedure can be carried out for $(Z_n^N)_1$ relatively simply, and $(Z_n^N)_2$ is negligible.

Now we begin the proof by showing that the random variables $U_m = U_m(H_k)$ are well-defined, i.e. that under the condition (1.13) for every $\varepsilon > 0$ there exists an $L = L(\varepsilon)$ such that

$$J = \mathbf{E}\left[\sum_{n=-L_1}^{L'_1} a_n H_k(Y_{n+m}) - \sum_{n=-L_2}^{L'_2} a_n H_k(Y_{n+m})\right]^2 < \varepsilon \tag{2.1}$$

for all $L_1, L'_1, L_1, L'_2 > L$.

The following estimates hold true:

$$J \leqq 2\mathbf{E}\left(\sum_{n=-L_1}^{-L_2-1} a_n H_k(Y_{n+m})\right)^2 + 2\mathbf{E}\left(\sum_{n=L'_1+1}^{L'_2} a_n H_k(Y_{n+m})\right)^2$$

$$\leqq 2\sum_{p=L}^{\infty}\sum_{n=L}^{\infty} (|a_p|\,|a_n| + |a_{-p}|\,|a_{-n}|)\,|r(p-n)|^k.$$

Since $|r(n)| \leqq K(|n|+1)^{-\alpha+\delta}$ with arbitrary $\delta > 0$ and $K = K(\delta)$ for all n, and $|a_n| \leqq K|n|^{-\beta-1}$ for all $n \neq 0$ with an appropriate $K > 0$, to justify (2.1) it is enough to show that

$$I = \sum_{l=L}^{\infty}\sum_{n=L}^{\infty} l^{-(\beta+1)} n^{-(\beta+1)} (|l-n|+1)^{-k\alpha+\delta} < \varepsilon \tag{2.2}$$

for sufficiently large $L = L(\varepsilon) > 0$.

The following estimation can be made:

$$\sum_{n=L}^{\infty} n^{-(\beta+1)}(|l-n|+1)^{-k\alpha+\delta}$$

$$\leqq C\left[l^{-k\alpha+\delta}\sum_{n=1}^{l/2} n^{-(\beta+1)} + l^{-(\beta+1)}\sum_{n=l/2+1}^{2l} (|n-1|+1)^{-k\alpha+\delta}\right.$$

$$\left. + \sum_{n=2l+1}^{\infty} n^{-\beta-1-k\alpha+\delta}\right]. \tag{2.3}$$

(The letters C, K, etc. will denote appropriate constants from now on. The same letter may denote different constants in different formulae.)

In case $\beta > 0$ it can immediately be seen that the expression in (2.3) is finite, and then a simple substitution in (2.2) shows that relation (2.2) holds. In case $\beta \leqq 0$, let us first observe that $\beta + 1 + k\alpha - \delta > 1$ for small $\delta > 0$, because $\beta + 1 + k\alpha \geqq 3 - 2\left(1 - \beta - \frac{k}{2}\alpha\right) > 1$ by (1.13). Hence the expression in (2.3) can be estimated by $C\cdot l^{-\beta-k\alpha+\delta}$ in this case. Substituting this estimate into (2.3) we get that

$$I\leqq C\sum_{l=L}^{\infty} l^{-3+2(1-\beta-k/2\alpha)+\delta}<\varepsilon,$$

as we claimed. We remark that the condition $k\alpha<1$ was needed only in the estimation of the second term in (2.3). It can be substituted by the condition $\beta>-\frac{1}{2}$ which is a consequence of (1.13).

Now we express Z_n^N by means of multiple Wiener-Itô integrals. We can write

$$Y_n=\int \exp(inx)\, Z_G(dx),$$

where G is the spectral measure of the stationary sequence Y_n, $n=\ldots-1, 0, 1, \ldots$, and Z_G is the random spectral measure corresponding to it. By the definition of U_m and the Itô-formula (see e.g. formulae 4.14 and 4.15 in [1]) the identities

$$U_m=\int \sum_{j=-\infty}^{\infty} a_j \exp[i(m+j)(x_1+\ldots+x_k)]\, Z_G(dx_1)\ldots Z_G(dx_k)$$

and

$$\begin{aligned} Z_n^N=A_N^{-1}\int &\exp[inN(x_1+\ldots+x_k)] \\ &\cdot\sum_{j=-\infty}^{\infty} a_j \exp[ij(x_1+\ldots+x_k)] \\ &\cdot\sum_{l=0}^{N-1} \exp[(il(x_1+\ldots+x_k)]\, Z_G(dx_1)\ldots Z_G(dx_k) \end{aligned} \tag{2.4}$$

hold true.

We want to prove that for arbitrary positive integer p and real numbers $c_{-p}, c_{-p+1}, \ldots, c_p$ the sequence $\sum_{l=-p}^{p} c_l Z_l^N$ tends in distribution to the random variable $\sum_{l=-p}^{p} c_l Z_l^*$. For this end we change variables in formula (2.4) with the substitution $Nx_j=y_j$ (cf. Proposition 4.2 in [1]). This substitution indicates $\sum_{l=-r}^{p} c_l Z_l^N$ has the same distribution as

$$\begin{aligned} \sum_{l=-p}^{p} c_l \tilde{Z}_l^N=\sum_{l=-p}^{p} c_l \int &\exp[il(x_1+\ldots+x_k)]\, \bar{K}_N(x_1+\ldots+x_k) \\ &\cdot Z_{G_N}(dx_1)\ldots Z_{G_N}(dx_k), \end{aligned} \tag{2.5}$$

where

$$\bar{K}_N(x)=K_N(x)\, M_N(x)$$

with

$$K_N(x)=\frac{\exp(ix)-1}{N\left[\exp\left(i\frac{1}{N}x\right)-1\right]}, \tag{2.6}$$

$$M_N(x)=N^{\beta}\sum_{j=-\infty}^{\infty} a_j \exp\left(i\frac{j}{N}x\right) \tag{2.7}$$

and the measure G_N is defined by the relation

$$G_N(A) = \frac{N^\alpha}{L(N)} G\left(\frac{A}{N}\right) \tag{2.8}$$

for every measurable set A on the real line. The infinite sum in (2.7) may not convergence pointwise. Nevertheless, formula (2.7) is meaningful if we consider the limit in the infinite sum on its right hand side in L_2 sense with respect to the measure

$$|K_N(x_1+\ldots+x_k)|^2 G_N(dx_1)\ldots G_N(dx_k).$$

(And we have to consider it so in order to prove that the Wiener-Itô integral in (2.5) is meaningful.)

Indeed, since

$$K_N(x_1+\ldots+x_k) = \frac{1}{N} \sum_{j=0}^{N-1} \exp\left[-i\frac{j}{N}(x_1+\ldots+x_k)\right], \tag{2.9}$$

to make (2.7) meaningful, we have to show that

$$\sum_{l=-L_1}^{L_2} \sum_{n=-L_1}^{L_2} a_n a_l \sum_{j_1=0}^{N-1} \sum_{j_2=0}^{N-1} r^k(n-l+j_1-j_2)$$

has a limit for every N as $L_1, L_2 \to \infty$.

Since

$$\sum_{j_1=0}^{N-1} \sum_{j_2=0}^{N-1} |r^k(n-l+j_1-j_2)| \leqq C(N,\delta)(|l-n|+1)^{-k\alpha+\delta}$$

for every $\delta > 0$, and integers N, l, n, this fact follows immediately from (2.2).

In Proposition 1 of paper [3] it has been proved that

$$G_N([0,x]) \to G_0([0,x]) = (\alpha D)^{-1} x^\alpha \quad \text{for all } x \geqq 0. \tag{2.10}$$

(Actually, the norming factor $(\alpha D)^{-1}$ was determined in remark 1.2 of [3].)

Let us define the measures μ_N on $\mathscr{B}^k$, on the Borel σ-algebra of the k-dimensional Euclidean space, by the formula

$$\mu_N(A) = \int_A |\bar{K}_N(x_1+\ldots+x_k)|^2 G_N(dx_1)\ldots G_N(dx_k),$$

$$A \in \mathscr{B}^k, \quad N = 0, 1, 2, \ldots.$$

(This formula is meaningful also for $N=0$, since $\bar{K}_0$ is defined in (1.9).)

In Sect. 3 we shall prove the following

Lemma. *If the sequence a_n satisfies the conditions of the Theorem then the sequence of measures μ_N tends weakly to the measure μ_0 as $N \to \infty$, where the constants b and c in the definition of M_0 are determined in the same way as in the Theorem.*

By aid of this lemma and relation (2.10) we want to make a limiting procedure which leads to

$$\sum_{l=-p}^{p} c_p \tilde{Z}_l^N \xrightarrow{\mathcal{D}} \sum_{l=-p}^{p} c_1 \int \exp[il(x_1+\ldots+x_k)]\, \bar{K}_0(x_1+\ldots+x_k) \cdot Z_{G_0}(dx_1)\ldots Z_{G_0}(dx_k), \tag{2.11}$$

where $\xrightarrow{\mathcal{D}}$ denotes convergence in distribution. Relation (2.11) implies the Theorem. In order to see it, one has to observe that the expression on the right-hand side of (2.11) has the same distribution as $\sum_{l=-p}^{p} c_l Z_l^*$. This follows from the rule of change of variables in Wiener-Itô integrals with the substitution $G_0(dx)=D^{-1}|x|^{\alpha-1}\,dx$.

We shall prove (2.11) in an indirect way. Instead of the random variables $\tilde{Z}_l^N$ we shall work with some new random variables $\tilde{\tilde{Z}}_l^N$ defined below. These random variables $\tilde{\tilde{Z}}_l^N$ are close to $\tilde{Z}_l^N$ and are easier to handle.

First we define two numerical sequences b'_n and c'_n, which are close to the sequences b_n and c_n defined in (1.14) and (1.15), and which have some nice properties. Let $f(x)$ be a twice differentiable funciton with support in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $f(x)=1$ for $|x|\leqq\pi/4$, $f(x)$ is monotone decreasing for $x>0$ and monotone increasing for $x\leqq 0$.

Let

$$b'_n=\frac{1}{2\pi}\, C(\beta) \int_{-\pi}^{\pi} \exp(-inx)\,|x|^{\beta} f(x)\,dx, \quad \text{if } -1<\beta<1,\ \beta\neq 0$$

$$b'_n=b_n \quad \text{if } \beta=0,$$

$$c'_n=\frac{1}{2\pi}\, C'(\beta) \int_{-\pi}^{\pi} \exp(-inx)\,|x|^{\beta} f(x)\cdot i\,\operatorname{sign} x\,dx, \quad \text{if } -1<\beta<1,$$

where

$$C(\beta)=2[C(1)+C(2)]\,\Gamma(-\beta)\sin\frac{(\beta+1)\pi}{2} \quad \text{if } -1<\beta<1,\ \beta\neq 0$$

$$C'(\beta)=2[C(1)-C(2)]\,\Gamma(-\beta)\cos\frac{(\beta+1)\pi}{2} \quad \text{if } -1<\beta<1,\ \beta\neq 0,$$

and

$$C'(0)=2\pi\cdot C(1).$$

We shall prove in Sect. 3 that

$$b'_n=\frac{C(1)+C(2)}{2}\,|n|^{-\beta-1}+O(|n|^{-2}) \quad \text{if } -1<\beta<1,\ \beta\neq 0 \tag{2.12}$$

and

$$c'_n=\frac{C(1)-C(2)}{2}\,|n|^{-\beta-1}\operatorname{sign} n+O(|n|^{-2}) \quad \text{if } -1<\beta<1.$$

(Although relation (2.12) seems to be well-known among specialists in Fourier analysis we could not trace it in the literature. Therefore we shall prove it.)

As a consequence of (2.12) we obtain that

$$\sum_{n=-\infty}^{\infty} b'_n \exp(inx) = C(\beta)\,|x|^\beta f(x) = C(\beta)\,|x|^\beta, \qquad -1<\beta<1,\ \beta \neq 0, \quad (2.13)$$

and

$$\sum_{n=-\infty}^{\infty} c'_n \exp(inx) = i\,C'(\beta)\,|x|^\beta f(x)\,\operatorname{sign} x = i\,C'(\beta)\,|x|^\beta \operatorname{sign} x, \qquad -1<\beta<1 \quad (2.14)$$

for $|x| \leqq \pi/4$ with the only expectation $x=0$ in the case $-1<\beta<0$.

Indeed, the Fourier series (2.13) and (2.14) are convergent with the only exception $x=0$ if $-1<\beta \leqq 0$, hence the Fejér theorem, e.g., implies that they agree with $C(\beta)\,|x|^\beta$ and $i\,C'(\beta)\,|x|^\beta \operatorname{sign} x$ respectively.

The substitution $x=0$ into (2.13) gives that

$$\sum_{n=-\infty}^{\infty} b'_n = 0 \qquad \text{if } 1>\beta>0.$$

On the other hand, $b'_n = b'_{-n}$ and $c'_n = -c'_{-n}$. Hence the sequence $a'_n = b'_n + c'_n$ also satisfies the conditions of the Theorem. Let us define the random variables

$$\tilde{\tilde{Z}}_l^N = \int \exp[il(x_1+\ldots+x_k)]\,\bar{K}'_N(x_1+\ldots+x_k)\,Z_{G_N}(dx_1)\ldots Z_{G_N}(dx_k),$$
$$l = \ldots -1, 0, 1, \ldots,$$

where

$$\bar{K}'_N(x) = K_N(x)\,M'_N(x)$$

with

$$M'_N(x) = N^\beta \sum_{j=-\infty}^{\infty} a'_j \exp\left[i\frac{j}{N}x\right]. \quad (2.15)$$

First we show with the aid of the Lemma that

$$\mathbf{E}(\tilde{Z}_l^N - \tilde{\tilde{Z}}_l^N)^2 \to 0 \qquad \text{for all } l \text{ as } N \to \infty. \quad (2.16)$$

Write

$$\mathbf{E}(\tilde{Z}_l^N - \tilde{\tilde{Z}}_l^N)^2 = \frac{1}{k!}\int |\bar{K}_N(x_1+\ldots+x_k) - \bar{K}'_N(x_1+\ldots+x_k)|^2\,G_N(dx_1)\ldots G_N(dx_k)$$
$$= \frac{1}{k!}\,\mu''_N(R^k),$$

where the measure μ''_N is defined in the same way as μ_N, only $M_N(x)$ is replaced by $M''_N(x) = N^\beta \sum_{j=-\infty}^{\infty} (a_j - a'_j) \exp\left(i\frac{j}{N}x\right)$. Since the sequence $a_n - a'_n$ also satisfies the conditions of the Theorem, with $C(1)=C(2)=0$ in this case, the Lemma gives that $\mu''_N(R^k) \to 0$. Therefore relation (2.16) holds.

Because of (2.16), it is enough to show that

$$\sum_{l=-p}^{p} c_l \tilde{\tilde{Z}}_l^N \xrightarrow{\mathscr{D}} \sum_{l=-p}^{p} c_l \int \exp[il(x_1+\ldots+x_k)] \bar{K}_0(x_1+\ldots+x_k) Z_{G_0}(dx_1)\ldots Z_{G_0}(dx_k) \tag{2.17}$$

in order to prove the Theorem.

Let us assume for a while that $\beta \neq 0$. It can be seen by comparing formulae (2.13), (2.14), (2.15) and (1.10) that

$$M'_N(x) = M_0(x) \quad \text{if } |x| < \frac{\pi}{4} N \quad \text{and} \quad x \neq 0,$$

where the constants b and c in M_0 are defined the same way as in the Theorem. Consequently, given any $A>0$ and $\delta>0$

$$\hat{K}_N(x_1+\ldots+x_k) \to \hat{K}_0(x_1+\ldots+x_k) \tag{2.18}$$

on the set $B=B(A, \delta)$, and the convergence in (2.18) is uniform if

$$B=B(A,\delta)=\{(x_1,\ldots,x_k), |x_j| \leqq A, j=1,2,\ldots,k, \; |x_1+\ldots+x_k| > \delta\} \tag{2.19}$$

$$\hat{K}_N(x) = \sum_{l=-p}^{p} c_l \exp(ilx) K_N(x) M'_N(x)$$

and

$$\hat{K}_N(x) = \sum_{l=-p}^{p} c_l \exp(ilx) \bar{K}_0(x).$$

Let us further observe that the Lemma implies that

$$\mu_N^* \to \mu_0^* \quad \text{weakly as } N \to \infty, \tag{2.20}$$

where

$$\mu_N^*(A) = \int_A |\hat{K}_N(x_1+\ldots+x_k)|^2 G_N(dx_1)\ldots G_N(dx_k), \quad A \in \mathscr{B}^k, \; N=1,2,\ldots$$

and

$$\mu_0^*(A) = \int_A |\hat{K}_0(x_1+\ldots+x_k)|^2 G_0(dx_1)\ldots G_0(dx_k), \qquad A \in \mathscr{B}^k.$$

Now we would like to deduce (2.17) from the relations (2.10), (2.18) and (2.20) by the aid of Lemma 3 in [3]. This lemma cannot be applied directly since $\hat{K}_0(x_1+\ldots+x_k)$ is not continuous in the points $(x_1, \ldots, x_k)$ where $x_1+\ldots+x_k = 0$ if $\beta<0$. However, it is not difficult to prove (2.17) by slightly changing the argument of this lemma. Let us first observe that because of (2.20) and the absolute continuity of the measure μ_0^* with respect to the Lebesgue measure $\mu_0^*(R^k)<\infty$, and there exist some $B>0$ and $\delta>0$ such that $\mu_0^*(R^k - B(A, \delta)) < \varepsilon$, where $B=B(A,\delta)$ is defined in (2.19). Because of (2.20) the relation $\mu_N(R^k - B(A,\delta))<\varepsilon$ also holds for $N>N_0=N_0(\varepsilon, A)$. The last relation means that

$$\frac{1}{k!}[\mathbf{E}\int \hat{K}_N(x_1+\ldots+x_k)\, I_{R^k-B}(x_1,\ldots,x_k)\, Z_{G_N}(dx_1)\ldots Z_{G_N}(dx_k)]^2<\varepsilon \quad (2.21)$$

for $N=0$ or $N>N(\varepsilon)$, where $I_A(\cdot)$ denotes the indicator function of the set A. Since $\hat{K}_0$ is continuous on the set $B=B(A,\delta)$, and relations (2.10) and (2.18) hold, the functions $\hat{K}_N$, $N=0,1,2,\ldots$ can be approximated on B by elementary functions just the same way as in Lemma 3 of [3]. This approximation together with (2.21) implies

$$\begin{aligned}&\int \hat{K}_N(x_1+\ldots+x_k)\, Z_{G_N}(dx_1)\ldots Z_{G_N}(dx_k)\\ &\quad \xrightarrow{\mathscr{D}} \int \hat{K}_0(x_1+\ldots+x_k)\, Z_{G_0}(dx_1)\ldots Z_{G_0}(dx_k)\end{aligned}$$

which is a rewriting of (2.17).

The case $\beta=0$ can be discussed similarly. We have to remark that

$$M'_N(x_1+\ldots+x_k)\to M_0(x_1+\ldots+x_k) \quad \text{uniformly}$$

on every set $B=B(A,\delta)$, $A>0$, $\delta>0$, because of the condition $\sum_{n=-\infty}^{\infty} b_n<\infty$. Hence the argument applied in the case $\beta\neq 0$ can be repeated without any change. The proof of the Theorem using the Lemma is now completed.

3. Proof of the Lemma of the Corollary and of Formula (2.12)

Proof of the Lemma. Let us define the following modified Fourier transformation of the measures μ_N, $N=1,2,\ldots$

$$\varphi_N(t_1,\ldots,t_k)=\int_{R^k} \exp\left[\frac{i}{N}(j_1x_1+\ldots+j_kx_k)\right]\mu_N(dx_1,\ldots,dx_k), \quad (3.1)$$

where the integers $j_1,j_2,\ldots,j_k$ are determined by the relation

$$\frac{j_1}{N}\leq t_1<\frac{j_1+1}{N},\ \ldots,\ \frac{j_k}{N}\leq t_k<\frac{j_k+1}{N}. \quad (3.2)$$

We shall show that

$$\lim_{N\to\infty}\varphi_N(t_1,\ldots,t_k)=\varphi_0(t_1,\ldots,t_k) \quad \text{for all } t_1,t_2,\ldots,t_k \quad (3.3)$$

and $\varphi_0(t_1,\ldots,t_k)$ is a continuous function. As the measure μ_N is concentrated on the cube $[-N\pi, N\pi)^k$ formula (3.3) and Lemma 2 of [3] imply that the measures μ_N tend weakly to a measure $\bar{\mu}_0$ whose Fourier transform is $\varphi_0(t_1,\ldots,t_k)$. First we show that $\bar{\mu}_0$ is actually μ_0, i.e., that (3.3) implies the Lemma. This could be done by calculating the Fourier transform of μ_0. But since in this case we would have to tackle some inconvenient convergence problems we chose another way. We define an auxiliary stationary process in the following way.

Let Y_n, $n=\dots-1,0,1,\dots \mathbf{E}Y_0=0$, $\mathbf{E}Y_0^2=1$ be a stationary Gaussian sequence with spectral density $C|x|^{\alpha-1}$, $-\pi\leqq x<\pi$, and $C=\alpha/2\pi^{-\alpha}$. Let $X_n=H_k(Y_n)$ and $U_m=U_m(H_k)=\sum_{n=-\infty}^{\infty} a'_n X_{m+n}$ where the sequence a'_n agrees with the sequence a'_n defined in the proof of the Theorem.

Let us observe that

$$r(n)=C\int_{-\pi}^{\pi}\exp(inx)\,|x|^{\alpha-1}\,dx\sim CDn^{-\alpha}\qquad \text{as } n\to\infty.$$

Hence the process U_n belongs to the class of processes investigated in the Theorem. Thus, relation (3.3) implies that the measures $\bar{\mu}_N$,

$$\bar{\mu}_N(A)=\int_{A\cap[-N\pi,N\pi)^k}|M'_N(x_1+\dots+x_k)|^2\,|K_N(x_1+\dots+x_k)|^2 \cdot G_0(dx_1)\dots G_0(dx_k),\qquad A\in\mathscr{B}^k$$

tend weakly to the measure $\bar{\mu}_0$ as $N\to\infty$, where G_0 is defined in (2.10), M'_N in (2.15) and K_N in (2.6). (To see why the last relation holds one has to observe that $G_N(x)=\lambda_N G_0(x)$ for $|x|<N\pi$ with a sequence $\lambda_N\to 1$ as $N\to\infty$ if the auxiliary process U_n is considered.) We want to show it is actually the measure μ_0 that the measures μ_N converge to. In Sect. 2, we have already seen that

$$|M'_N(x_1+\dots+x_k)|^2\,|K_N(x_1+\dots+x_k)|^2 \to |M_0(x_1+\dots+x_k)|^2\,|K_0(x_1+\dots+x_k)|^2$$

uniformly on every bounded closed subset of R^k separated from the hyperplane $x_1+\dots+x_k=0$. Moreover, since $|K_N(x)|\leqq 1$ for all x on every bounded set $B\subset R^k$,

$$|M'_N(x_1+\dots+x_k)|^2\,|K_N(x_1+\dots+x_k)|^2\leqq K+K|x_1+\dots+x_k|^{2\beta}$$

with an appropriate $K>0$ if $N>N_0=N_0(B)$. (We wrote the constant term K in the last inequality only to include the case $\beta=0$.) Hence to prove $\bar{\mu}_N\to\mu_0$ (which implies also $\mu_N\to\mu_0$) it is sufficient to show that for all $A>0$ and $\varepsilon>0$ there exists a $\delta>0$ such that

$$\int_{U(A,\delta)}[1+|x_1+\dots+x_k|^{2\beta}]\,|x_1|^{\alpha-1}\dots|x_k|^{\alpha-1}\,dx_1,\dots,dx_k<\varepsilon, \tag{3.4}$$

where the set $U(A,\delta)$ is defined by

$$U(A,\delta)=\{(x_1,\dots,x_k),\ |x_j|\leqq A, j=1,2,\dots k,\ |x_1+\dots+x_k|<\delta\}.$$

To prove (3.4) let us first observe that (3.3) together with Lemma 2 in [3] imply that $\sup_N \mu_N(R^k)<\infty$.

Hence

$$\int_{|x_1+\dots+x_k|<1}|x_1+\dots+x_k|^{2\beta}\,|x_1|^{\alpha-1}|x_k|^{\alpha-1}\,dx_1\dots dx_k=K<\infty.$$

Therefore the homogeneity of the integrands implies that

$$\int_{U(A,\delta)} |x_1+\dots+x_k|^{2\beta}|x_1|^{\alpha-1}\dots|x_k|^{\alpha-1}dx_1\dots dx_k$$
$$<\int_{|x_1+\dots+x_k|<\delta} |x_1+\dots+x_k|^{2\beta}|x_1|^{\alpha-1}\dots|x_k|^{\alpha-1}dx_1\dots dx_k=K\delta^{2\beta+k\alpha},$$

and

$$\int_{U(A,\delta)} |x_1|^{\alpha-1}\dots|x_k|^{\alpha-1}dx_1\dots dx_k<K'\delta^{k\alpha}.$$

These inequalities imply (3.4) because $2\beta+k\alpha>0$, $k\alpha>0$ by (1.13).

Now we turn to the proof of (3.3). By writing $K_N(x_1+\dots+x_k)$ in the form (2.9) it can be seen that

$$\begin{aligned}\varphi_N&(t_1,\dots,t_k)\\ &=\frac{N^{k\alpha+2\beta-2}}{L(N)^k}\sum_{u_1,u_2=-\infty}^{\infty} a_{u_1}a_{u_2}\sum_{v_1,v_2=0}^{N-1}\prod_{l=1}^{k} r(u_1+v_1-u_2-v_2+j_1)\\ &=\frac{N^{k\alpha+2\beta-2}}{L(N)^k}\sum_{u=-\infty}^{\infty}\sum_{v=-\infty}^{\infty}\gamma_u(N)\,\gamma_v(N)\,r(u-v+j_1)\dots r(u-v+j_k),\end{aligned}\tag{3.5}$$

where

$$\gamma_u(N)=\sum_{k=-u}^{N-u-1} a_k.\tag{3.6}$$

Let us remark that with a slight modification in the argument which shows that (2.7) is meaningful it can be proved that the middle term in (3.5) is absolutely convergent. Hence the rearrangement made in (3.5) is legitimate. Let us define the functions

$$P_N(x,y,t_1,\dots,t_k)=N^{2\beta}\gamma_{[Nx]}(N)\,\gamma_{[Ny]}(N)\sum_{l=1}^{k}\frac{N^{\alpha}}{L(N)}r([Nx]-[Ny]+[Nt_1])\tag{3.7}$$

Formula (3.1) can be rewritten in the form

$$\varphi_N(t_1,\dots,t_k)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}P_N(x,y,t_1,\dots,t_k)\,dx\,dy.\tag{3.8}$$

We claim that

$$N^{\beta}\gamma_{[Nx]}(N)\to\gamma(x)\quad\text{as } N\to\infty,\tag{3.9}$$

where

$$\gamma(x)=\begin{cases}\dfrac{C(1)}{\beta}[|x|^{-\beta}-(1+|x|)^{-\beta}] & \text{if } x<0\\[2mm] -\dfrac{C(1)+C(2)}{2\beta}[x^{-\beta}+(1-x)^{-\beta}]+\dfrac{C(1)-C(2)}{2\beta}[x^{-\beta}-(1-x)^{-\beta}] & \text{if } 0<x<1\\[2mm] -\dfrac{C(2)}{\beta}[x^{-\beta}-(x-1)^{-\beta}] & \text{if } x>1\end{cases}$$

for $\beta \neq 0$, and

$$\gamma(x)=\begin{cases} C(1)\,[\log|x|-\log(1+|x|)] & \text{if } x<0\\ C(2)\,[\log x-\log(1-x)]+b & \text{if } 0<x<1\\ C(2)\,[\log x-\log(x-1)] & \text{if } x>1\end{cases}$$

for $\beta=0$.

In checking (3.9) the case $0<x<1$ deserves some special attention. In this case one has to write $a_n=b_n+c_n$ and then exploit the relation $c_n=-c_{-n}$, and that in the case $\beta>0$ the relation

$$\sum_{k=-u}^{N-u-1} b_k=-\sum_{k=-\infty}^{-u-1} b_k-\sum_{k=N-u}^{\infty} b_k$$

holds.

Moreover, the inequality

$$|N^\beta \gamma_{[Nx]}(N)|\leqq \bar{\gamma}(x)=\gamma_1(x)+\gamma_2(x)+\gamma_3(x)+\gamma_4(x) \tag{3.10}$$

holds true, where

$$\begin{aligned}\gamma_1(x)&=C|x|^{-(\beta+1)}I(x\leqq -\tfrac{1}{2})\\ \gamma_2(x)&=C(|x|^{-\beta}+1)\,I(|x|\leqq \tfrac{1}{2})\\ \gamma_3(x)&=C(|x-1|^{-\beta}+1)\,I(\tfrac{1}{2}<x<2)\\ \gamma_4(x)&=C|x|^{-(\beta+1)}I(x\geqq 2)\end{aligned}$$

with an appropriate $C>0$ if $\beta \neq 0$, and

$$\begin{aligned}\gamma_1(x)&=\frac{C}{|x|}I(x\leqq -\tfrac{1}{2})\\ \gamma_2(x)&=C|x|^{-\varepsilon}I(|x|\leqq \tfrac{1}{2})\\ \gamma_3(x)&=C|x-1|^{-\varepsilon}I(\tfrac{1}{2}<x<2)\\ \gamma_4(x)&=\frac{C}{x}I(x\geqq 2)\end{aligned}$$

with an arbitrary $\varepsilon>0$ and $C=C(\varepsilon)>0$ if $\beta=0$. ($\bar{\gamma}(x)=\infty$ if $x=0$ or $x=1$ and $\beta\geqq 0$.)

One can prove (3.10) similarly to (3.9) by exploiting the inequality $|a_n|\leqq K(|n|+1)^{-\beta-1}$ with an appropriate $K>0$. In checking (3.10) one has to be more careful in the regions $x\in\left[-\frac{C}{N},\frac{C}{N}\right)$ and $x\in\left(1-\frac{C}{N},1+\frac{C}{N}\right)$. Inequality (3.10) holds also in these regions, since

$$|N^\beta \gamma_N(u)|\leqq \begin{cases} CN^\beta & \text{if } \beta>0\\ C & \text{if } \beta<0\\ C\log N & \text{if } \beta=0\end{cases} \tag{3.10'}$$

for all u.

On the other hand

$$\frac{N^\alpha}{L(N)} r([Nx]-[Ny]+[Nt_1]) \to |x-y+t_1|^{-\alpha} \quad \text{as } N\to\infty,\ l=1,2,\dots,k.$$

This formula together with (3.9) imply that in formula (3.7) a limit can be taken which gives

$$P_N(x,y,t_1,\dots,t_k) \to P_0(x,y,t_1,\dots,t_k) = \gamma(x)\,\gamma(y) \prod_{l=1}^{k} |x-y+t_1|^{-\alpha}.$$

We are going to show that a formal limiting procedure in (3.8) is legitimate, i.e.,

$$\lim_{N\to\infty} \varphi_N(t_1,\dots,t_k) = \varphi_0(t_1,\dots,t_k) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_0(x,y,t_1,\dots,t_k)\,dx\,dy \qquad (3.11)$$

and the integral in (3.11) is finite.

Karamata's theorem (see e.g. [4]) implies that for every $\varepsilon>0$ there exists a constant $K=K(\varepsilon)>0$ such that

$$L(uN) \leqq K L(N)\, u^{-\varepsilon} \quad \text{for } 0<u<1$$
$$L(uN) \leqq K L(N)\, u^{\varepsilon} \quad \text{for } u\geqq 1.$$

These inequalities together with the relation $|r(n)|\leqq 1$ for all n imply that

$$\frac{N^\alpha}{L(N)} r([Nx]-[Ny]+[Nt_l]) \leqq K(\varepsilon)[|x-y+t_l|^{-\alpha+\varepsilon} + |x-y+t_l|^{-\alpha-\varepsilon}].$$

(Relation $|r(n)|\leqq 1$ was needed to show that the above inequality holds also in the case $|x-y+t_l|<CN^{-1}$).

The last inequality together with (3.10) imply that

$$|P_N(x,y,t_1,\dots t_k)| \leqq K\bar{\gamma}(x)\,\bar{\gamma}(y) \prod_{l=1}^{k} [|x-y+t_l|^{-\alpha-\varepsilon} + |x-y+t_l|^{-\alpha+\varepsilon}]$$
$$= \bar{P}(x,y,t_1,\dots,t_k).$$

Hence by the dominated convergence theorem it is enough to prove that

$$J(t_1,\dots,t_k) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{P}(x,y,t_1,\dots,t_k)\,dx\,dy < \infty$$
$$\text{for arbitrary } t_1,\dots,t_k \qquad (3.12)$$

in order to prove (3.11).

In proving (3.12) it is enough to restrict ourselves to the case $t_1=t_2=\dots t_k =t$, since Hölder's inequality implies that

$$J(t_1,\dots,t_k) \leqq \left[\prod_{l=1}^{k} J(t_l,\dots,t_l)\right]^{1/k}.$$

Let us first consider the integral

$$I(y,t)=\int_{-\infty}^{\infty}\bar{\gamma}(x)[|x-y+t|^{-k(\alpha+\varepsilon)}+|x-y+t|^{-k(\alpha-\varepsilon)}]\,dx.$$

We claim that

$$I(y,t)\leqq B(y,t)=B_1(y-t)+B_2(y,t) \tag{3.13}$$

with

$$B_1(y)=K[(|y|+1)^{-\beta-k(\alpha-\varepsilon)}+(|y|+1)^{-k(\alpha-\varepsilon)}]$$
$$B_2(y,t)=KI(|t-y|\leqq 2)[1+|y-t|^{1-\beta-k(\alpha+\varepsilon)}+|y-t-1|^{1-\beta-k(\alpha+\varepsilon)}]$$

where the constant K may depend on $\alpha, \beta, k, \varepsilon$ and t. We assume that $\varepsilon>0$ is sufficiently small, and $\beta\neq 0$. The case $\beta=0$ is similar but simpler. Let us introduce the notation

$$I_j(y,t)=\int \gamma_j(x)[|x-y+t|^{-k(\alpha+\varepsilon)}+|x-y+t|^{-k(\alpha-\varepsilon)}]\,dx, \qquad j=1,2,3,4.$$

To prove (3.13) it is enough to show that

$$I_j(y,t)\leqq B(y,t), \qquad j=1,2,3,4.$$

Let us observe that $\beta+1+k\alpha=\left(\beta+\frac{k}{2}\alpha-1\right)+\frac{k}{2}\alpha+2>1$ by (1.13). Hence in case $|t-y|>2$ the following estimates hold true:

$$\begin{aligned}
I_1(y,t)&\leqq C\int_{-\infty}^{-1/2}|x|^{-\beta-1}|x-y+t|^{-k(\alpha-\varepsilon)}\,dx\\
&=C\left[\int_{-\infty}^{-2|t-y|}+\int_{-2|t-y|}^{-\frac{1}{2}|t-y|}+\int_{-\frac{1}{2}|t-y|}^{-\frac{1}{2}}\right]\\
&\leqq C'\left[\int_{-\infty}^{-2|t-y|}|x|^{-\beta-1-k(\alpha-\varepsilon)}\,dx+\int_{-2|t-y|}^{-\frac{1}{2}|t-y|}|t-y|^{-\beta-1}|x-y+t|^{-k(\alpha-\varepsilon)}\,dx\right.\\
&\quad\left.+\int_{-\frac{1}{2}|t-y|}^{-\frac{1}{2}}|x|^{-\beta-1}|t-y|^{-k(\alpha-\varepsilon)}\,dx\right]\\
&\leqq C''|t-y|^{-k(\alpha-\varepsilon)}[1+|t-y|^{-\beta}]\leqq B_1(y-t),
\end{aligned}$$

and

$$I_2(y,t)\leqq C|t-y|^{-k(\alpha-\varepsilon)}\int_{-\frac{1}{2}}^{\frac{1}{2}}(|x|^{-\beta}+1)\,dx<C'|t-y|^{-k(\alpha-\varepsilon)}\leqq B_1(y-t).$$

In case $|t-y|\leqq 2$

$$I_1(y,t)\leqq C\left[\int_{-\infty}^{-4}|x|^{-\beta-1-k(\alpha-\varepsilon)}\,dx+\int_{-4}^{-\frac{1}{2}}|x-y+t|^{-k(\alpha+\varepsilon)}\,dx\right]<C'<B_2(y,t),$$

and

$$\begin{aligned}
I_2(y,t) &\leqq C \int_{-\frac{1}{2}}^{\frac{1}{2}} (|x|^{-\beta}+1)\,|x-y+t|^{-k(\alpha+\varepsilon)}\,dx \\
&\leqq C\Big[\int_{|x|<\frac{1}{2}|t-y|} + \int_{\frac{1}{2}|t-y|<|x|<2|t-y|} + \int_{2|t-y|<|x|<\frac{1}{2}}\Big] \\
&\leqq C'\Big[\int_{|x|<\frac{1}{2}|t-y|} (|x|^{-\beta}+1)\,|y-t|^{-k(\alpha+\varepsilon)}\,dx \\
&\quad + \int_{\frac{1}{2}|t-y|<|x|<2|t-y|} [|y-t|^{-\beta}+1]\,|x-y+t|^{-k(\alpha+\varepsilon)}\,dx \\
&\quad + \int_{2|t-y|<|x|<\frac{1}{2}} (|x|^{-\beta}+1)\,|x|^{-k(\alpha+\varepsilon)}\,dx\Big] \\
&\leqq C''[1+|t-u|^{1-k(\alpha+\varepsilon)-\beta}] \leqq B_2(y,t).
\end{aligned}$$

The terms $I_3(y,t)$ and $I_4(y,t)$ can be estimated similarly. To prove (3.12) with the help of (3.13) it is enough to show that

$$\int_{-\infty}^{\infty} \bar{\gamma}(y)\,B(y,t)\,dt < \infty. \tag{3.14}$$

Inequality (3.14) holds because, if ε is sufficiently small, the function $\bar{\gamma}(y)\,B(y,t)$ tends to zero faster than $|y|^{-c}$ with some $c>1$ in plus and minus infinity, and its finite singularities are smaller than $|y|^{-c'}$ with some $c'<1$. These facts follow from the inequalities

$$2\beta+k\alpha>0,\ \beta+k\alpha>0 \quad \text{and} \quad \beta<1,\ k\alpha+2\beta<2,\ k\alpha+\beta<2$$

which are consequences of (1.13).

To complete the proof of the Lemma, it suffices to show that $\varphi_0(t_1, \dots, t_k)$ is a continuous function. This follows from the following consideration: The above made estimations imply that for all $\varepsilon>0$ and $t_1, \dots t_k$ there exists an $A>0$, $\delta>0$ and a neighbourhood $B(t_1, \dots, t_k) \subset R^k$ of the point $(t_1, \dots, t_k) \in R^k$ such that

$$\int_{R^2-D(A,\delta)} P_0(x, y, s_1, \dots, s_k)\,dx\,dy < \varepsilon \qquad \text{for all } (s_1, \dots, s_k) \in B(t_1, \dots, t_k)$$

where

$$\begin{aligned}
D(A,\delta) &= D(A, \delta, t_1, \dots, t_k) \\
&= [-A, A] \times [-A, A] - \bigcup_{j=1}^{k} \{(x,y)\colon |x-t_j|<\delta,\ |y-t_j|<\delta\}.
\end{aligned}$$

Then the continuity of the function $P_0(x, y, t_1, \dots, t_k)$ in all of its variables on the set $D \times B(t_1, \dots, t_k)$ implies the continuity of φ_0 in the point $(t_1, \dots, t_k)$. The Lemma is proven.

Proof of the Corollary. Since $\mathbf{E} H_k(X_n)\,H_l(X_m)=0$ for all n, m if $k \neq l$,

$$\mathbf{E}\Big(\sum_{l=k+1}^{\infty} d_l Z_n^N(H_l)\Big)^2 = \sum_{l=k+1}^{\infty} d_l^2\,\mathbf{E}(Z_n^N(H_l))^2.$$

On the other hand

$$\mathbf{E}(Z_n^N(H_l))^2 = \frac{1}{l!} \sum_{u=-\infty}^{\infty} \sum_{v=-\infty}^{\infty} \gamma_u(N)\, \gamma_v(N)\, r(u-v)^l,$$

where $\gamma_u(N)$ is defined in (3.6).

We are going to show that

$$A_N^{-2}\mathbf{E}[Z_n^N(H) - d_k Z_n^N(H_k)]^2 = A_N^{-2} \sum_{l=k+1}^{\infty} d_l^2 \mathbf{E}(Z_n^N(H_l))^2 \to 0. \tag{3.15}$$

Formula (3.15) implies the Corollary.

The identity

$$\mathbf{E}(Z_n^N(H_l))^2 = (l!)^{-1} \varphi_N(0, \ldots, 0)$$

holds true, where φ_N is defined in (3.1), only we have to substitute k by l in it. In the Lemma we have proved that if $l > k$, $1 - \beta - \frac{1}{2}\alpha > 0$, $l\alpha < 1$ then

$$A_N^{-2}\mathbf{E} Z_n^N(H_l)^2 = A_N^{-2} O(N^{2-2\beta-l\alpha} L(N)^l) \to 0.$$

Hence, because of (1.17), in order to prove (3.15) it is enough to show that

$$\sum_{u=-\infty}^{\infty} \sum_{v=-\infty}^{\infty} \gamma_u(N), \gamma_v(N)\, r(u-v)^l \leqq \begin{cases} K(\varepsilon)\, N^{\varepsilon} & \text{if } \beta \geqq \frac{1}{2} \\ K(\varepsilon)\, N^{1-2\beta+\varepsilon} & \text{if } \beta < \frac{1}{2} \end{cases}$$

for every $\varepsilon > 0$ and l satisfying the relation

$$1 - \beta - \tfrac{1}{2}\alpha \leqq 0 \quad \text{or} \quad l\alpha \geqq 1, \tag{3.16}$$

where $K(\varepsilon) > 0$ depends on ε, but not on l. (Let us remark that $1 - \beta - \frac{k}{2}\alpha > 1 - \beta$, therefore $A_N^2 > N^{1-2\beta+\varepsilon}$ for small ε.) The inequality $|r(n)| < Cn^{-\alpha+\delta}$ holds for every n, $n \neq 0$, and $\delta > 0$ with an appropriate $C = C(\delta)$, hence

$$|r(n)|^l \leqq \begin{cases} n^{-1+\varepsilon} & \text{if } \beta < \frac{1}{2} \\ n^{-2+2\beta+\varepsilon} & \text{if } \beta \geqq \frac{1}{2} \end{cases}$$

for every $\varepsilon > 0$, $|n| > C(\varepsilon)$ and l satisfying (3.16).

Because of the last estimate and (3.10),

$$\Big| \sum_{|u-v| > C(\varepsilon)}\!\!\sum \gamma_u(N)\, \gamma_v(N)\, r(u-v)^l \Big| \leqq N^{\lambda - 2\beta + 2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \bar{\gamma}(u)\, \bar{\gamma}(v)\, |u-v|^{\lambda}\, du\, dv$$

with $\lambda = \varepsilon - 1$ if $\beta < \frac{1}{2}$, and $\lambda = -2 + 2\beta + \varepsilon$ if $\beta \geqq \frac{1}{2}$. But in the proof of (3.12) actually we have proved that the last integral is finite. Because of the inequalities

$$|r(n)| \leqq 1, \; |\gamma_u(N)\, \gamma_v(N)| \leqq \tfrac{1}{2}(\gamma_u^2(N) + \gamma_v^2(N)),$$

it is enough to show that

$$\sum\sum_{|u-v|\leq C(\varepsilon)} [\gamma_u^2(N)+\gamma_v^2(N)] \leqq 4C(\varepsilon) \sum_{u=-\infty}^{\infty} \gamma_u^2(N)$$
$$\leqq \begin{cases} K(\varepsilon)\,N^{\varepsilon} & \text{if } \beta \geqq \frac{1}{2} \\ K^{\varepsilon} N^{1-2\beta+\varepsilon} & \text{if } \beta < \frac{1}{2} \end{cases} \tag{3.17}$$

in order to complete the proof of (3.15).

Inequality (3.17) can be deduced from (3.10) and (3.10)′.

In the case $\beta<\frac{1}{2}$, $\beta\neq 0$

$$\sum_{u=-\infty}^{\infty} \gamma_u^2(N) < N^{1-2\beta} \int_{-\infty}^{\infty} \bar{\gamma}^2(x)\,dx \leqq K N^{1-2\beta}$$

and in the case $\beta\geqq\frac{1}{2}$

$$\sum_{x=-\infty}^{\infty} \gamma_u^2(N) \leqq K \sum_{n=-2N}^{2N} (|n|+1)^{-2\beta} + N^{1-2\beta} \int_{|x|>2} \bar{\gamma}^2(x)\,dx < K'.$$

(We exploited that (1.13) implies that $\beta>-\frac{1}{2}$, therefore $\bar{\gamma}^2(x)$ tends to zero fast enough in plus and minus infinity.) In case $\beta=\frac{1}{2}$ the same argument shows that $K\log N$ is a good upper bound in (3.17). The case $\beta=0$ can be discussed with some slight changes. The proof of the Corollary is now complete.

Proof of Formula (2.12). Let us first consider the case $-1<\beta<0$. We can write, applying the substitution $nx=y$,

$$I(n)=\int \exp(inx)\,|x|^{\beta} f(x)\,dx = n^{-\beta-1}\int \exp(iy)\,|y|^{\beta} f\left(\frac{y}{n}\right) dy.$$

It is easy to see that

$$\lim_{n\to\infty} \int \exp(iy)\,|y|^{\beta} f\left(\frac{y}{n}\right) dy = \int_{-\infty}^{\infty} \exp(iy)\,|y|^{\beta}\,dy.$$

We want to investigate the speed of convergence in the last formula. Let us consider the expression

$$I(N,n)=\int \exp(iy)\,|y|^{\beta}\left[f\left(\frac{y}{n}\right)-f\left(\frac{y}{N}\right)\right] dy.$$

for arbitrary N, n, $N>n>0$. Integrating by parts, and exploiting the fact that $f\left(\frac{y}{n}\right)-f\left(\frac{y}{N}\right)=0$ for $y<\frac{\pi}{4}n$, we get

$$I(N,n) = -i \int_{|y|>\frac{\pi}{4}n} \exp(iy)\left[|y|^{\beta}\left(\frac{1}{n}f'\left(\frac{y}{n}\right)-\frac{1}{N}f'\left(\frac{y}{N}\right)\right)\right.$$
$$\left.+\beta\,|y|^{\beta-1}\left[f\left(\frac{y}{n}\right)-f\left(\frac{y}{N}\right)\right]\right] dy.$$

$f'(y/N)$ and $f'(y/n)$ are functions of bounded variation, and they are vanishing at large y. Hence for $y>0$ they can be written as the difference of two monotone decreasing functions with a compact support. A similar decomposition holds for $y<0$. These properties yield that

$$\left| \int_{|y|>\frac{\pi}{4}n} \exp(iy)\frac{1}{n} f'\left(\frac{y}{n}\right) |y|^\beta \, dy \right| \leqq K n^{\beta-1}$$

and

$$\left| \int_{|y|>\frac{\pi}{4}n} \exp(iy)\frac{1}{N} f'\left(\frac{y}{N}\right) |y|^\beta \, dy \right| \leqq K n^{\beta-1}.$$

Quite similarly

$$\left| \int_{|y|>\frac{\pi}{2}n} |y|^{\beta-1} \left[f\left(\frac{y}{n}\right) - f\left(\frac{y}{N}\right) \right] \exp(iy)\, dy \right| \leqq K n^{\beta-1}.$$

These estimates together imply that

$$I(N, n) \leqq K n^{\beta-1},$$

where the constant K does not depend on N. Letting N go to infinity we get that

$$I(n) = n^{-\beta-1} \int_{-\infty}^{\infty} \exp(iy)|y|^\beta \, dy + O(n^{-2}).$$

The relation

$$\int \exp(inx)|x|^\beta f(x) \operatorname{sign} x \, dx = n^{-\beta-1} \int_{-\infty}^{\infty} \exp(iy)|y|^\beta \operatorname{sign} y \, dy + O(n^{-2}),$$

if $-1<\beta<0$ can be proved in the same way.

If $1>\beta>0$ we get, on integrating by parts, that

$$\int \exp(inx)|x|^\beta f(x)\, dx = -\frac{i}{n} \int \exp(inx)[|x|^\beta f'(x) + \beta |x|^{\beta-1} f(x)]\, dx.$$

We have already seen that

$$-i\frac{\beta}{n} \int \exp(inx)|x|^{\beta-1} f(x)\, dx = -i\beta n^{-\beta-1} \int_{-\infty}^{\infty} \exp(ix)|x|^{\beta-1}\, dx + O(n^{-2}).$$

Integrating by parts again we get that

$$n^{-1} \int \exp(inx)|x|^\beta f'(x)\, dx = n^{-2} \cdot \int \exp(inx)[|x|^\beta f''(x) + |x|^{\beta-1} \beta f'(x)]\, dx = O(n^{-2}).$$

These calculations together yield that

$$\int \exp(inx)|x|^\beta f(x)\, dx = -i\beta \int_{-\infty}^{\infty} \exp(inx)|x|^{\beta-1}\, dx \cdot n^{-\beta-1} + O(n^{-2}) \qquad \text{for } 0<\beta<1.$$

Similary

$$\int \exp(inx)|x|^\beta \operatorname{sign} x \cdot f(x)\,dx = -i\beta \int_{-\infty}^{\infty} \exp(inx)|x|^{\beta-1} \operatorname{sign} x\,dx \cdot n^{-\beta-1} + O(n^{-2})$$

for $0<\beta<1$.

The estimate

$$\int \exp(inx) \operatorname{sign} x f(x)\,dx = 2in^{-1} + O(n^{-2})$$

can be proved similary to the case $-1<\beta<0$.

These relations together with the identities

$$\int_0^\infty t^{\beta-1} \exp(-it)\,dt = \Gamma(\beta)\exp(-\tfrac{1}{2}\pi i\beta) \quad \text{for } 0<\beta<1,$$

$$\Gamma(\beta+1)=\beta\Gamma(\beta) \quad \text{and} \quad \Gamma(1-\beta)\Gamma(\beta)=\pi \sin\pi\beta$$

imply formula (2.12).

4. Possible Generalizations, Comments

Remark 4.1. Condition (iii) in the Theorem and also in the Corollary can be substituted with the following weaker condition (iii)';

$$\text{(iii)}', \quad a_n=b_n+c_n, \quad n=\ldots-1,0,1,\ldots, \quad \sum_{n=-\infty}^{\infty} |b_n|<\infty, \quad c_n=-c_{-n},$$

$n=0,1,2,\ldots,$ and $c_n=C(1)n^{-1}+o(n^{-1})$ as $n\to\infty$. (Condition (1.14) is not assumed to be satisfied.)

In this case also $b=\sum_{n=-\infty}^{\infty} b_n$, $c=C(1)$ in the Theorem.

This strengthened form of the Theorem contains Theorem A as a special case.

Remark 4.2. The class of self-similar processes defined in (1.8) is only a special case of the class constructed by Dobrushin in [1]. The process

$$Z_n^* = \int \exp[in(x_1+\ldots+x_k)]\, K_0(x_1+\ldots+x_k)\, M(x_1,\ldots,x_k) \cdot Z_{G_0}(dx_1)\ldots Z_{G_0}(dx_k)\; n=\ldots-1,0,1,\ldots \tag{4.1}$$

is also self-similar if the following conditions are satisfied:

a1) $M(x_1,\ldots,x_k)=\overline{M(-x_1,\ldots,-x_k)}$,

a2) $M(x_{\pi(1)},\ldots,x_{\pi(k)})=M(x_1,\ldots,x_k)$ for all $\pi\in\Pi$,

where Π denotes the set of all permutations of the numbers $1,2,\ldots,k$.

a3) There exists a β such that

$$M(\gamma x_1,\ldots,\gamma x_k)=\gamma^\beta M(x_1,\ldots,x_k) \quad \text{for all } \gamma>0 \text{ and } x_1,\ldots,x_k\in R.$$

b) Z_{G_0} is the random spectral measure corresponding to the function $G_0(x) = |x|^\alpha$, with some $\alpha > 0$.

c) $\int |K_0(x_1 + \ldots + x_k)|^2 |M(x_1, \ldots, x_k)|^2 G_0(dx_1) \ldots G_0(dx_k) < \infty$.

The self-similarity parameter of the process Z_n^* defined in (4.1) is $\beta + \frac{k}{2}\alpha$. (The representation of the processes Z_n^* in the form (4.1) is not unique. The class of self-similar fields constructed by Dobrushin consists of the linear combination of the above defined processes.)

The aim of the present remark is to illuminate the content of the Theorem, and to indicate how to look for a large class of stationary sequences $\ldots U_{-1}, U_0, U_1, \ldots$ such that the sequences Z_n^N defined by (1.7) and these U_m tend to a sequence Z_n^* of the form (4.1).

Let $g(x)$ be a bounded function on the real line vanishing outside the interval $[-\pi, \pi)$ and such that $g(x) \geqq 0$, $g(x) = g(-x)$ for all x, $g(0) = 1$, and g is continuous in zero. Define the spectral measure $\hat{G}$,

$$\hat{G}(A) = \int_A g(x)\, G_0(dx), \qquad A \in \mathscr{B}.$$

Let $f: R^k \to R^1$ be a bounded function satisfying a1) and a2) and having the following properties: It is continuous in the origin, $f(0, \ldots, 0) = 1$, and

$$f(x_1, \ldots, x_k) = 0 \quad \text{if } |x_1 + \ldots + x_k| > \frac{\pi}{2}.$$

Set

$$\hat{M}(x_1, \ldots, x_k) = M(x_1, \ldots, x_k) f(x_1, \ldots, x_k)$$

and

$$U_m = \int \exp[im(x_1 + \ldots + x_k)]\, \hat{M}(x_1, \ldots, x_k) \cdot Z_{\hat{G}}(dx_1) \ldots Z_{\hat{G}}(dx_k), \qquad m = \ldots -1, 0, 1, \ldots. \tag{4.2}$$

Let us impose the following condition a4) on the function M.

a4) The set of points where M is discontinuous has zero Jordan measure, i.e. for every $\varepsilon > 0$ and $K > 0$ the intersection of this set with the cube $[-K, K]^k$ can be covered with finitely many rectangulars whose total volume is less than ε.

Now we formulate the following

Proposition. *The processes Z_n^N defined by* (4.2) *and* (1.7) *with the choice $A_N = N^{1-\beta-\frac{k}{2}\alpha}$ tend in distribution to the process Z_n^* defined in* (4.1) *provided that the function M satisfies the additional condition* a4).

Proof of the Proposition. Let us consider an arbitrary integer p and real numbers $c_{-p}, \ldots, c_p$. By change of variables it can be seen that the random variable $\sum_{l=-p}^{p} c_l Z_l^N$ has the same distribution as

$$\int J_N(x_1, \ldots, x_k)\, Z_{\hat{G}_N}(dx_1) \ldots Z_{\hat{G}_N}(dx_k) \tag{4.3}$$

where

$$J_N(x_1, \dots, x_k) = \sum_{l=-p}^{p} c_l \exp[il(x_1 + \dots + x_k)]$$
$$\cdot N^\beta \hat{M}_N\left(\frac{x_1}{N}, \dots, \frac{x_k}{N}\right) K_N(x_1 + \dots + x_k),$$

and

$$\hat{G}_N(A) = N^\alpha \hat{G}\left(\frac{A}{N}\right), \qquad A \in \mathscr{B}.$$

Let us introduce the notation

$$J_0(x_1, \dots, x_k) = \sum_{l=-p}^{p} c_l \exp[il(x_1 + \dots + x_k)] M_0(x_1, \dots, x_k) K_0(x_1 + \dots + x_k).$$

It is easy to see that the density function $\hat{g}_n$ of the measure $\hat{G}_N$ satisfies the inequality

$$\hat{g}_N(x) \leqq C \frac{d}{dx} G_0(x).$$

On the other hand

$$J_N(x_1, \dots, x_k)| \leqq C' |J_0(x_1, \dots, x_k)|, \qquad N = 1, 2, \dots$$

since property a3) holds for M, and either $|x_1 + \dots + x_k| > \frac{\pi}{2} N$, and $\hat{M}\left(\frac{x_1}{N}, \dots, \frac{x_k}{N}\right) = 0$ or $|x_1 + \dots + x_k| < \frac{\pi}{2} N$ and

$$|K_N(x_1 + \dots + x_k)| \leqq \tfrac{\pi}{2} |K_0(x_1 + \dots + x_k)|.$$

These relations together with property c) of the function M imply that for every $\varepsilon > 0$ there exists a $K = K(\varepsilon)$ such that

$$\mathbf{E}\left[\int J_N(x_1, \dots, x_k)\left[1 - \prod_{l=1}^{k} I_{\{|x_l| \leqq K\}}(x_l)\right] Z_{\hat{G}_N}(dx_1) \dots Z_{\hat{G}_N}(dx_k)\right]^2 < \varepsilon$$
$$\text{for all } N = 1, 2, \dots.$$

Let us still observe that

$$J_N(x_1, \dots, x_k) \to J_0(x_1, \dots, x_k)$$

uniformly on every bounded set where the function J_0 is bounded, and

$$\hat{G}_N([0, x]) \to G_0([0, x]) \qquad \text{for all } x.$$

Hence, because of property a4) of the function M, Lemma 3 in [3] with some modification proves the Proposition.

It is natural to expect that the statement of the Proposition remains valid if $\hat{M}$ and $\hat{G}$ are slightly perturbated. We explain what we mean by slight perturbation, and show that it may better explain the content of the Theorem.

Let us choose sufficiently smooth functions f and g in the definition of $\hat{M}$ and $\hat{G}$. Then

$$r(n)=\int \exp(inx)\,\hat{G}(dx)\sim n^{-\alpha} \quad \text{with} \quad a \quad D>0, \quad \text{if } \alpha \neq 1, 2, 3, \ldots \tag{4.4}$$

and

$$(2\pi)^{-k}\int \exp[i(n_1 x_1+\ldots+n_k x_k)]\,\hat{M}(x_1, \ldots, x_k)\,dx_1 \ldots dx_k$$
$$=a_n=b\left(\frac{n}{|n|}\right)|n|^{-\beta-1}+O(|n|^{-\lambda}) \tag{4.5}$$

if the function M is "nice enough", where $n=(n_1, \ldots, n_k)$.

$$|n|^2=\left(\sum_{l=1}^{k} n_l^2\right)^2, \qquad b(\cdot)$$

is a function on the unit sphere of R^k, and λ can be made arbitrary large by choosing a sufficiently smooth function f. Moreover it is natural to expect that in nice cases

$$\hat{M}(x_1, \ldots, x_k)=\sum_n a_n \exp[i(n_1 x_1+\ldots+n_k x_k)]$$
$$\text{for} \quad -\pi<x_l<\pi, \; l=1, 2, \ldots, k. \tag{4.6}$$

Generally the function M has a singularity in zero because of its homogeneity. Therefore the function $b(\cdot)$ in (4.5) cannot vanish everywhere, i.e. $a_n \neq o(|n|^{-\beta-1})$. The case $M(x_1, \ldots, x_k)\equiv 1$, when M is analytic, is exceptional. This exceptional behaviour of the constant function can explain the special role of the self-similar processes $\bar{Z}_n$ defined in (1.2), if we consider them rewritten in the form

$$\bar{Z}_n=\int \exp[in(x_1+\ldots+x_k)]\,K_0(x_1+\ldots+x_k)\,Z_{G_0}(dx_1)\ldots Z_{G_0}(dx_k)$$

with $G_0(x)=|x|^\alpha$ (Other natural candidates for an analytic function M like $M(x_1, \ldots, x_k)=(x_1+\ldots+x_k)^{2p}$, $p=1, 2, \ldots$ are excluded by condition c).

Now we formulate the following conjecture:

Let

$$a_n=b\left(\frac{n}{|n|}\right)|n|^{-\beta-1}+o(|n|^{-\beta-1}). \tag{4.7}$$

$$a_n=a_{n'} \quad \text{if } n=(n_1, \ldots, n_k) \quad \text{and} \quad n'=(n_{\pi(1)}, \ldots, n_{\pi(k)}), \qquad \pi\in\Pi$$

where the function $b(\cdot)$ is the same as in (4.5), and let G_0 be a spectral measure on $(-\pi, \pi)$ with the property

$$r(n)=\int \exp(inx)\,G(dx)=n^{-\alpha}L(n). \tag{4.8}$$

with a slowly varying function $L(\cdot)$.

Define with the help of this sequence a_n and measure G the process

$$U_m=\int \exp[im(x_1+\ldots+x_k)]\,\bar{M}(x_1, \ldots, x_k)\,Z_G(dx_1)\ldots Z_G(dx_k), \tag{4.9}$$

where

$$M(x_1, \dots, x_k) = \sum_n a_n \exp[i(n_1 x_1 + \dots + n_k x_k)].$$

If the function M "behaves nicely", and the sequence a_n satisfies some identities connected with the behaviour of M (e.g. $\sum a_n = 0$ if $M(0, \dots, 0) = 0$) then the processes Z_n^N defined by (4.9) and (1.7) with the choice $A_N = N^{1-\beta-\frac{k}{2}\alpha} L(N)^{\frac{k}{2}}$ tend in distribution to the process Z_n^* defined in (4.1) as $N \to \infty$.

Let us explain why it is natural to expect such a result. If we want to prove the conjecture by the method of this paper the crucial point is to prove the following statement:

Define the measures μ_N, $N = 0, 1, 2, \dots$ by the formulae:

$$\mu_0(A) = \int_A |K_0(x_1 + \dots + x_k)|^2 M(x_1, \dots, x_k)|^2 G_0(dx_1) \dots G_0(dx_k), \qquad A \in \mathscr{B}^k$$

$$\mu_N(A) = \int_A |K_N(x_1 + \dots + x_k)|^2 \bar{M}(x_1, \dots, x_k)|^2 G_N(dx_1) \dots G_N(dx_k),$$
$$A \in \mathscr{B}^k \qquad N = 1, 2, \dots$$

where $G_N(A) = N^\alpha / L(N) \cdot G\left(\frac{A}{N}\right)$.

Then the measures μ_N tend weakly to the measure μ_0 as $N \to \infty$.

In the proof of the Proposition we have proved this convergence in the special case $\bar{M} = \hat{M}$ and $\bar{G} = \hat{G}$. The convergence of the measures μ_N to μ_0 is equivalent to the convergence of the modified Fourier transform φ_N of μ_N defined in (3.1) to the Fourier transform φ_0 of $\mu_0 \cdot \varphi_N(t_1, \dots, t_k)$ can be expressed by the correlation functions $r(n)$ and the coefficients a_n as it was done in the proof of the Lemma. If the numbers $r(n)$ and a_n are defined by (4.4) and (4.5) then $\varphi_N \to \varphi_0$. So what we have to check is that a small perturbation of a_n and $r(n)$ does not change the convergence $\varphi_N \to \varphi$.

Let us now explain the relation between the conjecture and the Theorem. The processes defined in (1.8) and (4.1) agree if we choose

$$M(x_1, \dots, x_k) = b|x_1 + \dots + x_k|^\beta + ci|x_1 + \dots + x_k|^\beta \operatorname{sign}(x_1 + \dots + x_k).$$

Let us choose a function $f(x_1, \dots, x_k) = \bar{f}(x_1 + \dots + x_k)$ in the definition of $\hat{M}$. Then only the coefficients a_n, $n = (n_1, \dots, n_1)$ differ from zero in the expansion (4.6). Let us choose also the coefficients a_n in (4.7) so that $a_n \neq 0$ only if $n = (n_1, \dots, n_l)$. Then the Itô formula yields that

$$U_m = \sum_{l=-\infty}^{\infty} a_n H_k(Y_{n+m}), \qquad \text{where } Y_n = \frac{1}{r(0)} \int \exp(inx) Z_G(dx),$$
$$n = \dots -1, 0, 1, \dots, \quad \text{and} \quad r(0)^2 = G([-\pi, \pi)).$$

Hence the conjecture contains the Theorem as a special case. We remark that the Itô formula combined with an orthogonalization makes it always possible to express the random variables U_m defined in (4.9) as a functional of the above defined process Y_n, but this representation is rather complicated in the general case.

Remark 4.3. The method of this paper seems to work also in the case of stationary fields, i.e. in the case when the random variables are parametrized by the lattice points of the ν dimensional space, $\nu \geqq 2$. The random fields have a richer structure for large ν. In particuliar there are self-similar fields with a representation analogous to (4.1), for which the function M is a non-constant analytic function.

One can investigate also limit theorems for generalized fields. In case of generalized fields there are self-similar fields whose representation contains a non-constant analytic function M even in the one-dimensional case. It is natural to expect that such fields have a large range of attraction. We return to this question in a subsequent paper.

Remark 4.4. A natural generalization of the problem discussed in this paper is the following one: Let $H: R^r \to R^1$ be a function of r variables such that

$$\mathbf{E}H(Y_1, \ldots, Y_r) = 0, \qquad \mathbf{E}H^2(Y_1, \ldots, Y_r) < \infty,$$

where $Y_n, n = \ldots -1, 0, 1$ is a stationary Gaussian sequence $\mathbf{E}Y_0 = 0$, $\mathbf{E}Y_0^2 = 1$, satisfying (1.4).

Define the process

$$U_m = \sum_{m=-\infty}^{\infty} a_n H(Y_{n-m-1}, \ldots, Y_{n-m-r}), \qquad m = \ldots -1, 0, 1, \ldots$$

where the sequence a_n is the same as in the Theorem. Let the process Z_n^N be defined by (1.7) via this process U_m. We are interested whether the processes Z_n^N have a limit as $N \to \infty$.

With the help of the Itô formula the random variables $H(Y_{n-m-1}, \ldots, Y_{n-m-r})$ can be expressed in the form

$$H(Y_{n-m-1}, \ldots, Y_{n-m-r}) = \sum_l \int \exp[in(x_1 + \ldots + x_l)]\, g_l(x_1, \ldots, x_l) \cdot Z_G(dx_1) \ldots Z_G(dx_l) \tag{4.10}$$

where the function $g_l(x_1, \ldots, x_l)$ has the form

$$g_l(x_1, \ldots, x_l) = \sum_{\substack{1 \leqq s_j \leqq r \\ j = 1, 2, \ldots, l}} c_{s_1, \ldots, s_l} \exp[i(s_1 x_1 + \ldots + s_l x_l)],$$

and

$$c_{s_1, \ldots, s_l} \cdot = c_{s_{\pi(1)}, \ldots, s_{\pi(l)}} \quad \text{for an arbitrary}$$

permutation π of the numbers $1, 2, \ldots, l$. (See Remark (6.1) in [3] for an explanation how this representation can be obtained.)

Let us first consider the special case when the sum (4.10) contains only one term with an index k.

Let us assume that the conditions of the Theorem are satisfied with this k. A natural modification in the proof of the Theorem shows that this new process Z_n^N tends in distribution to the process $(\sum c_{s_1,\ldots,s_k}) Z_n^*$, where the process $Z_n^* = Z_n^*(\alpha, \beta, k, b, c)$ and the norming constants A_N are the same as in the Theorem. A slight modification in the proof of the Corollary shows that the same result holds if the sum (4.10) contains finitely many terms, i.e. in this case only the smallest index counts. The expression in (4.10) is a finite sum if and only if H is a polynomial. Probably the condition about the finiteness of the sum (4.10) can be dropped, but we were unable to prove this.

Remark 4.5. The self-similar fields Z_n^* defined in (1.8) can be represented also by means of the original Wiener-Itô integral defined in [5]. It can be done by the help of Lemma 6.1 in [10].

$$Z_n^* = D(\alpha)^k \int \left[\int_{-\infty}^{\infty} |n+u+t_1|^{\frac{\alpha+1}{2}} \ldots |n+u+t_k|^{\frac{\alpha+1}{2}} N(u)\, du \right] W(dt_1) \ldots W(dt_k) \tag{4.11}$$

with

$$N(u) = bA(\beta)(|y|^{-\beta} \operatorname{sign} y - |y-1|^{-\beta} \operatorname{sign}(y-1)) + cA'(\beta)[|y|^{-\beta} - |y-1|^{-\beta}]$$

if $\beta \neq 0$, and

$$N(u) = b \cdot I_{\{0 \leqq u < 1\}}(u) + c \cdot A'(0)[\ln|y-1| - \ln|y|] \quad \text{if } \beta = 0,$$

where $D(\alpha)$, $A(\beta)$, $A'(\beta)$ are appropriate constants, and $W(\cdot)$ denotes the white noise process.

To verify (4.11) one has to show that the integrand (with respect the white noise) in (4.11) is the Fourier transform of the integrand in (1.8). We give a short informal proof of this relation which, however can be made rigorous.

The identity

$$\bar{K}_0(t) = \int_{-\infty}^{\infty} \exp(itu)\, N(u)\, du \tag{4.12}$$

holds. Formula (4.12) is meaningless in the usual sense, but it is meaningful and correct if we interpret it in the following way: The generalized function $\bar{K}_0$ is the Fourier transform of the generalized function N. With the substitution $t = x_1 + \ldots + x_k$ in (4.12) we get that

$$\begin{aligned} &\int \exp\{[i(t_1 x_1 + \ldots + t_k x_k) + n(x_1 + \ldots + x_k)]\} \\ &\quad \cdot \bar{K}_0(x_1 + \ldots + x_k) |x_1|^{\frac{\alpha-1}{2}} \ldots |x_k|^{\frac{\alpha-1}{2}} dx_1 \ldots dx_k \\ &= \int N(u) \left\{ \prod_{l=1}^{k} \int \exp[i(t_l + n + u)\, x_l]\, |x_l|^{\frac{\alpha-1}{2}} dx_l \right\} du \\ &= D(\alpha)^k \int N(u) \prod_{l=1}^{k} |t_l + n + u|^{-\frac{\alpha+1}{2}} du, \end{aligned}$$

as we claimed.

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