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On Renormalizing Gaussian Fields

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Summary. We investigate the Gaussian self-similar fields and their Gaussian domain of attraction. Both discrete and generalized fields are considered.

1. Introduction

In recent time several papers investigated the construction of self-similar fields and their domain of attraction. In the general case this problem is very hard, but if we restrict ourselves to the Gaussian case it becomes much simpler. The reason for this simplicity is that the distribution of a Gaussian field (with zero expectation) is completely determined by its correlation function, or in the case of a stationary field by its spectral measure. The problems about self-similarity property and domain of attraction can be formulated in terms of spectral measures in a natural way. Nevertheless these problems, which we are going to investigate in this paper lead to some not completely trivial analytical problems.

First we consider generalized fields. We recall some definitions.

The set of random variables $X(\varphi)$, $\varphi \in \mathcal{S}$, where $\mathcal{S}$ denotes the Schwartz space of infinitely many times differentiable rapidly decreasing functions on the ν-dimensional Euclidean space R^ν, is a ν-dimensional generalized field if

a) $X(c_1\varphi_1+c_2\varphi_2)=c_1X(\varphi_1)+c_2X(\varphi_2)$ for all real numbers c_1, c_2 and all $\varphi_1, \varphi_2 \in \mathcal{S}$.

b) $X(\varphi_n)\to X(\varphi)$ stochastically if $\varphi_n\to\varphi$ in the topology of $\mathcal{S}$.

The field X is called stationary if $X(\varphi)\overset{\Delta}{=}X(T_t\varphi)$ for all $\varphi\in\mathcal{S}$ and $t\in R^\nu$, where $T_t\varphi(x)=\varphi(x+t)$, and $\overset{\Delta}{=}$ denotes equality in distribution. The field X is Gaussian if $X(\varphi)$ is Gaussian for all $\varphi\in\mathcal{S}$. It is self-similar with self-similarity parameter α if $t^{-\alpha}X(\varphi_t)\overset{\Delta}{=}X(\varphi)$ for all $\varphi\in\mathcal{S}$ and $t>0$ with $\varphi_t(x)=\varphi\left(\frac{x}{t}\right)$. We say that the generalized field X has a large scale (short scale) limit with

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normalizing factor A_t if for all $\varphi \in \mathcal{S}$

$$A_t^{-1} X(\varphi_t) \xrightarrow{\mathcal{D}} X_0(\varphi) \quad \text{as } t \to \infty \ (t \to 0),$$

where $\xrightarrow{\mathcal{D}}$ denotes convergence in distribution.

It has been proved (see [2]) that the large scale (short scale) limit of a stationary generalized field is always a stationary generalized field. If the limiting field has self-similarity parameter α then the norming constant A_t must be chosen as $A_t = t^\alpha L(t)$, where $L(\cdot)$ is a slowly varying function at infinity (at zero). Obviously the large scale (short scale) limit of a Gaussian field is again Gaussian.

We shall assume throughout this paper that the generalized fields we are considering have zero expectation, i.e. $EX(\varphi)=0$ for all $\varphi \in \mathcal{S}$. Then their distribution is determined by their correlation function $R(\varphi, \psi) = EX(\varphi)X(\psi)$; $\varphi, \psi \in \mathcal{S}$. If X is a stationary field then by the Bochner-Schwartz theorem (see e.g. [4]), there is a unique σ-finite measure G on R^ν such that

$$R(\varphi, \psi) = \int \tilde{\varphi}(x) \bar{\tilde{\psi}}(x)\, G(dx), \tag{1.1}$$

where $\tilde{}$ denotes Fourier transform. The measure G, which is called the spectral measure of the field X, has the properties $G(A) = G(-A)$ and

$$\int (1+|x|)^{-r}\, G(dx) < \infty \tag{1.2}$$

with some $r > 0$.

The Gaussian stationary generalized self-similar fields are completely described in [1] by means of their spectral measure. A Gaussian stationary generalized field is self-similar with a self-similarity parameter α, $\alpha \leq \nu$ if and only if its spectral measure G satisfies the relation $G(tA) = t^{2(\nu-\alpha)} G(A)$ for all $t > 0$ and $A \in \mathcal{B}^\nu$. In case $\alpha = \nu$ this means that G is concentrated in the origin. If $\alpha > \nu$ then the only Gaussian self-similar field is the trivial one, i.e. $X(\varphi) \equiv 0$ for all $\varphi \in \mathcal{S}$.

Now we are interested in the following question: Which stationary Gaussian generalized fields have a large scale (short scale) limit? This question will be answered in the following Theorem 1. First we recall that a sequence of locally finite measures μ_n on R^ν tends vaguely to a locally finite measure μ_0 (in notation $\mu_n \xrightarrow{v} \mu_0$) if and only if $\int f(x)\, \mu_n(dx) \to \int f(x)\, \mu_0(dx)$ for all continuous functions f with a bounded support.

Theorem 1. *Let X be a Gaussian stationary generalized field with spectral measure G. It has a large scale (short scale) limit with the normalizing factor $A_t = t^\alpha L(t)$, where $L(\cdot)$ is a slowly varying function at infinity (in zero), if and only if the measures G_t*

$$G_t(A) = t^{2(\nu-\alpha)} L^{-2}(t)\, G(t^{-1}A), \qquad A \in \mathcal{B}^\nu \tag{1.3}$$

tend vaguely to a measure G_0 as $t \to \infty$, $(t \to 0)$. If the limit G_0 exists then it has the homogeneity property

$$G_0(A) = t^{2(\nu-\alpha)} G_0(t^{-1}A) \quad \textit{for all } A \in \mathcal{B}^\nu \textit{ and } t > 0, \tag{1.4}$$

and it is the spectral measure of the limiting field.

If a sequence of measures G_t defined by (1.3) tends vaguely to a measure G_0 which satisfies (1.4) as $t \to \infty$ then

$$\frac{G(\varepsilon A)}{G_0(\varepsilon A)} = L^{-2}\left(\frac{1}{\varepsilon}\right) \frac{G_{1/\varepsilon}(A)}{G_0(A)} \sim L^{-2}\left(\frac{1}{\varepsilon}\right) \quad \text{as } \varepsilon \to 0$$

for a set A with zero G_0 boundary, and if $G_t \xrightarrow{v} G_0$ as $t \to 0$ then

$$\frac{G\left(\frac{1}{\varepsilon}A\right)}{G_0\left(\frac{1}{\varepsilon}A\right)} \sim L^{-2}(\varepsilon) \quad \text{as } \varepsilon \to 0.$$

Hence, roughly speaking, Theorem 1 states that a Gaussian stationary generalized random field with a spectral measure G has a large scale (short scale) limit field with a spectral measure G_0 if and only if the spectral measure G behaves similarly to the spectral measure G_0 in a neighbourhood of zero (infinity).

Now we turn to the investigation of discrete fields. Let X_n, $n \in \mathscr{Z}^\nu$, be a stationary Gaussian field, where $\mathscr{Z}^\nu$ denotes the lattice of points with integer coordinates in R^ν. We assume throughout this paper that $EX_n = 0$. Introduce the notation

$$B_n^N = \{j \in \mathscr{Z}^\nu;\ N n_k \leqq j_k < N(n_k+1),\ k=1,\dots,\nu\}, \qquad n \in \mathscr{Z}^\nu,\ N=1,2,\dots,$$

where the subscript k denotes the k-th coordinate of a vector. Given a stationary field X_n, $n \in \mathscr{Z}^\nu$, we define the fields

$$Z_n^N = \frac{1}{A_N} \sum_{p \in B_n^N} X_p, \qquad n \in \mathscr{Z}^\nu \tag{1.5}$$

for all $N = 1, 2, \dots$, where the norming constants A_N are appropriately chosen. We say that the field X_n has a large scale limit Z_n^*, $n \in \mathscr{Z}^\nu$, if the finite dimensional distributions of the fields Z_n^N defined in (1.5) tend to those of the field Z_n^*. A stationary field X_n, $n \in \mathscr{Z}^\nu$, is called self-similar with self-similarity parameter α if

$$(Z_{n_1}^N, \dots, Z_{n_k}^N) \stackrel{\Delta}{=} (X_{n_1}, \dots, X_{n_k})$$

with $A_N = N^\alpha$ for all $N = 1, 2, \dots, k = 1, 2, \dots, n_j \in \mathscr{Z}^\nu$, $j = 1, \dots, k$. It can be seen that, just like in the case of generalized fields, under some slight regularity conditions the large scale limit of a random field must be self-similar. If the limiting field has self-similarity parameter α then the norming constant A_N in the definition of Z_n^N must be of the form $A_N = N^\alpha L(N)$, where $L(\cdot)$ is a slowly varying function. We remark without proof, that if the random field X_n satisfies a certain continuity property to be defined below then it has the above mentioned properties.

Definition. Let X_n, $n \in \mathscr{Z}^\nu$, be a discrete stationary random field such that $EX_n = 0$, $EX_n^2 < \infty$. Set

$$D_N^2 = E[(\sum_{p \in B_0^N} X_p)^2], \qquad N = 1, 2, \ldots.$$

The field X_n satisfies the continuity property if for all $1 \leqq k \leqq \nu$ and $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ and an $N_0 = N_0(\varepsilon)$ such that for all rectangles $P \subset \mathscr{Z}^\nu$ of the form $P = [L_1, M_1] \times \ldots \times [L_\nu, M_\nu]$ with the properties $0 \leqq M_j - L_j \leqq N$, $j = 1, \ldots, \nu$, $M_k - L_k \leqq \delta N$, $N \geqq N_0$, the relation

$$E[(\sum_{p \in P} X_p)^2] < \varepsilon D_N^2$$

holds true.

The heuristic content of the above continuity property is the following. If the set B_n^N in the sum $\sum_{p \in B_n^N} X_p$ is slightly perturbated then the sum $\sum X_p$ changes relatively little.

We are interested in the description of the stationary Gaussian self-similar fields and their Gaussian domain of attraction. We restrict ourselves mainly to the fields satisfying the continuity property.

The correlation function $r(n) = EX_0 X_n$ of a discrete stationary random field can be written in the form

$$r(n) = \int e^{i(n,x)} G(dx), \qquad n \in \mathscr{Z}^\nu,$$

where G is a finite even measure on the torus $[-\pi, \pi)^\nu$. The measure G is called the spectral measure of the field X_n, $n \in \mathscr{Z}^\nu$.

Let $\hat{G}$ be a σ-finite even measure on R^ν with the following two properties:

$$\hat{G}(tA) = t^{2(\nu-\alpha)} \hat{G}(A) \qquad \text{for all } A \in \mathscr{B}^\nu \text{ and } t > 0 \tag{1.6}$$

and

$$\int \prod_{j=1}^{\nu} \frac{1 - \cos x_j}{x_j^2} \hat{G}(dx) < \infty. \tag{1.7}$$

Then, as it is proved in [1] or [5], the measure G defined on the torus $[-\pi, \pi)^\nu$ by the formula

$$G(E) = \sum_{q \in \mathscr{Z}^\nu} \int_E \prod_{j=1}^{\nu} \frac{2(1 - \cos x_j)}{(x_j + 2\pi q_j)^2} \hat{G}(dx + 2\pi q), \; E \subset [-\pi, \pi)^\nu \tag{1.8}$$

is the spectral measure of a Gaussian self-similar field with self-similarity parameter α. We shall prove the following

Theorem 2. *A ν-dimensional discrete Gaussian stationary random field is self-similar with self-similarity parameter α and it satifies the continuity property if and only if its spectral measure can be written in the form* (1.8), *where $\hat{G}$ is an even measure on R^ν satisfying* (1.6) *and* (1.7). *The random field determines the measure G and $\hat{G}$ uniquely.*

We shall prove Theorem 2 by the help of the following Theorem 3 which describes when a stationary Gaussian random field satisfying the continuity property has a large scale limit. The formula $\mu_N \xrightarrow{w} \mu_0$ will indicate weak convergence.

Let X_n, $n \in \mathscr{Z}^\nu$ be a stationary Gaussian random field with a spectral measure G. Define the measures G_N and μ_N, $N=1,2,\dots$ by the formulas

$$G_N(A) = N^{2\nu-2\alpha} L^{-2}(N)\, G\left(\frac{A}{N}\right), \qquad A \in \mathscr{B}^\nu \tag{1.9}$$

$$\mu_N(A) = \int_A \prod_{j=1}^{\nu} \frac{1-\cos x_j}{N^2\left[1-\cos\frac{x_j}{N}\right]}\, G_N(dx), \qquad A \in \mathscr{B}^\nu, \tag{1.10}$$

where $L(\cdot)$ is an appropriately chosen slowly varying function. We formulate the following

Theorem 3. *a) The following two statements are equivalent:*

(i) *The field X_n satisfies the continuity property and it has a large scale limit with $A_N = N^\alpha L(N)$.*

(ii) *There exists a measure μ_0 such that $\mu_N \xrightarrow{w} \mu_0$.*

The limiting Gaussian field Z_n^*, $n \in \mathscr{Z}^\nu$ is self-similar with self-similarity parameter α, and its correlation function is defined by the formula

$$E Z_j^* Z_{j+n}^* = \int \exp[i(n,x)]\, \mu_0(dx). \tag{1.11}$$

b) The relation $\mu_N \xrightarrow{w} \mu_0$ implies that there exists a locally finite measure G_0 such that $G_N \xrightarrow{v} G_0$. The measure G_0 has the homogeneity property

$$G_0(A) = t^{2(\nu-\alpha)} G_0(t^{-1}A) \quad \textit{for all } t>0 \textit{ and } A \in \mathscr{B}^\nu. \tag{1.12}$$

Moreover

$$\int \prod_{j=1}^{\nu} \frac{1-\cos x_j}{x_j^2}\, G_0(dx) < \infty \tag{1.13}$$

and

$$\mu_0(A) = \int_A \prod_{j=1}^{\nu} 2\,\frac{1-\cos x_j}{x_j^2}\, G_0(dx), \qquad A \in \mathscr{B}^\nu. \tag{1.14}$$

c) In the case $\nu=1$ and $\alpha>0$ the relation $G_N \xrightarrow{v} G_0$ implies that $\mu_N \xrightarrow{w} \mu_0$, where μ_0 is defined in (1.14). *In case $\nu \geqq 2$ it may happen that $G_N \xrightarrow{v} G_0$, but the relation $\mu_N \xrightarrow{w} \mu_0$ does not hold even if $\alpha>0$.*

By comparing the correlation function of the limiting fields in Theorem 3 defined by formulas (1.11), (1.12) and (1.14) with the self-similar fields appearing in Theorem 2 one can see that the large scale limit of a field which satisfies the continuity property is a self-similar field also satisfying the continuity property. This fact may indicate the importance of self-similar fields with continuity property.

Let us compare Theorems 1 and 3. In both cases the relation $G_N \xrightarrow{v} G_0$ is a necessary condition for the existence of a large scale limit. In the case of a generalized field this relation is also sufficient. In the case of discrete fields with $\nu \geqq 2$ however the somewhat stronger condition $\mu_N \xrightarrow{w} \mu_0$ is needed.

In this paper we do not intend to give a complete description of discrete Gaussian stationary self-similar fields. We only show through an example that there exist discrete Gaussian fields which do not satisfy the continuity property.

Let $\dots \xi_{-1}, \xi_0, \xi_1, \dots$ be a sequence of independent standard Gaussian random variables. Then $X_n = \xi_{n+1} - \xi_n$, $n = \dots -1, 0, 1, \dots$ is a self-similar sequence with self-similarity parameter zero, and it does not satisfy the continuity property. In higher dimensions such fields can be constructed also with positive self-similarity parameter. Indeed, let ξ_n, $n = \dots -1, 0, 1, \dots$, be a self-similar Gaussian sequence with self-similarity parameter α, $\alpha > 0$. Let $\xi_{n,k}$, $k = \dots -1, 0, 1, \dots$ be independent copies of this sequence, and define the two-dimensional field $X_{n,k} = \xi_{n,k+1} - \xi_{n,k}$, $k, n = \dots -1, 0, 1, \dots$ This field is self-similar with self-similarity parameter α, and it does not satisfy the continuity property.

This paper consists of three sections. Section 2 contains the proof of the theorems with the help of a lemma. This lemma is proved in Sect. 3.

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2. Proof of the Theorems

Proof of Theorem 1. First we show that if the field X has a large scale (short scale) limit then the measures G_t tend vaguely to a measure G_0 as $t \to \infty$ $(t \to 0)$.

Choose a function $\varphi \in \mathcal{S}$, and define the measures $\mu_{t,\varphi}$,

$$\mu_{t,\varphi}(A) = \int_A |\tilde{\varphi}(x)|^2 \, G_t(dx), \qquad A \in \mathcal{B}^\nu$$

for all $t > 0$. Observe that

$$\begin{aligned} \lim \int e^{i(s,x)} \mu_{t,\varphi}(dx) &= \lim A_t^{-2} E X(\varphi_t) X(T_s \varphi_t) \\ &= R_\varphi(s) \quad \text{as } t \to \infty \ (t \to 0), \end{aligned} \tag{2.1}$$

where $R_\varphi(s) = E X_0(\varphi) X_0(T_s \varphi)$ and X_0 is the limit field. Moreover, since $R_\varphi(s)$ is a continuous function in s, relation (2.1) implies that $\mu_{t,\varphi} \xrightarrow{w} \mu_\varphi$ with an appropriate measure μ_φ as $t \to \infty$ $(t \to 0)$. This relation holds for all $\varphi \in \mathcal{S}$, hence $G_t \xrightarrow{v} G_0$.

Now we show that if $G_t \xrightarrow{v} G_0$ as $t \to \infty$ $(t \to 0)$ then G_0 satisfies the homogeneity property (1.4). Let f be an arbitrary continuous function with a compact support. We claim that

$$\int f(x) \, G_0(dx) = c^{2\nu - 2\alpha} \int f(cx) \, G_0(dx) \qquad \text{for all } c > 0. \tag{2.2}$$

Indeed,

$$\lim \int f(x) \, G_t(dx) = \int f(x) \, G_0(dx)$$

and

$$\begin{aligned} \lim \int f(x) \, G_t(dx) &= \lim c^{2\nu - 2\alpha} \left(\frac{L(t)}{L(ct)} \right)^2 \int f(cx) \, G_{ct}(dx) \\ &= c^{2\nu - 2\alpha} \int f(cx) \, G_0(dx) \end{aligned}$$

as $t \to \infty$ $(t \to 0)$. The last two relations imply (2.2), and since the continuous function f with a compact support can arbitrarily be chosen, (2.2) implies relation (1.4).

Finally we are going to show that if $G_t \xrightarrow{v} G_0$ then

$$\lim A_t^{-2} EX(\varphi_t)^2 = \lim \int |\tilde{\varphi}(x)|^2 G_t(dx) = \int |\tilde{\varphi}(x)|^2 G_0(dx). \tag{2.3}$$

This relation completes the proof of Theorem 1.

Since $\tilde{\varphi} \in \mathcal{S}$ (the Fourier transform of a function $\varphi \in \mathcal{S}$ also belongs to $\mathcal{S}$) and G_0 is a homogeneous measure, we have

$$\int_{|x|>K} |\tilde{\varphi}(x)|^2 G_0(dx) < \varepsilon$$

for $K > K(\varepsilon)$. On the other hand

$$\int_{|x| \leqq K} |\tilde{\varphi}(x)|^2 G_t(dx) \to \int_{|x| \leqq K} |\tilde{\varphi}(x)|^2 G_0(dx)$$

for arbitrary $K > 0$. Hence to prove formula (2.3) it is enough to show that

$$\int_{|x|>K} |\tilde{\varphi}(x)|^2 G_t(dx) < \varepsilon$$

if K is sufficiently large for all $t > 2$ $(t < 1)$. Since $|\tilde{\varphi}(x)| < C_l |x|^{-l}$ for all $l > 0$ and $x \in R^\nu$ it is enough to prove that

$$\int_{|x|>K} |x|^{-l} G_t(dx) < \varepsilon \qquad \text{for all } t > 2 \ (t < 1),$$

where l is chosen sufficiently large (independently of t and ε). Let us remark that $G_t \xrightarrow{v} G_0$ implies that

$$G_s(|x| \leqq 1) \leqq B \qquad \text{for all } s \geqq 2 \ (s \leqq 1) \tag{2.4}$$

with an appropriate $B > 0$. Let us first consider the case when the large scale limit exists.

Set $L = \log K$. We can write

$$\begin{aligned} I(t) &= \int_{|x|>K} |x|^{-l} G_t(dx) \leqq \sum_{j \geqq L} 2^{-jl} G_t(2^j < |x| \leqq 2^{j+1}) \\ &= \sum_{\substack{j \geqq L \\ 2^{j+1} \leqq t}} + \sum_{\substack{j > L \\ 2^{j+1} > t}} = \Sigma_1 + \Sigma_2. \end{aligned}$$

Because of relation (2.4)

$$\begin{aligned} \Sigma_1 &\leqq \sum_{\substack{j \geqq L \\ t \cdot 2^{-j-1} \geqq 1}} 2^{-jl} t^{2\nu - 2\alpha} L^{-2}(t) (t \cdot 2^{-j-1})^{2\alpha - 2\nu} L^2(t \cdot 2^{-j-1}) G_{t2^{-j-1}}(|x| \leqq 1) \\ &\leqq B\, 2^{2\nu - 2\alpha} \sum_{\substack{j \geqq L \\ t \cdot 2^{-j-1} \geqq 1}} 2^{-j(l + 2\nu - 2\alpha)} \frac{L^2(t\, 2^{-j-1})}{L^2(t)} \leqq \frac{\varepsilon}{2} \end{aligned}$$

if first l and then L (i.e. K) are chosen sufficiently large. (The choice of l is independent of ε.) In the last step of this estimation we have exploited that

$$\frac{L^2(t\cdot 2^{-j-1})}{L^2(t)} \leqq C_\delta 2^{\delta j}$$

for all $t>2$, $t\cdot 2^{-j-1}\geqq 1$ and $\delta>0$, which, in turn easily follows from Karamata's theorem.

To estimate Σ_2 observe that because of (1.2) for all sufficiently large $p>0$ there exists a $C_p>0$ such that

$$G(|x|<t^{-1}2^{j+1})\leqq C_p t^{-p} 2^{jp} \quad \text{if } t\cdot 2^{-j-1}\leqq 1.$$

Therefore

$$\Sigma_2 \leqq C_p \sum_{j\geqq L} 2^{-jl} t^{2\nu-2\alpha} L^{-2}(t) 2^{jp} t^{-p}.$$

Let us choose p so large that $t^{2\nu-2\alpha-p}L^{-2}(t)\leqq 1$ for all $t\geqq 2$, and let $l>2p$. Then

$$\Sigma_2 \leqq C_p \sum_{j\geqq L} 2^{-j\frac{l}{2}} \leqq \frac{\varepsilon}{2}$$

if L is chosen sufficiently large. In the case when the short scale limit exists the proof is simpler. In this case we may write

$$2^{-jl}G_t(2^j<|x|\leqq 2^{j+1}) = 2^{-jl}t^{2(\nu-\alpha)}L^{-2}(t)(t\cdot 2^{-j})^{2(\alpha-\nu)}$$

$$L^2(t\cdot 2^{-j})\, G_{t\cdot 2^{-j}}(1<|x|<2) \leqq C\cdot 2^{-\frac{j}{2}l}$$

if $t<1$, $j\geqq 1$ and L is chosen sufficiently large. Summing up these inequalities for $j>L$ we get that $I(t)\leqq\varepsilon$ if L (i.e. K) is chosen sufficiently large. Theorem 1 is proved.

Now we turn to the proof of Thorem 3. First we formulate two lemmas which will be needed during the proof. We introduce the following notation: Given an $x\in R^\nu$ its integer part $[x]$ is the vector $n\in\mathscr{Z}^\nu$ satisfying the inequalities $x_j-1<n_j\leqq x_j$, $j=1,\dots,\nu$.

Lemma 1. *Let $\mu_1,\mu_2,\dots$ be a sequence of finite measures on R^ν such that $\mu_N(R^\nu \setminus [-C_N\pi, C_N\pi]^\nu)=0$ with some sequence $C_N\to\infty$. Define the functions*

$$\varphi_N(t)=\int_{R^\nu}\exp\left[\left(i\frac{[tC_N]}{C_N},x\right)\right]\mu_N(dx).$$

If for all $t\in R^\nu$ the sequence $\varphi_N(t)$ tends to a function $\varphi_0(t)$ continuous at the origin then the sequence μ_N weakly tends to a finite measure μ_0. The function φ_0 is the Fourier transform of μ_0.

Lemma 2. *For all $\nu\geqq 1$ there exists a $B=B(\nu)>0$ and an $N_0=N_0(\nu)$ such that for all $N>N_0$ and $x\in[-N\pi,N\pi)^\nu$*

$$\frac{1}{N}\sum_{l=[\frac{N}{2}]}\prod_{j=1}^{\nu}\frac{1-\cos\frac{l}{N}x_j}{N^2\left(1-\cos\frac{x_j}{N}\right)} \geqq B\prod_{j=1}^{\nu}\frac{1}{1+x_j^2}. \tag{2.5}$$

Lemma 1 coincides with Lemma 2 in [3]. We prove Lemma 2 in Sect. 3.

Proof of Theorem 3. a) Let X_n, $n\in\mathscr{Z}^\nu$, satisfy the continuity property, and let it have a large scale limit Z_n^* with normalization $A_N=N^\alpha L(N)$. We show that there exists a measure μ_0 such that $\mu_N \xrightarrow{w} \mu_0$. Set

$$\varphi_N(t)=\int \exp\left[i\left(\frac{[Nt]}{N},x\right)\right]\mu_N(dx),\qquad t\in R^\nu.$$

Since the measure μ_N is concentrated in $[-N\pi, N\pi)^\nu$ it is enough to prove, because of Lemma 1, that the limit $\varphi_0(t)=\lim_{N\to\infty}\varphi_N(t)$ exists for all $t\in R^\nu$, and it is continuous at zero. Observe that

$$\begin{aligned}\varphi_N(t)&=\int \exp\left[i\left(\frac{[Nt]}{N},x\right)\right]\prod_{j=1}^{\nu}\frac{1}{N^2}\left|\sum_{p=0}^{N-1}\exp\left(i\frac{p}{N}x_j\right)\right|^2 G_N(dx)\\&=N^{-2\alpha}L^{-2}(N)\,E[(\sum_{l\in B_0^N}X_l)(\sum_{p\in B_0^N+[Nt]}X_p)],\end{aligned}\tag{2.6}$$

where

$$B+m=\{x;\ x=y+m,\ y\in B\}.$$

Let us choose a sufficiently small $\eta>0$ of the form $\eta=\frac{1}{M}$, where M is an integer. Define the set $A(t,\eta)$, $t\in R^\nu$

$$A(t,\eta)=\{n;\ n\in\mathscr{Z}^\nu,\ t_j+\frac{\eta}{2}\leqq n_j\eta\leqq t_j+1-\tfrac{3}{2}\eta \text{ for all } j=1,\dots,\nu\}.$$

Set

$$C(N,t,\eta)=\bigcup_{n\in A(t,\eta)}B_n^{[\eta N]}$$

and

$$D(N,t,\eta)=(B_0^N+[Nt])\setminus C(N,t,\eta).$$

(Our aim with the definition of the set $C(N,t,\eta)$ was the following. We wanted to fill the cube $B_0^N+[Nt]$ almost completely with the union of non-overlapping cubes $B_n^{[N\eta]}$ in such a way that the set of the subscripts n of the cubes $B_n^{[N\eta]}$ which are contained in this union does not depend on N.)

Observe that for $N>2\eta^{-1}$ $C(N,t,\eta)\subset B_0^N+[Nt]$. We define the functions

$$\bar\varphi_N(t)=\frac{1}{N^{2\alpha}L^2(N)}E[(\sum_{j\in C(N,0,\eta)}X_j)(\sum_{l\in C(N,t,\eta)}X_l)].$$

Then there exists a function $\bar\varphi_0$ such that

$$\lim_{N\to\infty}\bar\varphi_N(t)=\lim_{N\to\infty}\eta^{2\alpha}E(\sum_{\substack{l\in A(0,\eta)\\p\in A(t,\eta)}}Z_l^{[N\eta]}Z_p^{[N\eta]})=\eta^{2\alpha}\sum_{\substack{l\in A(0,\eta)\\p\in A(t,\eta)}}EZ_p^*Z_l^*=\bar\varphi_0(t).\tag{2.7}$$

On the other hand

$$\varphi_N(t)=\frac{1}{N^{2\alpha}L^2(N)}E([\sum_{j\in C(N,0,\eta)}X_j+\sum_{j\in D(N,0,\eta)}X_j][\sum_{l\in C(N,t,\eta)}X_l+\sum_{l\in D(N,t,\eta)}X_l])$$

if $N > 2\eta^{-1}$. Hence

$$\limsup |\bar{\varphi}_N(t) - \varphi_N(t)| = \limsup \left| \frac{1}{N^{2\alpha} L^2(N)} \Big(E \sum_{j \in D(N,0,\eta)} X_j \sum_{l \in B_0^N + [Nt]} X_l + E \sum_{j \in D(N,t,\eta)} X_j \sum_{l \in B_0^N} X_l - E \sum_{j \in D(N,0,\eta)} X_j \sum_{l \in D(N,t,\eta)} X_l \Big) \right|. \tag{2.8}$$

The sets $D(N,t,\eta)$ and $D(N,0,\eta)$ can be represented as the union of at most 2ν rectangles in such a way that the length of the edges of these rectangles is less than or equal to N, and the length of one of these edges is less than ηN. Hence the continuity property of the field X_n implies that

$$N^{-2\alpha} L^{-2}(N) E\Big(\sum_{j \in D(N,0,\eta)} X_j \Big)^2 < \varepsilon \tag{2.9}$$

$$N^{-2\alpha} L^{-2}(N) E\Big(\sum_{j \in D(N,t,\eta)} X_j \Big)^2 < \varepsilon \tag{2.9'}$$

if $\eta < \eta(\varepsilon)$ and $N > N_0(\varepsilon)$. Relations (2.8), (2.9), (2.9)′ and the Schwarz inequality imply that

$$\limsup |\bar{\varphi}_N(t) - \varphi_N(t)| < \varepsilon \tag{2.10}$$

for $\eta < \eta(\varepsilon)$, where η does not depend on t. Relations (2.7) and (2.10) imply that the limit $\varphi_0(t) = \lim \varphi_N(t)$ exists. Moreover, since $\bar{\varphi}_0(t)$ is constant for $|t| < \frac{\eta}{2}$ hence the function φ_0 is continuous at zero. Then Lemma 1 implies that $\mu_N \xrightarrow{w} \mu_0$.

Let us now assume that $\mu_N \xrightarrow{w} \mu_0$. Then there exists a Gaussian stationary field Z_n^*, $n \in \mathscr{Z}^\nu$ whose correlation function is defined by (1.11). Moreover

$$\lim_{N \to \infty} E Z_0^N Z_n^N = \lim_{N \to \infty} \int e^{i(n,x)} \mu_N(dx) = \int e^{i(n,x)} \mu_0(dx) = E Z_0^* Z_n^*,$$

therefore Z_n^*, $n \in \mathscr{Z}^\nu$, is the large scale limit of the field X_n. We shall prove with the help of Lemma 2 that the field X_n satisfies the continuity property.

Let $P = [L_1, M_1] \times \ldots \times [L_\nu, M_\nu]$, $P \subset \mathscr{Z}^\nu$ be a rectangle such that $0 \leqq M_1 - L_1 \leqq \delta N$, $0 \leqq M_j - L_j \leqq N$, $j = 2, \ldots, N$. We have to show that

$$\frac{1}{N^{2\alpha} L^2(N)} E\Big(\sum_{p \in P} X_p \Big)^2 < \varepsilon$$

if $\delta < \delta(\varepsilon)$. We can write

$$\frac{1}{N^{2\alpha} L^2(N)} E\Big(\sum_{p \in P} X_p \Big)^2 = \int \prod_{j=1}^{\nu} \frac{1 - \cos\left(\frac{M_j - L_j}{N} x_j \right)}{N^2 \left(1 - \cos \frac{x_j}{N} \right)} G_N(dx)$$

$$\leqq C \int \prod_{j=1}^{\nu} \min\left(\frac{1}{x_j^2}, \left(\frac{M_j - L_j}{N} \right)^2 \right) G_N(dx)$$

$$\leqq 2^\nu C \int \frac{\delta^2}{1 + \delta^2 x_1^2} \prod_{j=2}^{\nu} \frac{1}{1 + x_j^2} G_N(dx) = 2^\nu C \Big[\int_{|x_1| < K} + \int_{|x_1| > K} \Big].$$

Because of Lemma 2 and the compactness of the sequence of measures μ_N the following estimates hold true:

$$\begin{aligned}\int_{\{|x_1|>K\}} &\leqq \int_{\{|x_1|>K\}} \prod_{j=1}^{\nu} \frac{1}{1+x_j^2} G_N(dx) \\ &\leqq \frac{1}{BN} \sum_{l=\left[\frac{N}{2}\right]}^{N} \int_{\{|x_1|>K\}} \prod_{j=1}^{\nu} \frac{1-\cos\frac{l}{N}x_j}{N^2\left(1-\cos\frac{x_j}{N}\right)} G_N(dx) \\ &= \frac{1}{BN} \sum_{l=\left[\frac{N}{2}\right]}^{N} \frac{N^{2\nu-2\alpha}}{l^{2\nu-2\alpha}} \frac{L^2(N)}{L^2(l)} \int_{\left\{|x_1|>\frac{N}{l}K\right\}} \prod_{j=1}^{\nu} \frac{1-\cos x_j}{l^2\left(1-\cos\frac{x_j}{l}\right)} G_l(dx) \\ &\leqq \frac{C}{N} \sum_{l=\left[\frac{N}{2}\right]}^{N} \mu_l\left(|x_1|>\frac{N}{l}K\right) \leqq \varepsilon\end{aligned}$$

if $K>K(\varepsilon)$.

Observe that if $|x|\leqq K$ and $\delta<\frac{1}{K^2}$ then

$$\frac{\delta^2}{1+\delta^2x^2} \leqq \frac{\delta}{1+x^2}.$$

By using this estimate together with Lemma 2 the following estimate can be obtained:

$$\int_{\{|x_1|<K\}} <\delta\int\prod_{j=1}^{\nu}\frac{1}{1+x_j^2}G_N(dx) \leqq \frac{\delta}{BN}\sum_{l=\left[\frac{N}{2}\right]}^{N}\frac{N^{2\nu-2\alpha}L^2(N)}{l^{2\nu-2\alpha}L^2(l)}\mu_l(R^\nu)<\varepsilon,$$

if $\delta>0$ is sufficiently small. Part a) of Theorem 3 is proved.

b) Let $\mu_N \xrightarrow{w} \mu_0$. Choose a non-negative continuous function h with a support in $-\left[\frac{\pi}{2},\frac{\pi}{2}\right)^\nu$. Then

$$\int h(x)\,G_N(dx)=\int h_N(x)\,\mu_N(dx)\to\int h(x)\prod_{j=1}^{\nu}\frac{x_j^2}{2(1-\cos x_j)}\mu_0(dx)$$

as $N\to\infty$ with

$$h_N(x)=h(x)\prod_{j=1}^{\nu}\frac{N^2\left(1-\cos\frac{x_j}{N}\right)}{1-\cos x_j},$$

since $h_N(x)\to h(x)\prod_{j=1}^{\nu}\frac{x_j^2}{2(1-\cos x_j)}$ uniformly as $N\to\infty$. Let us define the measures G_t, $G_t(A)=\frac{t^{2\nu-2\alpha}}{L^2(t)}G\left(\frac{A}{t}\right)$, $A\in\mathscr{B}^\nu$ for all $t>0$. It is easy to see that

$$\lim_{t\to\infty}\int h(x)\,G_t(dx)=\int h(x)\prod_{j=1}^{\nu}\frac{x_j^2}{2(1-\cos x_j)}\mu_0(dx) \tag{2.11}$$

since

$$\limsup \frac{G_t(A)}{G_{[t]}(A)} \leqq 1, \qquad \liminf \frac{G_t(A)}{G_{[t]+1}(A)} \leqq 1,$$

and relation (2.11) is already proved for the case when t takes only integer values.

Let us now consider a non-negative continuous function h with a compact support. Let this support be contained in a rectangle $[-B, B]^\nu$. Let us choose a number $b > B$. Then

$$\begin{aligned} \lim_{t\to\infty} \int h(x)\, G_t(dx) &= \lim_{t\to\infty} b^{2\alpha-2\nu} \frac{L^2(bt)}{L^2(t)} \int h(bx)\, G_{tb}(dx) \\ &= b^{2\alpha-2\nu} \int h(bx) \prod_{j=1}^{\nu} \frac{x_j^2}{2(1-\cos x_j)}\, \mu_0(dx). \end{aligned} \tag{2.12}$$

Relation (2.12) means in particular that $\lim\limits_{t\to\infty} \int h(x) G_t(dx)$ exists for all continuous functions h with a compact support, and this implies that there exists a measure G_0 such that $G_t \xrightarrow{v} G_0$. Applying relation (2.12) to the function $h(sx)$, $s>0$, with the constant $s^{-1}b$ instead of b we get that

$$\lim_{t\to\infty} \int h(x)\, G_t(dx) = \lim_{t\to\infty} s^{2\nu-2\alpha} \int h(sx)\, G_t(dx).$$

Taking limit in the last relation we get that

$$\int h(x)\, G_0(dx) = s^{2(\nu-\alpha)} \int h(sx)\, G_0(dx).$$

This relation implies the homogeneity property (1.12) since it holds for all $s>0$ and all continuous function h with compact support.

Let f be a continuous function with compact support. The relation $\mu_N \xrightarrow{w} \mu_0$ implies that

$$\lim_{N\to\infty} \int f(x)\, \mu_N(dx) = \int f(x)\, \mu_0(dx). \tag{2.13}$$

On the other hand since $G_N \xrightarrow{v} G_0$, and

$$\prod_{j=1}^{\nu} \frac{1-\cos x_j}{N^2 \left(1 - \cos \dfrac{x_j}{N}\right)} \to \prod_{j=1}^{\nu} 2 \frac{1-\cos x_j}{x_j^2}$$

uniformly in all bounded regions

$$\begin{aligned} \lim_{N\to\infty} \int f(x)\, \mu_N(dx) &= \lim_{N\to\infty} \int f(x) \prod_{j=1}^{\nu} \frac{1-\cos x_j}{N^2\left(1-\cos\dfrac{x_j}{N}\right)}\, G_N(dx) \\ &= \int f(x) \prod_{j=1}^{\nu} 2 \frac{1-\cos x_j}{x_j^2}\, G_0(dx). \end{aligned} \tag{2.14}$$

Relations (2.13) and (2.14) imply (1.14). Since the measure μ_0 is finite, relation (1.13) also holds. Part b) of Theorem 3 is proved.

c) If $G_N \xrightarrow{v} G_0$ then $\mu_N \xrightarrow{v} \mu_0$, where μ_N is defined in (1.10) and μ_0 in (1.14). To prove that in case $\nu=1$, $\alpha>0$ even the relation $\mu_N \xrightarrow{w} \mu_0$ holds we have to show that the sequence of measures μ_N is uniformly tight, i.e. $\mu_N(|x|>K)<\varepsilon$ for all $N=1,2,\ldots$ if $K>K(\varepsilon)$. We can write

$$\mu_N(|x|>K)=\int_{N\pi\geqq x>K} \frac{1-\cos x}{N^2\left(1-\cos\frac{x}{N}\right)} G_N(dx)$$
$$\leqq C \int_{N\pi\geqq x>K} \frac{1}{x^2} G_N(dx) \leqq \sum_{j=L}^{[\log N]} \frac{1}{2^{2j}} G_N(2^j \leqq |x| < 2^{j+1}),$$

where $L=[\log K]$. On the other hand

$$G_N(2^j \leqq |x| < 2^{j+1}) \leqq G_N(|x|<2^{j+1})$$
$$\leqq \frac{N^{2\nu-2\alpha}}{L^2(N)} \frac{L^2([N\cdot 2^{-j}]+1)}{([N 2^{-j}]+1)^{2-2\alpha}} G_{([N2^{-j}]+1)}([-1,l])$$
$$\leqq C' 2^{j[(2-2\alpha)+\delta]},$$

where $\delta>0$ can be chosen arbitrary small, if C' is sufficiently large, since the sequence $G_M([-1,1])$, $M=1,2,\ldots$ is bounded. Let $\delta<2\alpha$. Then the above estimations imply that

$$\mu_N(|x|>K) \leqq C' \sum_{j=L}^{\infty} 2^{j(\delta-2\alpha)} < \varepsilon.$$

for all $N=1,2,\ldots$ if K and therefore L is sufficiently large.

In the case $\nu \geqq 2$ we show an example where $G_N \xrightarrow{v} G_0$ but the relation $\mu_N \xrightarrow{w} \mu_0$ does not hold.

Let us consider a spectral measure G satisfying relations (1.6), (1.7) and (1.8) with some $0<\alpha<\nu-1$. Such a G exists. (See e.g. [1].) Let us fix a point $b=(b_1,0,\ldots,0)\in R^\nu$, $0<b_1<\pi$, and let the measure ρ be concentrated in the points b and $-b$, $\rho(b)=\rho(-b)=1$. Set $G'=G+\rho$. Then $G_N \xrightarrow{v} G_0$ and $G'_N \xrightarrow{v} G_0$ with $A_N=N^{2(\nu-\alpha)}$. On the other hand if μ_N is defined by formula (1.10) with the measure G'_N then

$$\limsup \mu_N(R^\nu) \geqq \limsup N^{2\nu-2\alpha-2}\rho(b)=\infty,$$

and therefore the relation $\mu_N \xrightarrow{w} \mu_0$ does not hold.

Now we turn to the

Proof of Theorem 2. Let Z_n^*, $n\in\mathscr{Z}^\nu$, be a Gaussian self-similar field with self-similarity parameter α which satisfies the continuity property. Then, as its distribution coincides with that of its large scale limit with $A_N=N^\alpha$ Theorem 3 implies that its correlation function is defined by formulas (1.11), (1.12), (1.13) and (1.14). Hence its spectral measure satisfies relations (1.6) (1.7) and (1.8) as it was claimed. Conversely we have to show that if the spectral measure of the Gaussian random field Z_n^* satisfies formulas (1.6), (1.7) and (1.8) then this field is self-similar with self-similarity parameter α, and it satisfies the continuity property. The following calculation proves the self-similarity property.

$$\frac{1}{N^{2\alpha}} E\big[\big(\sum_{j\in B_0^N} Z_j^*\big)\big(\sum_{p\in B_n^N} Z_p^*\big)\big] = \int \frac{e^{iN(n,x)}}{N^{2\alpha}} \prod_{j=1}^{\nu} \frac{1-\cos N x_j}{1-\cos x_j} G(dx)$$
$$= \int_{R^\nu} \frac{e^{i(n,Nx)}}{N^{2\alpha-2\nu}} \prod_{j=1}^{\nu} \frac{2(1-\cos N x_j)}{(N x_j)^2} \hat{G}(dx) = \int e^{i(n,x)} G(dx) = E Z_0^* Z_n^*.$$

Let us define the measures G_N, μ_N, $N=1,2,\ldots$, and μ_0 by formulas (1.9), (1.10) and (1.14) with $L(N)\equiv 1$, where G is the spectral measure of the field Z_n^*, and G_0 coincides with the measure $\hat{G}$ appearing in (1.6). By part a) of Theorem 3 to prove that the field Z_n^* satisfies the continuity property it is enough to show that $\mu_N \xrightarrow{w} \mu_0$. Let $A \subset [-N\pi, N\pi)^\nu$. Then

$$\mu_N(A) = N^{2\nu-2\alpha} \int_{\frac{A}{N}} \prod_{j=1}^{\nu} \frac{1-\cos N x_j}{N^2(1-\cos x_j)} G(dx)$$
$$= \sum_{t\in\mathscr{Z}^\nu} \int_{\frac{A}{N}} N^{2\nu-2\alpha} \prod_{j=1}^{\nu} 2\frac{1-\cos N x_j}{N^2(x_j+2\pi t_j)^2} \hat{G}(dx+2\pi t)$$
$$= \sum_{t\in\mathscr{Z}^\nu} \int_{A} \prod_{j=1}^{\nu} 2\frac{1-\cos x_j}{(x_j+2\pi N t_j)^2} \hat{G}(dx+2\pi N t)$$
$$= \sum_{t\in\mathscr{Z}^\nu} \mu_0(A+2\pi t N) = \mu_0(A) + \sum_{t\in\mathscr{Z}^\nu\setminus\{0\}} \mu_0(A+2\pi t N).$$

The second term in the last expression can be bounded by $\mu_0(R^\nu \setminus [-N\pi, N\pi)^\nu$ which tends to zero as $N\to\infty$ by condition (1.7). Hence $\mu_N \xrightarrow{w} \mu_0$ as we claimed. Finally observe that the field Z_n^* determines uniquely its spectral measure G, and since $G_N \xrightarrow{v} \hat{G}$ with $G_N(A) = N^{2\nu-2\alpha} G\left(\frac{A}{N}\right)$ also the measure $\hat{G}$. Theorem 2 is proved.

Remark. We considered self-similar fields whose spectral measures were given by formulas (1.6), (1.7) and (1.8). Relation (1.7) can be replaced by

$$\int \prod_{j=1}^{\nu} \frac{1}{1+x_j^2} \hat{G}(dx) < \infty. \tag{2.15}$$

It is clear that relation (2.15) implies (1.7). We show with the help of Lemma 2 that relations (1.6) and (1.7) imply (2.15). Let us choose a sufficiently large N, and let $\left[\frac{N}{2}\right] \leq l \leq N$. Let us define the cube $B(N) = \{x\in R^\nu;\ -N\pi \leq x_j < N\pi,\ j=1,\ldots,\nu\}$. Then

$$\int_{B(N)} \prod_{j=1}^{\nu} \frac{1-\cos\frac{l}{N}x_j}{N^2\left(1-\cos\frac{x_j}{N}\right)} \hat{G}(dx) \leq C \int \prod_{j=1}^{\nu} \frac{1-\cos x_j}{x_j^2} \hat{G}(dx)$$

for all $\left[\frac{N}{2}\right] \leq l \leq N$, where the constant C does not depend on N. Summing up these inequalities, and applying Lemma 2 we get that

$$\int_{B(N)} \prod_{j=1}^{\nu} \frac{1}{1+x_j^2} \hat{G}(dx) \leq \frac{C}{B} \int \prod_{j=1}^{\nu} \frac{1-\cos x_j}{x_j^2} \hat{G}(dx).$$

Since this inequality holds for arbitrary large N, relation (1.6) implies (2.15).

3. The Proof of Lemma 2

We shall deduce Lemma 2 from a result formulated below. This result is actually equivalent to Lemma 2. For all positive integers n we define the transformation T_n: $[-\pi,\pi)\to[-\pi,\pi)$ by the formula $T_n x = nx \bmod (2\pi)$, $x\in[-\pi,\pi)$. We formulate the following

Lemma 3. *For all positive integers k there exists a threshold $N_0=N_0(k)$ and some numbers $p=p(k)>0$, $\delta=\delta(k)>0$ such that for all $N>N_0$ and $x_1,\dots,x_k$, $\frac{1}{2N}\leqq|x_j|\leqq\pi$, $j=1,\dots,k$ we have*

$$\operatorname{Card} A(N)\geqq pN,$$

where

$$A(N)=A(N,x_1,\dots,x_k)=\left\{n;\ \frac{N}{2}<n<N,\ \textit{and}\ |T_n x_j|>\delta\ \textit{for all}\ j=1,\dots,k\right\}.$$

Proof of Lemma 2 via Lemma 3. Let us introduce the notation

$$J(l)=J(l,x_1,\dots,x_v)=\prod_{j=1}^{v}\frac{1-\cos\frac{l}{N}x_j}{N^2\left(1-\cos\frac{x_j}{N}\right)}$$

for $\left[\frac{N}{2}\right]\leqq l\leqq N$ and $|x_j|\leqq N\pi$, $j=1,\dots,v$.

Let $\left|\frac{x_1}{N}\right|>\frac{1}{2N},\dots,\left|\frac{x_k}{N}\right|>\frac{1}{2N}$ and $\left|\frac{x_{k+1}}{N}\right|\leqq\frac{1}{2N},\dots,\left|\frac{x_v}{N}\right|\leqq\frac{1}{2N}$. We are going to show that

$$J(l)\geqq C\prod_{j=1}^{v}\frac{1}{1+x_j^2} \tag{3.1}$$

for all $l\in A\left(N,\frac{x_1}{N},\dots,\frac{x_k}{N}\right)$ with some $C=C(v)>0$. Since $J(l)\geqq 0$ for all $\left[\frac{N}{2}\right]\leqq l\leqq N$ relation (3.1) and Lemma 3 together imply Lemma 2. We can write

$$\frac{1-\cos\frac{l}{N}x_j}{N^2\left(1-\cos\frac{1}{N}x_j\right)}\geqq\frac{1-\cos\delta}{5x_j^2}\geqq\frac{1-\cos\delta}{5}\,\frac{1}{1+x_j^2}$$

for $1\leqq j\leqq k$ and $l\in A\left(N,\frac{x_1}{N},\dots,\frac{x_k}{N}\right)$, and

$$\frac{1-\cos\frac{l}{N}x_j}{N^2\left(1-\cos\frac{1}{N}x_j\right)}\geqq\frac{1}{10}\,\frac{l^2}{N^2}\geqq\frac{1}{40}\geqq\frac{1}{40}\cdot\frac{1}{1+x_j^2}$$

for $k<j\leqq v$ and $\left[\frac{N}{2}\right]\leqq l\leqq N$. These inequalities imply (3.1) and therefore also Lemma 2.

Now we turn to the proof of Lemma 3. The idea of the proof is the following: It is well-known that the distribution of the sequence $T_n x$, $n=1,\dots,N$ tends to the uniform distribution as $N\to\infty$, *if* x/π is irrational. Therefore it is natural to expect that if x/π is relatively far from all rational numbers with small denominator then $|T_n x|>\delta$ for a very large proportion of the numbers $\frac{N}{2}\leqq n<N$. We shall prove this by adapting some ideas from the theory of concentration functions. Then we can reduce Lemma 3 to the special case when all $\frac{x_j}{\pi}$, $j=1,\dots,k$, are near to a rational number with a small denominator, and this latter case is relatively simple.

Proof of Lemma 3. In the proof we do not give sharp estimates. We shall deduce Lemma 3 from the following three statements: For all $\varepsilon>0$ there exists an $A=A(\varepsilon)$ and $N_0=N_0(A,\varepsilon)$ such that if $A>A(\varepsilon)$, $N>N_0$ (A,ε) then the following estimates hold:

(i) For all $x\in D_1(N)$

$$\operatorname{Card}\left\{n;\ 0\leqq n<N,\ |T_n x|<\frac{1}{2A}\right\}<\varepsilon N$$

where

$$D_1(N)=[-\pi,\pi)\setminus\bigcup_{l=1}^{A^2} D_0(l),$$

and

$$D_0(l)=D_0(l,N)=\left\{x;\ \left|x-2\pi\frac{j}{l}\right|\leqq\frac{A}{\sqrt{N}}\ \text{for some } j=0,\ \pm1,\dots,\pm\left(\left[\frac{l}{2}\right]+1\right)\right\}.$$

(ii) For all $x\in D_2(N)$

$$\operatorname{Card}\{n;\ 0<n\leqq N,\quad |T_n x|<A^{-5}\}<\varepsilon N$$

where

$$D_2(N)=\bigcup_{l=1}^{A^2} D_2(l,N),$$

and

$$D_2(l,N)=\left\{x;\ \frac{1}{2NA^2}<\left|x-2\pi\frac{j}{l}\right|<\frac{A}{\sqrt{N}}\ \text{for some } j=0,\ \pm1,\dots,\pm\left(\left[\frac{l}{2}\right]+1\right)\right\}.$$

(iii) If $x_1\in D_3(N),\dots,x_j\in D_3(N)$, $j\leqq k$, with

$$D_3(N)=[-\pi,\pi)\setminus\left(D_1(N)\cup D_2(N)\cup\left[-\frac{1}{2N},\frac{1}{2N}\right]\right)$$

then

$$\operatorname{Card}\left\{n;\ \frac{N}{2}\leqq n<N,\ |T_n x_m|>\frac{1}{2A^2}\ \text{for all } m=1,\dots,j\right\}>pN$$

with $p=p(j)=\frac{1}{3}\prod_{l=1}^{j}\left(1-\frac{1}{p_l}\right)$, where p_l denotes the l-th prime number.

We remark that statements (i) and (ii) are very similar to each other. We separated these two statements because their proofs are different. The estimates (i), (ii) and (iii) imply Lemma 3. Indeed if $x_l \in D_3(N)$ for $1 \leqq l \leqq j$, $x_l \in D_1(N) \cup D_2(N)$, $j < l \leqq k$, and ε is chosen in such a way that $p = p(k) - k\varepsilon > 0$ then there are at least pN indices $\frac{N}{2} < n < N$ such that $|nx_l| > \delta$ for all $l = 1, \ldots, k$ with $\delta = \min\left(\frac{1}{2A}, \frac{1}{A^5}, \frac{1}{2A^2}\right) = \frac{1}{A^5}$.

Proof of (i). Let us choose a sufficiently large $A > 0$ (independently of N), and define the function

$$f_A(u) = \begin{cases} 1 - A|u| & \text{for } |u| < \frac{1}{A} \\ 0 & \text{otherwise} \end{cases}$$

Fix a number x, and set

$$F_N(u) = \frac{1}{N} \operatorname{Card}\{j; j < N, T_j x < u\}.$$

We express the function $f_A(u)$ by its Fourier series:

$$f_A(u) = \frac{1}{A\pi} + \sum_{m=1}^{\infty} \frac{A}{2m^2\pi}\left(1 - \cos\frac{m}{A}\right)(e^{imu} + e^{-imu}).$$

The following estimate holds:

$$\operatorname{Card}\left\{n;\, n < N,\, |T_n x| \leqq \frac{1}{2A}\right\} < 2N \int f_A(u)\, F_N(du). \tag{3.2}$$

On the other hand

$$\begin{aligned} \int f_A(u) F_N(du) &= \frac{1}{A\pi} + \sum_{m=1}^{\infty} \frac{A\left(1 - \cos\frac{m}{A}\right)}{2m^2\pi} \int (e^{imu} + e^{-imu})\, F_N(du) \\ &= \frac{1}{A\pi} + \sum_{m=1}^{A^2} + \sum_{m=A^2+1}^{\infty} = \frac{1}{A\pi} + \Sigma_1 + \Sigma_2. \end{aligned}$$

Clearly

$$|\Sigma_2| \leqq \sum_{m=A^2+1}^{\infty} \frac{2A}{m^2\pi} \leqq \frac{1}{A}.$$

If $x \in D_1(N)$ and $1 \leqq m < A^2$, then

$$|\operatorname{Re} \int e^{imu} F_N(du)| = \left|\operatorname{Re} \frac{e^{iNmx} - 1}{N(e^{imx} - 1)}\right| \leqq \frac{4}{N(1 - \cos mx)} \leqq \frac{8}{A^2},$$

since $|mx - 2k\pi| > \frac{A}{\sqrt{N}}$ for all integers $k = 0, \pm 1, \pm 2, \ldots$. Therefore

$$\frac{1}{A\pi} + |\Sigma_1| \leqq \frac{1}{A\pi} + \sum_{m=1}^{\infty} \frac{2A}{m^2\pi} \cdot \frac{8}{A^2} \leqq \frac{10}{A}.$$

These estimates imply that for $x \in D_1(N)$

$$\int f_A(x)\, F_N(dx) \leqq \frac{11}{A}.$$

The last relation together with (3.2) imply that (i) holds if $A > \frac{22}{\varepsilon}$.

Proof of (ii). Let $x \in D_2(l, N)$, $1 \leqq l \leqq A^2$. Then there exists an integer j such that $\frac{1}{2NA^2} \leqq |y| \leqq \frac{A}{\sqrt{N}}$ for $y = x - 2\pi \frac{j}{l}$. Define $T_n^l y = ny \left(\operatorname{mod} \frac{2\pi}{l}\right)$ for $n = 0, 1, \ldots$. Obviously $|T_n x| < A^{-5}$ can hold only if $|T_N^l y| < A^{-5}$. Hence to prove (ii) it is enough to show that

$$\operatorname{Card}\{n;\, 0 < n < N,\, |T_n^l y| < A^{-5}\} < \varepsilon N \tag{3.3}$$

if $\frac{1}{2A^2 N} \leqq y \leqq \frac{A}{\sqrt{N}}$.

Set $M = \left(\frac{1}{[ly]} + 1\right)$. For arbitrary integer j the sequence $T_{j+1}^l y, \ldots, T_{j+M}^l y$ contains at most $\frac{2}{yA^5} + 2 < \frac{3}{yA^5}$ elements such that $|T_{j+p}^l y| < A^{-5}$. (Here we have exploited that in the sum $\frac{2}{yA^5} + 2$ the first term dominates since $\frac{1}{y} \geqq \frac{\sqrt{N}}{A}$, and we may choose N very large.) Since $l \leqq A^2$ the inequality $\frac{3}{A^5 y} \leqq \frac{3}{A^3 l y} \leqq \frac{4}{A^3} M$ holds. Put $\bar{N} = M\left(\left[\frac{N}{M}\right] + 1\right)$. The sequence $T_1^l y, \ldots, T_{\bar{N}}^l y$ contains at most

$$\frac{4}{A^3} M \left(\left[\frac{N}{M}\right] + 1\right) \leqq \frac{4}{A^3}(N + M) \leqq 4 \frac{(2A^2 + 1)}{A^3} N$$

elements such that $|T_n^l y| < A^{-5}$. Since $\bar{N} \geqq N$ this relation implies that for sufficiently large A relation (3.3) and hence (ii) hold.

Proof of (iii). If $x_1, \ldots, x_j \in D_3(N)$ then there exist some rational numbers $\frac{r_l}{s_l}$, $l = 1, \ldots, j$, such that $1 \leqq s_l \leqq A^2$, $-\frac{s_l}{2} \leqq r_l \leqq \frac{s_l}{2}$, $r_l \neq 0$, r_l and s_l are relatively primes, and $\left|x_l - 2\pi \frac{r_l}{s_l}\right| \leqq \frac{1}{2A^2 N}$, $l = 1, \ldots, j$. This implies that $\left|n x_l - 2\pi n \frac{r_l}{s_l}\right| \leqq \frac{1}{2A^2}$ if $n \leqq N$, and $|T_n x_l| \geqq \frac{1}{s_l} - \frac{1}{2A^2} \geqq \frac{1}{2A^2}$ if $n r_l \neq 0 \pmod{s_l}$. Hence to prove (iii) it is enough to show that the proportion of n, $\frac{N}{2} \leqq n \leqq N$ satisfying the relations

$$r_1 n \neq 0 \pmod{s_1}, \ldots, r_j n \neq 0 \pmod{s_j} \tag{3.4}$$

simultaneously is greater than $2p(j)$. We may assume that $s_1, \ldots, s_j$ are (distinct) prime numbers. Otherwise the proportion of the numbers satisfying (3.4) can be decreased by substituting s_l with one of its prime divisors. For large N the proportion of the numbers n, $\frac{N}{2} \leqq n \leqq N$ satisfying (3.4) is asymptotically

$$\prod_{l=1}^{j}\left(1-\frac{1}{s_l}\right) \geqq \prod_{l=1}^{j}\left(1-\frac{1}{p_l}\right) = 3\,p(j)$$

Statement (iii) and hence also Lemma 3 is proved.

References

1. Dobrushin, R.L.: Gaussian and their subordinated generalized fields. Ann. of Probab. **7**, 1–28 (1979)
2. Dobrushin, R.L.: Automodel generalized random fields and their renorm group, in Multicomponent random systems. New York: Dekker 1980
3. Dobrushin, R.L., Major, P.: Non-central limit theorems for non-linear functional of Gaussian fields. Z. Wahrscheinlichkeitstheorie verw. Gebiete **50**, 27–52 (1979)
4. Gelfand, I.M., Vilenkin, N.Ja.: Generalized functions IV. Some application of harmonic analysis. New York: Academic Press 1964
5. Sinai, Ja.G.: Automodel probability distributions. Theory Probability Applications **21**, 273–320 (1976)

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