INTRODUCTION TO TOPOLOGY

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Preface

These notes are based on a course given at the Budapest Semester in Mathematics in 2002. It is intended for a one semester undergraduate course. For readers who are interested in more details I suggest the Topology of Munkres which covers much more material than can be covered in a one semester course. The * sign at some homework problem indicates that I think it is more difficult. It may not apply for you.

Contents

1. Introduction—What is Topology?

Topology is elastic (rubber-sheet) geometry. It studies properties of shapes unchanged under deformations. As the joke says: Topologist is the person who cannot distinguish the teacup from the doughnut. I list some problems leading to the field of Topology:

Problems 1.1.

(1) The two-pancake problem: Suppose you have two pancakes on a plate. Find a straight cut halving both.
(2) The sleeping snake: Our snake sleeps in a tubelike bed having the same length as the snake has. The snake goes to sleep straight but has nightmares and wakes up all twisted. Show that there is a point on the snake which wakes up at the spot it went to sleep.
(3) A polynomial of odd order has a root.
(4) Is there a good map of Earth? If it is one piece, then it is extremely distorted at the poles. Otherwise you need two pieces.
(5) When can you untie a knot? For a mathematician a knot has no ends: a knot is the image of a circle in space without self-intersections.
(6) How to distinguish knots?
(7) Can you brush a hedgehog? In—not yet defined—mathematical terms: Is there a nowhere zero vector field on the sphere? An other equivalent question: Is it possible that the wind blows at every point on the surface of Earth at the same time? We assume that the Earth is a big sphere and the direction of the wind is always tangential (horizontal).
(8) The Dirac experiment: If you rotate an object 360° then things (or your hand) get twisted. Do it once more and the strings get untwisted.

Homework for Section 1. At this point we don't even have the proper definitions, so I would like you to think intuitively and find out how far you can get. Later in this course we will formulate these problems in precise mathematical terms and—hopefully—solve them.

(1) Solve the 1-pancake problem.
(2) * Solve the 2-pancake problem.
(3) Solve the sleeping snake problem and formulate the sleeping amoeba problem.
(4) Prove that $x^5 - 7x^3 + x^2 - 2$ has a root.
(5) * Prove that any polynomial of odd order has a root.
(6) Find a wind distribution on Earth with two (one) no-wind spot.

2. Continuity and Metric Spaces

The single most important notion underlying all these problems is continuity. We are surrounded by continuous functions. That is why our life is reasonably safe and predictable. What do we mean by continuous in everyday life?

(1) There are no jumps: if I buy a little more gas it costs a little more.
(2) Small changes have small effect. More precisely small change in the input results in a small change in the output: if I throw a ball with a little different speed or angle, it lands at a little different position.

(3) No missing value: If someone crosses a bridge then at a certain moment he has to be at the middle of the bridge.

Exercise 2.1. You may try to find discontinuous functions in everyday life.

To define continuity we need a notion of "small". Traditionally it is done by measuring. It leads to the notion of *metric spaces*: We want to measure the distance of two points.

Definition 2.2. *A metric space is a pair (X, d), where X is a set and $d : X \times X \to \mathbb{R}$ is a distance function, i.e. it satisfies the following properties:*

(1) (positivity) $d(x, y) \geq 0$ *and* $d(x, y) = 0 \iff x = y$.
(2) (symmetry) $d(x, y) = d(y, x)$
(3) (triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$.

Exercise 2.3. You may try to find examples where one of the three conditions are not satisfied.

Examples 2.4. (of metric spaces)

(1) The set of real numbers $\mathbb{R}$ with the distance function: $d(x, y) = |x - y|$. This is the most important example. Positivity and symmetry are clear, for $|x - z| \leq |x - y| + |y - z|$ notice that if y is in the middle then we have an equality, otherwise $|x - z| \leq |x - y|$ or $|x - z| \leq |y - z|$.
(2) The discrete metric spaces: X is any set and
$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases}$$
(3) The real plane $\mathbb{R}^2$ with the distance function:
$$d(P_1, P_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2},$$
where $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ are points of the plane. This is sometimes called the *round metric*. (Check the triangle inequality!).
(4) The real plane $\mathbb{R}^2$ with the distance function:
$$d(P_1, P_2) = |x_1 - x_2| + |y_1 - y_2|,$$
the *taxicab metric*: the shortest way in NY city.
(5) The real plane $\mathbb{R}^2$ with the distance function:
$$d(P_1, P_2) = \max\{|x_1 - x_2|, |y_1 - y_2|\},$$
the *maximum* or *square metric*.
(6) $X = \{f : [0, 1] \to \mathbb{R} \text{ continuous}\}$ and $d(f_1, f_2) = \int_0^1 |f_1(t) - f_2(t)| \mathrm{d}t$ (the so called *L^1-metric*).
(7) $X = \{f : [0, 1] \to [0, 1]\}$ and $d(f_1, f_2) = \sup_{0 \leq t \leq 1} |f_1(t) - f_2(t)|$ (the so called *supremum metric*).

Constructions 2.5. We can build new metric spaces out of old ones:

(1) Subspace:
Take any subset $Y \subset X$ of a metric space (X, d) and define a metric on Y by restricting the metric d to Y: $d_Y := d|_Y$. (Show that the discrete metric space with 3 elements is a subspace of $(\mathbb{R}^2, d_{\text{round}})$.)

(2) Product of two metric spaces:
If (X, d_X) and (Y, d_Y) are metric spaces then we can put a metric on the set $X \times Y$. In fact we have several choices:
(a) $d_{\text{round}}((x_1, y_1), (x_2, y_2)) = \sqrt{d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2}$,
(b) $d_{\text{cab}}((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)$,
(c) $d_{\max}((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}$.

(3) Product of infinitely many metric spaces:
The previous construction can be generalized to the case of countably many metric spaces with some boundedness assumption. (Try to define d_{round} and d_{cab}.) The third metric works even for uncountably many spaces:
Suppose that you have metric spaces $\{(X_i, d_i) : i \in I\}$ with the property that $d_i(x, y) \leq 1$ for all $i \in I$ and $x, y \in X_i$. Then
$$d_{\sup}(f, g) = \sup_{i \in I}\{d_i(f(i), g(i))\}$$
is a metric on the product set $\prod_{i \in I} X_i$. Recall that an element of the product set $\prod_{i \in I} X_i$ is a function on the index set I such that $f(i) \in X_i$ for all $i \in I$. Notice that the function space in Example 2.4.7 is a product space.

These are powerful constructions. It is not difficult (but tricky) to prove that every bounded metric space—i.e. $d(x, y) < K$ for some constant K—can be obtained as a subspace of some product of copies of the interval $[0, K]$.

The paradox of measuring: We want to use measuring—the distance function—to define continuity but "small" cannot be defined rigorously. But we are not far from resolving this conflict by defining *topological space*. Nevertheless this paradox makes the definition of continuity quite awkward: If you tell what you mean on "small" in the range of the function then I tell what I mean on "small" in the domain of the function.

Definition 2.6. *A map $f : X \to Y$ of two metric spaces (X, d_X) and (Y, d_Y) is continuous at the point $x_1 \in X$ if for every given $\epsilon > 0$ there is a $\delta > 0$ such that $d_X(x_1, x_2) < \delta$ implies that $d_X(f(x_1), f(x_2)) < \epsilon$.*
f is said to be continuous if it is continuous at every point $x_1 \in X$.

There is a more geometric way to think about continuity.

Definition 2.7. *The open ball centered at x with radius r in the metric space (X, d_X) is*
$$B_r(x) = \{z \in X : d_X(x, z) < r\}.$$

Remark 2.8. So f is continuous at x iff for every open ball $B_\epsilon(f(x))$ around $f(x)$ there is an open ball $B_\delta(x)$ around x such that $f(B_\delta(x)) \subset B_\epsilon(f(x))$.

Remark 2.9. The name "ball" might be misleading. Try to draw a picture of a ball in the different metrics on $\mathbb{R}^2$ defined in Examples 2.4. This explains the names of these metrics.

Homework for Section 2.

(1) Check the triangle inequality for the round metric 2.5.2a.
(2) Check the triangle inequality for the taxicab metric 2.5.2b.
(3) Check the triangle inequality for the maximum metric 2.5.2c.
(4) Show that $d_{\min}((x_1, y_1), (x_2, y_2)) = \min\{d_X(x_1, x_2), d_Y(y_1, y_2)\}$ is not a metric.

(5) * Show that that every bounded—i.e. $d(x, y) < K$ for some constant K—metric space can be obtained as a subspace of some product of copies of the interval $[0, K]$ (use the supremum metric!).
(6) Find a subspace of $\mathbb{R}^3$ which is isometric to the discrete metric on four points.
(7) Describe the open ball $B_1(0)$ for the metrics d_{round}, d_{cab} and $d_{\max}$.
(8) Let (X, d_X), (Y, d_Y) and (Z, d_Z) be metric spaces and $f : X \to Y$, $g : Y \to Z$ continuous functions. Show that the composition $g \circ f : X \to Z$ is continuous.
(9) Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is continuous at $x = 1$ by finding a δ for any given ϵ.

3. Continuity and Topological Spaces

You can rephrase the paradox in a different way: We want elastic geometry, so size shouldn't matter. Then why do we use it?

Now we try to find a better definition for continuity. We try to replace balls with a less metric concept. Let (X, d_X) be a metric space what we denote by X for short.

Definition 3.1. *The set $U \subset X$ is open if it is the union of open balls.*

Proposition 3.2. *$U \subset X$ is open iff for all $x \in U$ there is an $\epsilon > 0$ such that $B_\epsilon(x) \subset U$.*

We may call a set satisfying this condition *safe* since if we are standing in U we always have a safety zone: if we move less than ϵ distance we are still in U. So we show that a set is open iff it is safe.

Proof. $\Leftarrow$ $x \in B_\epsilon(x)$ so U is the union of these open balls.

$\Rightarrow$ Enough to show that an open ball is safe. So let $y \in B_\epsilon(x)$. Then by the triangle inequality $B_\delta(y) \subset B_\epsilon(x)$ if $\delta = \epsilon - d_X(x, y)$. □

Theorem 3.3. *Let (X, d_X) and (Y, d_Y) be metric spaces. Then a map $f : X \to Y$ is continuous iff for all $U \subset Y$ the preimage $f^{-1}(U) \subset X$ is open.*

Proof. $\Rightarrow$ Let $x \in f^{-1}(U)$. Then $f(x)$ is in U which is open by assumption. So there is an open ball $B_\epsilon(f(x)) \subset U$. Continuity of f implies that there is an open ball $B_\delta(x)$ such that $f(B_\delta(x)) \subset B_\epsilon(f(x))$ so $B_\delta(x) \subset f^{-1}(U)$. Since x was arbitrary we showed that $f^{-1}(U)$ is open.

$\Leftarrow$ Since any open ball is open by the assumption $f^{-1}(B_\epsilon(f(x))$ is open, too. $x \in f^{-1}(B_\epsilon(f(x))$, so there is a δ such that $B_\delta(x) \subset f^{-1}(B_\epsilon(f(x))$ in other words $f(B_\delta(x)) \subset B_\epsilon(f(x)$. □

Proposition 3.4. *Union of open sets is open and if U_1 and U_2 are open then $U_1 \cap U_2$ is open, too.*

Proof. The first statement is obvious, for the second let $x \in U_1 \cap U_2$ then we have two balls centered at x, one in U_1 and the other one in U_2. The one with the smaller radius is in $U_1 \cap U_2$. □

Notice that it was the triangle inequality which made it possible to show that the intersection of two open sets is open. Now we are in the position to get rid of the metric and make the open sets the fundamental objects.

Definition 3.5. *Let $\mathcal{U}$ be a collection of subsets of the set X. They define the open sets of a* **topology** *on X if the following conditions are satisfied:*

(1) $\emptyset,\ X \in \mathcal{U}$.
(2) $U_1,\ U_2 \in \mathcal{U}$ *implies that* $U_1 \cap U_2 \in \mathcal{U}$.
(3) if $U_i \in \mathcal{U}$ *for every* $i \in I$ *then* $\bigcup_{i\in I} U_i \in \mathcal{U}$.

The collection $\mathcal{U}$ is usually called *topology* on X and a pair $(X, \mathcal{U})$ is called a *topological space*.

Definition 3.6. *Let* $f : X \to Y$ *is a map between topological spaces* $(X, \mathcal{U}_X)$, $(Y, \mathcal{U}_Y)$. f *is* continuous *if for every* $U \in \mathcal{U}_Y$ *we have* $f^{-1}U \in \mathcal{U}_X$. *(In other words if preimage of open sets is open.)*

Definition 3.7. *The topological spaces* $(X, \mathcal{U}_X)$ *and* $(Y, \mathcal{U}_Y)$ *are* homeomorphic *(* $(X, \mathcal{U}_X) \cong (Y, \mathcal{U}_Y)$ *in notation) if there is a continuous bijective function* $f : X \to Y$ *such that the inverse function* f^{-1} *is continuous, too. Such a function is called* homeomorphism.

Homeomorphic spaces are indistinguishable in Topology. One main objective of Topology is to decide whether two topological spaces are homeomorphic or not.
We can define continuity at a point, we imitate the $\epsilon - \delta$-definition:

Definition 3.8. $N \subset X$ *is a* neighborhood *of* $x \in X$ *if there is an open set* $U \in \mathcal{U}_X$ *such that* $x \in U \subset N$.

Warning: Munkres defines neighborhood what others—and these notes—call open neighborhood.

Definition 3.9. $f : X \to Y$ *is* continuous at the point $x \in X$ *if for every neighborhood* N *of* $f(x)$ *there is a neighborhood* M *of* x *such that* $f(M) \subset N$.

Exercise 3.10. Show that $f : X \to Y$ is continuous iff it is continuous at every point $x \in X$.

3.1. **Examples of Topological Spaces.** The main examples for topological spaces come from metric spaces. We have defined open sets in a metric space in Definition 3.1. To avoid confusion we can call them metrically open sets. Proposition 3.4 shows that metrically open sets in a metric space satisfy the axioms in Definition 3.5. For a metric space (X, d) we denote the topological space defined by the metrically open sets by $(X, \mathcal{U}_d)$.
Not every topological space comes from a metric space. One traditional question of topology was to find sufficient and necessary conditions for a topological space to be "metrizable". I will give a partial answer to this question.

Examples 3.11.
(1) The *trivial* topological space on a set X: $\mathcal{U}_{\text{triv}} = \{\emptyset, X\}$.
(2) The *discrete* topological space on a set X: $\mathcal{U}_{\text{dis}} = \mathcal{P}(X)$ i.e. every set is open.

Different topologies on the same set. We have already found three different topologies on the set $\mathbb{R}$. It is easy to see that $\mathcal{U}_{\text{triv}} \subset \mathcal{U}_{\text{d}} \subset \mathcal{U}_{\text{dis}}$. So we say that $\mathcal{U}_{\text{triv}}$ is *smaller* or *coarser* than the usual topology $\mathcal{U}_{\text{d}}$ and $\mathcal{U}_{\text{dis}}$ is *bigger* or *finer* than $\mathcal{U}_{\text{d}}$.

Exercise 3.12. Show that if $\mathcal{U}_1, \mathcal{U}_2$ are topologies on the set X then $\mathcal{U}_1$ is bigger than $\mathcal{U}_2$ iff the identity map is continuous from $(X, \mathcal{U}_1)$ to $(X, \mathcal{U}_2)$.

Closed sets. Defining closed sets in a topological space is very easy:

Definition 3.13. *Let* $(X, \mathcal{U})$ *be a topological space.* $Z \subset X$ *is called* closed *if* $X \setminus Z \in \mathcal{U}$.

We usually denote the set of closed sets by $\mathcal{Z}$.

Proposition 3.14. *Let $\mathcal{Z}$ be the collection of closed subsets of the topological space X. Then the following conditions are satisfied:*

(1) $\emptyset,\ X \in \mathcal{Z}$.
(2) $Z_1,\ Z_2 \in \mathcal{Z}$ implies that $Z_1 \cup Z_2 \in \mathcal{Z}$.
(3) if $Z_i \in \mathcal{Z}$ for every $i \in I$ then $\bigcap_{i\in I} Z_i \in \mathcal{Z}$.

Proof. Apply the De Morgan identities:

$$X \setminus (\bigcup_{i\in I} Z_i) = \bigcap_{i\in I}(X \setminus Z_i), \qquad X \setminus (\bigcap_{i\in I} Z_i) = \bigcup_{i\in I}(X \setminus Z_i).$$

□

This allows us to give an alternative definition of topological space on a set X by giving a collection $\mathcal{Z}$ of subsets of X satisfying properties (1) to (3) of Proposition 3.14. Also continuity can be described via closed sets:

Proposition 3.15. *$f : X \to Y$ is continuous iff preimage of closed sets is closed.*

Proof is left to the reader.

Homework for Section 3.

(1) Prove Proposition 3.15.
(2) List all topological spaces on the set $X = \{1, 2\}$. In other words list all collections of subsets of X, which satisfy the axioms in 3.5.
(3) Show that $(\mathbb{R}^2, d_{\text{round}})$ and $(\mathbb{R}^2, d_{\max})$ have the same open sets, i.e. the topologies $\mathcal{U}_{d_{\text{round}}}$ and $\mathcal{U}_{d_{\max}}$ are the same.
(4) Show that $(X, \mathcal{U}_{\text{triv}})$ is not metrizable if X has at least 2 points.
(5) Define $\bar{A} = \bigcap\{Z \in \mathcal{Z} : Z \supset A\}$. The subset $\bar{A}$ is called the *closure* of A. Show that
  (a) $\bar{A}$ is the smallest closed set containing A.
  (b) $\bar{\bar{A}} = \bar{A}$.
  (c) $A \subset B$ implies $\bar{A} \subset \bar{B}$.
(6) Show that $x \in \bar{A} \Leftrightarrow$ every neighborhood of x intersects A.
(7) Show that for any $A \subset X$ there is a biggest open set in A. This set is usually called the *interior* of A, denoted by $\operatorname{int} A$.
(8) The set $\bar{A} \setminus \operatorname{int} A$ is called the *boundary* of A, denoted by ∂A. Show that ∂A is closed.
(9) Prove Exercise 3.10.
(10) Prove Exercise 3.12.
(11) Let $f : X \to Y$ be a continuous function and $f(X) \subset A \subset Y$. Show that $f : X \to A$ is also continuous.

4. Further Examples of Topological Spaces

Generating a Topological Space. Suppose that $\mathcal{S} \subset \mathcal{P}(X)$ and you want to know in which topologies these sets are open. They are certainly open in $\mathcal{U}_{\text{disc}}$ so the natural question is which is the smallest topology in which they are open.

Proposition 4.1. *For any $\mathcal{S} \subset \mathcal{P}(X)$ the smallest topology $\mathcal{U}_\mathcal{S}$ such that $\mathcal{S} \subset \mathcal{U}_\mathcal{S}$ is*

$$\mathcal{U}_\mathcal{S} = \{\bigcup_{i\in I} B_i : B_i \in \mathcal{B}_\mathcal{S}\} \text{ where } \mathcal{B}_\mathcal{S} = \{\bigcap_{i=1}^{n} S_i : S_i \in \mathcal{S}\}.$$

Remark 4.2. You may wonder why $X \in \mathcal{U}_\mathcal{S}$. We can agree that the intersection of no sets is the whole space X, or if you don't like this convention, just simply add the condition that $\bigcup\{S \in \mathcal{S}\} = X$ i.e. $\mathcal{S}$ is a *cover* of X.

Proof. The axioms of topology in Definition 3.5 implies that any topology $\mathcal{U} \supset \mathcal{S}$ has to be bigger than $\mathcal{U}_\mathcal{S}$ so we only have to verify that $\mathcal{U}_\mathcal{S}$ is a topology.

(1) $\emptyset, X \in \mathcal{U}_\mathcal{S}$ by the convention on empty union and intersection.
(2) Suppose that $U_k \in \mathcal{U}_\mathcal{S}$ for $k \in K$. Then $U_k = \bigcup_{i\in I_k} B_i$ for $B_i \in \mathcal{B}_\mathcal{S}$ so $\bigcup_{k\in K} U_k = \bigcup_{i\in I} B_i$ where $I = \bigcup_{k\in K} I_k$.
(3) Let $U, V \in \mathcal{U}_\mathcal{S}$. Then $U = \bigcup_{i\in I} B_i$ and $V = \bigcup_{j\in J} B_j$ where $B_i, B_j \in \mathcal{B}_\mathcal{S}$. By the distributivity for $\cup$ and $\cap$ we get:

$$U \cap V = \bigcup_{i\in I, j\in J} B_i \cap B_j.$$

It is clear from the definition of $\mathcal{B}_\mathcal{S}$ that $B_i \cap B_j$ is also in $\mathcal{B}_\mathcal{S}$, so $U \cap V \in \mathcal{U}_\mathcal{S}$.

□

Definition 4.3.

(1) A set $\mathcal{S} \subset \mathcal{P}(X)$ generating the topology $\mathcal{U}$ is called a subbasis *for $\mathcal{U}$ if $\mathcal{S}$ is a cover of X (See Remark 4.2).*
(2) A set $\mathcal{B} \subset \mathcal{P}(X)$ is called a basis *for the topology $\mathcal{U}$ if $\mathcal{U} = \{\bigcup_{i\in I} B_i : B_i \in \mathcal{B}\}$ and $\mathcal{B}$ is a cover of X.*

Exercise 4.4. $\mathcal{B} \subset \mathcal{P}(X)$ is a basis for a topology iff $B_1, B_2 \in \mathcal{B}$ implies the existence of $B_i \in \mathcal{B}$ for $i \in I$ such that $B_1 \cap B_2 = \bigcup_{i\in I} B_i$ and $\mathcal{B}$ is a cover of X.

Defining topologies by giving a basis or a subbasis for them is a useful method:

Examples 4.5.

(1) Metric topologies: If (X, d) is a metric space then $\mathcal{B} := \{B_\epsilon(x) : x \in X, \epsilon > o\}$. Proposition 3.4 shows that $\mathcal{B}$ is a basis for the metric topology on (X, d).
(2) Discrete topology: $\{\{x\} : x \in X\}$ is a basis for $\mathcal{U}_{\text{dis}}$.
(3) $\{U \subset X : X \setminus U$ is a finite set$\}$ is a basis for the *cofinite* topology $\mathcal{U}_{\text{cof}}$.
(4) $\{U \subset X : X \setminus U$ is a countable set$\}$ is a basis for the *cocountable* topology $\mathcal{U}_{\text{coco}}$.
(5) $\{(a, b] : -\infty \leq a < b \leq \infty\}$ is a basis for the *left* topology $\mathcal{U}_{\text{left}}$ on the real line $\mathbb{R}$ (also called the *Sorgenfrey line*).
(6) Let $(X, <)$ be an ordered set, and let $(a, \infty) = \{x \in X : a < x\}$ and $(-\infty, a) = \{x \in X : a > x\}$. Then

$$\{(a, \infty), (-\infty, a) : a \in X\} \cup \{X\}$$

is a subbasis for the *order-topology* $\mathcal{U}_<$.

Subbasis is also a useful tool when you want to check the continuity of a function:

Proposition 4.6. *Suppose that $\mathcal{S}$ is a subbasis for the topological space Y. then $f : X \to Y$ is continuous iff $f^{-1}(S)$ is open for all $S \in \mathcal{S}$.*

The proof is left to the reader.

Constructing new topologies from old ones.

Definition 4.7. Subspace topology*: If $(X, \mathcal{U})$ is a topological space and $A \subset X$ then $\mathcal{U}_A = \{U \cap A : U \in \mathcal{U}\}$ is a topology on A.*

Exercise 4.8. Show that $\mathcal{U}_A$ is the smallest topology for which the inclusion $i : A \to X$ is continuous.

Definition 4.9. Product topology*: If $(X, \mathcal{U}_X)$ and $(Y, \mathcal{U}_Y)$ are topological spaces then let $\mathcal{U}_{X \times Y}$ be the smallest topology on the product set $X \times Y$ for which the projection maps $\pi_X : (x, y) \mapsto x$ and $\pi_Y : (x, y) \mapsto y$ are continuous. In other words $\mathcal{U}_{X \times Y}$ is generated by the subbasis*

$$\{\pi_X^{-1}(U) : U \in U_X\} \cup \{\pi_Y^{-1}(U) : U \in U_Y\}.$$

Proposition 4.10. *The function $f : Z \to X \times Y$ is continuous iff the coordinate functions $\pi_X \circ f$ and $\pi_Y \circ f$ are continuous.*

Proof. $\Rightarrow$ follows from the definition since the projections π_X and π_Y are continuous. $\Leftarrow$ By Proposition 4.6 it is enough to check the preimages of elements of a subbase to prove continuity. But

$$f^{-1}\big(\pi_X^{-1}(U)\big) = (\pi_X \circ f)^{-1}(U)$$

and the right hand side is open by assumption. Replacing X with Y in the equation above we finished the proof. $\square$

Proposition 4.10 can be used to show the following universal property of the product:

Theorem 4.11. *Let X and Y be topological spaces. Then there exists a space T unique up to homeomorphism such that there are continuous maps $t_X : T \to X$ and $t_Y : T \to Y$ such that for any continuous maps $f_X : Z \to X$, $f_Y : Z \to Y$ there is a unique continuous map $h : Z \to T$ with the property: $f_X = t_X \circ h$ and $f_Y = t_Y \circ h$.*

The statement of the Theorem can be summarized with a so called *commutative diagram*:

$$\begin{tikzcd}
X & T \arrow[l, "t_X"'] \arrow[r, "t_Y"] & Y \\
& Z \arrow[ul, "f_X"] \arrow[u, dashed, "h"] \arrow[ur, "f_Y"'] &
\end{tikzcd}$$

The commutativity of such a diagram means that if there are two ways of getting from one point to the other following the arrows then the corresponding compositions of maps are equal.

Proof. By Proposition 4.10 $T = X \times Y$, $t_X = \pi_X$ and $t_Y = \pi_Y$ have the required property so the existence part is proved.

Unicity: Assume that there is an other space T' having the universal property with the corresponding maps $t'_X : T' \to X$ and $t'_Y : T' \to Y$. Then using the universal property of T we get the diagram:

(1)
$$\begin{tikzcd} X & T \arrow[l, "t_X"'] \arrow[r, "t_Y"] & Y \\ & T' \arrow[ul, "t'_X"] \arrow[u, dashed, "h"] \arrow[ur, "t'_Y"'] & \end{tikzcd}$$

Similarly using the universal property of T' we get the diagram:

(2)
$$\begin{tikzcd} X & T' \arrow[l, "t'_X"'] \arrow[r, "t'_Y"] & Y \\ & T' \arrow[ul, "t_X"] \arrow[u, dashed, "h"] \arrow[ur, "t_Y"'] & \end{tikzcd}$$

Now we show that the diagram

(3)
$$\begin{tikzcd} X & T \arrow[l, "t_X"'] \arrow[r, "t_Y"] & Y \\ & T \arrow[ul, "t_X"] \arrow[u, "h \circ h'"] \arrow[ur, "t_Y"'] & \end{tikzcd}$$

commutes so by the unicity of the vertical map $h \circ h' = \mathrm{Id}_T$. But $t_X \circ h \circ h' = t'_X \circ h'$ by the first diagram which is equal to t_X by the second diagram. Similarly you can get that $t_Y \circ h \circ h' = t_Y$. Replacing T with T' we get that $h' \circ h = \mathrm{Id}_{T'}$ so $T \cong T'$. □

This theorem provides us an alternative definition of the product which is very useful in applications since it reflects the basic property of the product.

Similarly you can define the product of any number of topological spaces by looking for the smallest topology on the product set for which all the projection maps are continuous.

Homework for Section 4.

(1) What are the continuous maps from $(\mathbb{R}, \mathcal{U}_{\text{triv}})$ to $(\mathbb{R}, \mathcal{U}_{\text{d}})$?
(2) Show that every map is continuous from $(X, \mathcal{U}_{\text{dis}})$ and every map is continuous to $(X, , \mathcal{U}_{\text{triv}})$.
(3) Prove Exercise 4.4: $\mathcal{B} \subset \mathcal{P}(X)$ is a basis for a topology iff it is closed for finite intersection i. e. $B_1, B_2 \in \mathcal{B}$ implies the existence of $B_i \in \mathcal{B}$ for $i \in I$ such that $B_1 \cap B_2 = \bigcup_{i \in I} B_i$.
(4) Find a basis for the order-topology in Example 4.5.
(5) Which previously defined topologies on the real line are smaller than $(\mathbb{R}, \mathcal{U}_{\text{cof}})$?
(6) Which previously defined topologies on the real line are smaller than $(\mathbb{R}, \mathcal{U}_{\text{left}})$?
(7) Prove Exercise 4.8: Show that $\mathcal{U}_A$ is the smallest topology for which the inclusion $i : A \to X$ is continuous.
(8) Show that the open rectangles $\{U \times V : U \in \mathcal{U}_X,\ V \in \mathcal{U}_Y\}$ provide a basis for the product topology $\mathcal{U}_{X \times Y}$.
(9) Show that the metric topology on the real plane $\mathbb{R}^2$ coincides with the product topology on $\mathbb{R} \times \mathbb{R}$. (More generally metric product is compatible with topological product.)

(10) * Show that the product of finitely many discrete topological spaces is discrete, but $\prod_{i=1}^{\infty} X_i$ where X_i is the discrete topology on $\{0,1\}$ is not discrete.
(11) Show that $(0,1) \cong \mathbb{R}$.
(12) Prove Exercise 4.6, that it is enough to check continuity for a subbasis.
(13) Let $Y_x = \{(x,y) : y \in Y\} \subset X \times Y$ (usually called a *slice*) Show that $Y_x \cong Y$.

5. Intermediate Value Theorem and Connectedness

One of the properties of continuous function we listed at the beginning of this course was the "no missing value" property. One of the first theorems in calculus is the

Theorem 5.1 (Intermediate Value Theorem)**.** *If $f : [0,1] \to \mathbb{R}$ is a continuous function, then $[f(0), f(1)] \subset f([0,1])$.*

There is a straightforward way to generalize this property:

Definition 5.2. *The topological space $(X, \mathcal{U})$ has the* Intermediate Value (IMV) *property if for any continuous function $f : X \to \mathbb{R}$ the image $f(X)$ is an interval (possibly infinite).*

Exercise 5.3. Find examples of topological spaces with and without the IMV property.

We can give a simpler description of the IMV property:

Definition 5.4. *The topological space $(X, \mathcal{U})$ is connected if for any nonempty $U, V \in \mathcal{U}$ the property $U \cup V = X$ implies $U \cap V \neq \emptyset$.*

Exercise 5.5. Show that $(X, \mathcal{U})$ is connected iff for any nonempty F, G closed sets if $F \cup G = X$ then $F \cap G \neq \emptyset$.

Exercise 5.6. Show that $(X, \mathcal{U})$ is connected iff the only open and close sets (sometimes called clopen) are $\emptyset$ and X.

Theorem 5.7. *The topological space $(X, \mathcal{U})$ has the IMV property iff it is connected.*

Proof. $\Rightarrow$ Suppose that X doesn't have the IMV property. Then $\exists$ a continuous function f and real numbers $a < z < b$ such that $a, b \in f(X)$ but $z \notin f(X)$. Then the open decomposition $X = f^{-1}\big((-\infty, z)\big) \cup f^{-1}\big((z, \infty)\big)$ shows that X is not connected.

$\Leftarrow$ Suppose we have a decomposition $X = U_1 \cup U_2$ to nonempty disjoint open sets. Then

$$f(x) = \begin{cases} 1 & \text{if } x \in U_1 \\ 2 & \text{if } x \in U_2 \end{cases}$$

is a continuous function demonstrating that X doesn't have the IMV property. $\square$

Theorem 5.8. *The closed interval $[0,1]$ is connected.*

Proof. Suppose we have a decomposition $[0,1] = U \cup V$ to disjoint open sets. We can assume that $1 \in V$. Let $x = \sup U$. Then $x \in U$ since $U = [0,1] \setminus V$ is closed as well. But U is open, too, so it has to contain a small interval around x. So x has to be equal to 1. Contradiction. $\square$

In fact the connectedness of $[0,1]$ is *equivalent* with the existence of supremum. Consequently at the definition of real numbers we can replace the Cantor axiom with declaring $[0,1]$ to be connected.

Exercise 5.9. Show that the connected subsets of $\mathbb{R}$ are the intervals (possibly infinite).

The most general and also the simplest version of the IMV-type theorems is the following:

Theorem 5.10. *The continuous image of a connected space is connected.*

Proof. The preimage of a clopen subset is clopen. □

Exercise 5.11. Prove Theorem 5.1 using Exercise 5.9 and Theorem 5.10.

Definition 5.12. *Let $a \sim b$ if a and b are elements of the topological space X and they belong to a connected subset of X.*

The following exercise shows that $\sim$ is an equivalence relation (you can use it to prove the transitivity of $\sim$):

Exercise 5.13. Suppose that $A_i \subset X$ are connected and the pairwise intersections $A_i \cap A_j$ are not empty. Prove that $\bigcup A_i$ is connected.

Definition 5.14. *The equivalence classes of $\sim$ are called the* connected components *of the topological space X.*

Exercise 5.13 implies that the connected components are indeed connected. There is a similar concept we will use more frequently later:

Definition 5.15. *The topological space $(X, \mathcal{U})$ is path connected (or arcwise connected) if for any two points a and b in X there is a continuous map $f : [0, 1] \to X$ (called path) such that $f(0) = a$ and $f(1) = b$.*

Proposition 5.16. *A path connected space is connected.*

Proof. The unit interval is connected so by Theorem 5.10 $f([0, 1])$ is also connected. So if $(X, \mathcal{U})$ is path connected then X has only one connected component so X is connected. □

Example 5.17. The following subspace of the plane is connected but not path connected:

$$X := \{(0, 0)\} \cup \{x, \sin(1/x) : x \in (0, \pi]\}.$$

Hint: Show that any path $f : [0, 1] \to X$ connecting $(0, 0)$ with $(\pi, 0)$ contains all the points of X. Then check continuity of f at 0.

Remark 5.18. Similarly to Definition 5.14 you can define the *path components* of a topological space.

Homework for Section 5.

(1) Show that if $A \subset X$ is connected then $\overline{A}$ is connected, too.
(2) Show that the connected components are closed.
(3) Show that $\mathbb{R} \not\cong \mathbb{R}^2$ with their usual topology.
(4) Show that $\mathbb{R} \not\cong (0, 1]$.
(5) What are the continuous maps from $(\mathbb{R}, \mathcal{U}_{\mathrm{d}})$ to $(\mathbb{R}, \mathcal{U}_{\mathrm{dis}})$?
(6) Which previously defined topologies are connected on $\mathbb{R}$?
(7) Prove Exercise 5.5: Show that $(X, \mathcal{U})$ is connected iff for any nonempty F, G closed sets if $F \cup G = X$ then $F \cap G \neq \emptyset$.
(8) Prove Exercise 5.6: Show that $(X, \mathcal{U})$ is connected iff the only clopen sets are $\emptyset$ and X.

(9) Prove Exercise 5.13: Show that if $A_i \subset X$ are connected and the pairwise intersections $A_i \cap A_j$ are not empty, then $\bigcup A_i$ is connected.
(10) Prove Exercise 5.9: Show that the connected subsets of $\mathbb{R}$ are the intervals (possibly infinite).
(11) Prove Exercise 5.11: Prove Theorem 5.1 using Exercise 5.9 and Theorem 5.10.
(12) * Prove the existence of supremum for subsets of the closed interval using that it is connected.
(13) * Show that the space of Example 5.17 is connected but not path connected.
(14) Show that if $U \subset \mathbb{R}^n$ is open and connected then also path connected.
(15) Show that the space X is connected iff every continuous function $f : X \to \{1, 2\}$ (with the discrete topology on $\{1, 2\}$) is constant.
(16) Show that the product of two connected spaces is connected and the product of two path connected spaces is path connected. (Hint: You can use the slice lemma Homework (13) in Section 4.)

6. Separation Axioms

If we want to deal with only "reasonable" (for example metrizable) topological spaces we need some restrictions.

Definition 6.1. *$(X, \mathcal{U})$ is T_0 if for any two points $x \neq y$ there is an open set containing one of them but not the other.*

Exercise 6.2. The topological space $(X, \mathcal{U})$ is T_0 if every $f : Y \to X$ continuous function—where Y is a trivial topological space with at least two elements—is constant.

Definition 6.3. *The topological space $(X, \mathcal{U})$ is T_1 if for any two points $x \neq y$ there is an open set containing x but not y and another open set containing y but not x.*

Exercise 6.4. $(X, \mathcal{U})$ is T_1 iff every point is closed.

Definition 6.5. *The topological space $(X, \mathcal{U})$ is T_2 or* Hausdorff *if for any two points $x \neq y$ there are disjoint open sets U and V with $x \in U$ and $y \in V$.*

Exercise 6.6. Every metric space is Hausdorff.

Hausdorff spaces behave well because sequences behave well:

6.1. **Limit of sequences.**

Definition 6.7. *Let $\{x_n : n \in \mathbb{N}\}$ is a sequence in the topological space $(X, \mathcal{U})$. We say that $\{x_n\}$ converges to $x \in X$ ($x_n \to x$ or x is the limit of the sequence $\{x_n\}$) if for every open set U containing x the set $\{n \in \mathbb{N} : x_n \notin U\}$ is finite.*

Exercise 6.8. If $x_n \to x$ and $\{x_n : n \in \mathbb{N}\} \subset A$ then $x \in \overline{A}$.

Proposition 6.9. *In a Hausdorff space a sequence has at most one limit.*

The proof is left to the reader. In applications you hardly ever meet topological spaces which are not Hausdorff. One notable exception is in algebraic geometry. Let $X = \mathbb{R}^n$ (or $\mathbb{C}^n$) and let $Z \subset X$ be closed if there are polynomials $\{p_i : i \in I\}$ of n variables such that $Z = \{x \in X : p_i(x) = 0 \text{ for all } i \in I\}$. Such sets are usually called algebraic sets. The corresponding topology is called the *Zariski* topology.

Exercise 6.10. Show that the algebraic sets in X satisfy the axioms for closed sets in a topology.

Homework for Section 6.

(1) Prove Exercise 6.2 that the topological space $(X,\mathcal{U})$ is T_0 if every continuous function $f : Y \to X$—where Y is a trivial topological space with at least two elements—is constant.
(2) Prove Exercise 6.4 that $(X,\mathcal{U})$ is T_1 iff every point is closed.
(3) Describe the sequences which have a limit in $\mathcal{U}_{\text{dis}}$, $\mathcal{U}_{\text{triv}}$, $\mathcal{U}_{\text{cof}}$ and $\mathcal{U}_{\text{coco}}$.
(4) Prove that if $f : X \to Y$ is continuous then $x_n \to x$ implies $f(x_n) \to f(x)$.
(5) Prove that if in a topological space limits are unique then it is a T_1 space.
(6) Find an example for a space in which limits are unique, but it is not Hausdorff.
(7) Show that the product of Hausdorff spaces is Hausdorff.
(8) Prove that X is Hausdorff iff the diagonal $\Delta = \{(x,x) : x \in X\} \subset X \times X$ is closed.
(9) Let $f : X \to Y$ be a continuous map. Show that if Y is T_1 then the level set
$$(f = c) = \{x \in X : f(x) = c\}$$
is closed.
(10) Let $f, g : X \to Y$ be continuous maps. Show that if Y is Hausdorff then $E = \{x \in X : f(x) = g(x)\}$ is closed in X.
(11) Show that $\mathrm{Id} : (\mathbb{R},\mathcal{U}_{\text{cof}}) \to (\mathbb{R},\mathcal{U}_{\text{dis}})$ is not continuous but $x_n \to x$ implies $f(x_n) \to f(x)$.
(12) Prove Exercise 6.8: If $x_n \to x$ and $\{x_n : n \in \mathbb{N}\} \subset A$ then $x \in \overline{A}$.
(13) Prove Exercise 6.10 on the Zariski topology.
(14) Prove that the standard topology on $X = \mathbb{R}^n$ (or $\mathbb{C}^n$) is bigger than the Zariski topology.

7. Countability Axioms

We know that for real functions continuity can be defined via convergent sequences:

Theorem 7.1. *$f : \mathbb{R} \to \mathbb{R}$ is continuous iff for every sequence $x_n \to x$ implies $f(x_n) \to f(x)$.*

Homework 4 in Section 6 says that $\Rightarrow$ is always true, but not the converse by Homework 11. So we introduce a class of topological spaces where the converse holds, too:

Definition 7.2. *The topological space $(X,\mathcal{U})$ is M_1 or first countable if for every $x \in X$ there exist open sets $\{U_n : n \in \mathbb{N}\}$ containing x such that for all $V \ni x$ open there is an n such that $U_n \subset V$.*

Theorem 7.3. *Suppose that $(X,\mathcal{U})$ is first countable. Then $f : X \to Y$ is continuous if for every sequence $x_n \to x$ implies $f(x_n) \to f(x)$.*

To simplify the language we introduce a definition:

Definition 7.4. *$\{N_i : i \in I\} \subset \mathcal{P}(X)$ is a neighborhood basis for $x \in X$ if all N_i is a neighborhood of x and for all N neighborhood of x there is an i such that $N_i \subset N$.*

So X is first countable iff every point has a countable neighborhood basis.

Proof of Theorem 7.3. Suppose that f is not continuous. Then there is an open set U such that $f^{-1}(U)$ is not open, i.e. there is an $x \in f^{-1}(U)$ which is in the closure of the complement i.e. for all $V \ni x$ open $V \setminus f^{-1}(U)$ is not empty. Let $\{U_n : n \in \mathbb{N}\}$ be a countable neighborhood basis of x. Then we can choose elements $x_n \in U_n \setminus f^{-1}(U)$. Then

(1) $x_n \to x$ since $x_n \in U_n$ so for any $V \ni x$ open there is an n such that $U_n \subset V$ so all but the first $n-1$ elements of the sequence $\{x_n\}$ is in V.
(2) $f(x_n) \not\to f(x)$ since $f(x_n) \notin U$.

□

Proposition 7.5. *Every metric space is first countable.*

Proof. $\{B_{1/n}(x) : n \in \mathbb{N}\}$ is a neighborhood basis of x. (For specialists: we used the Archimedean axiom here ;-) □

Definition 7.6. *The topological space $(X, \mathcal{U})$ is M_2 or second countable if it has a countable basis.*

Homework for Section 7.

(1) If X is first countable then $\bar{A} = \{x \in X : \exists a_n \in A \; a_n \to x\}$.

8. Compactness

Compactness is probably the most frequently used topological property. The basic example for compact space is the unit interval $[0,1]$. Let us list some properties related to compactness shared by $[0,1]$:

(1) Every continuous function $f : X \to \mathbb{R}$ is bounded.
(2) Every continuous function $f : X \to \mathbb{R}$ attains its maximum.
(3) Every sequence in X has a convergent subsequence.
(4) Every decreasing sequence of closed subsets of X has a nonempty intersection. (The Cantor Axiom)
(5) Every open cover of X has a finite subcover.

Here open cover means a collection $\mathcal{C}$ of open sets such that $\cup\mathcal{C} = X$ and a finite subcover is a finite subset $\mathcal{C}'$ of $\mathcal{C}$ such that $\cup\mathcal{C}' = X$.

For general topological spaces these properties are not equivalent.

Definition 8.1. *The topological space X is* compact *if every open cover of X has a finite subcover.*

Exercise 8.2. Show that for a topological space X a subset $A \subset X$ is compact in the subspace topology iff for every collection $\mathcal{C}$ of open subsets of X such that $\cup\mathcal{C} \supset A$ there is a finite subset $\mathcal{C}'$ of $\mathcal{C}$ such that $\cup\mathcal{C}' \supset A$. So we use the name open cover for $\mathcal{C}$ and the name finite subcover for $\mathcal{C}'$ in this situation as well.

Proposition 8.3. *Let Y be a Hausdorff space. Then every compact subset of Y is closed.*

Proof. Fix a $x \in X \setminus K$. Then for any $k \in K$ let $U_k \ni k$ and $V_k \ni x$ disjoint open sets provided by the Hausdorff property. Then $\bigcup_{k \in K} U_k$ is an open cover of K so it has a finite subcover $\{U_{k_1}, \ldots, U_{k_n}\}$. Then

$$W_x = \bigcap_{i=1}^{n} V_{k_i} \subset (X \setminus K)$$

is open (being the finite intersection of open sets). Construct such a W_x for all $x \in X \setminus K$. Then $X \setminus K = \bigcup_{x\in X\setminus K} W_x$ is open. $\square$

Exercise 8.4. Suppose that X is compact and $A \subset X$ is closed. Then A is compact.

Exercise 8.5. If K is a compact subspace of a metric space then K is bounded (i.e. there is a ball $B_N(x) \supset K$).

Exercise 8.6. It is not true that a closed and bounded subspace of a metric space is compact. Find an example.

Proposition 8.7. *The unit interval* $[0,1]$ *is compact.*

Proof. Let $\mathcal{C}$ be an open cover of $[0,1]$ and let

$$S = \{s \in [0,1] : \text{ there is a finite subcover } \mathcal{C}' \subset \mathcal{C} \text{ for } [0,s]\}.$$

Let $b = \sup S$. There is a $U \in \mathcal{C}$ such that $b \in U$. The set U is open, so if $b \neq 1$ then there is an $\epsilon > 0$ such that $(b-\epsilon, b+\epsilon) \subset U$. Since $b - \epsilon/2 \in S$ there is a finite subcover $\mathcal{C}'$ covering $[0, b-\epsilon/2]$ so $\mathcal{C}' \cup \{U\}$ is a finite subcover for $[0, b+\epsilon/2]$. So b had to be 1. $\square$

Proposition 8.8. *If* $f : X \to Y$ *is continuous and* X *is compact then* $f(X)$ *is compact.*

Proof. Preimage of an open cover is an open cover. $\square$

Corollary 8.9. *If* X *is compact then every continuous function* $f : X \to \mathbb{R}$ *is bounded.*

Proof. $f(X)$ is compact by 8.8 therefore bounded by 8.5. $\square$

Corollary 8.10. *If* X *is compact then every continuous function* $f : X \to \mathbb{R}$ *attains its maximum.*

Proof. $f(X)$ is compact therefore closed by 8.3. For any bounded set $A \subset \mathbb{R}$ we have that $\sup A \in \bar{A}$ so $\sup f(X) \in f(X)$. $\square$

Proposition 8.11. *If* X *is compact and* M_1 *then every sequence in* X *has a convergent subsequence.*

We prepare the proof with a definition and two lemmas:

Definition 8.12. $x \in X$ *is an* accumulation point *of* $A \subset X$ *if for every open set* $U \ni x$ *the set* $(U \setminus x) \cap A$ *is not empty (limit point in Munkres).*

The set of accumulation points of $A \subset X$ is denoted by A'.

Exercise 8.13. Show that $\bar{A} = A \cup A'$.

Lemma 8.14. *If* X *is compact then every infinite set has an accumulation point.*

Proof. Suppose that $A \subset X$ has no accumulation point. Then for all $x \in X$ there is an open set $U_x \ni x$ such that $(U_x \setminus x) \cap A$ is empty. $\{U_x : x \in X\}$ is an open cover of X so it has a finite subcover $\{U_{x_1}, \dots, U_{x_n} : x_i \in X\}$. Then $A \subset \{x_1, \dots, x_n\}$ so A is finite. $\square$

Lemma 8.15. *If* X *is* M_1 *and* $x \in X$ *is an accumulation point of* $A \subset X$ *then there is a sequence* $\{a_n\} \subset A$ *such that* $a_n \to x$.

Proof. Let $\{U_n : n \in \mathbb{N}\}$ be a countable neighborhood basis of x. Since x is an accumulation point of A we can pick points $a_n \in U_n \cap A$. As in the proof of Theorem 7.3 we can see that $a_n \to x$. □

These lemmas immediately imply Proposition 8.11. A space X with the property that every sequence in X has a convergent subsequence is called *sequentially compact.* So Proposition 8.11 says that a compact M_1 space is sequentially compact.

Proposition 8.16. *If X is compact then every decreasing sequence of closed subsets of X has a nonempty intersection.*

The proof is left to the reader.

Theorem 8.17. *The product of compact spaces is compact.*

The proof uses the Axiom of Choice and not so easy so we omit it.

Theorem 8.18. *If X is compact Hausdorff and M_2 then metrizable.*

This is a special case of Urison's metrization theorem. We omit the proof.

Homework for Section 8.

(1) Prove Exercise 8.4: Suppose that X is compact and $A \subset X$ is closed. Then A is compact.
(2) Prove Exercise 8.5: If K is a compact subspace of a metric space then K is bounded.
(3) Prove Exercise 8.6: find a subspace of a metric space which is closed and bounded but not compact.
(4) Prove the Basis theorem for compactness: Suppose that $\mathcal{B}$ is a basis for the topological space X. then X is compact iff for any cover $\mathcal{C} \subset \mathcal{B}$ there is a finite subcover. The theorem is true for subbasis, too. It is called the Alexander subbasis theorem. The proof is more difficult and uses the Axiom of Choice. The Alexander subbasis theorem implies Theorem 8.17 (this is not difficult).
(5) * Prove Theorem 8.17 for two spaces. Hint:
   (a) Use Homework (8) in Section 3 that the "rectangles"
   $$\{U \times V : U \text{ open in } X \text{ and } V \text{ open in } Y\}$$
   give a basis for the topology on $X \times Y$.
   (b) Use the slice lemma Homework (13) in Section 4 that the "slice"
   $$Y_x := \{(x, y) : y \in Y\} \subset X \times Y$$
   is homeomorphic to Y.
(6) * Let $X_S = \{0, 1\}$ with the discrete topology for $S \in \mathcal{P}(\mathbb{N})$. Then $X = \prod_{S \in \mathcal{P}(\mathbb{N})} X_S$ is compact by Theorem 8.17. Let
$$x_n(S) = \begin{cases} 0 & \text{if } n \in S \\ 1 & \text{if } n \notin S \end{cases}.$$
Show that $\{x_n\}$ doesn't have a convergent subsequence.
(7) Prove that if X is T_1 then x is an accumulation point of $A \subset X$ iff for every neighborhood N of x the intersection $A \cap N$ is infinite. Find a T_0 space which shows that the T_1 condition above is necessary.

(8) Let $f : X \to Y$ be a continuous bijection where X is compact and Y is Hausdorff. Show that f is a homeomorphism.

9. The notion of Homotopy and the Fundamental Group

Till now we treated functions in a stricter way than spaces. Two spaces are considered to be the same if you can "deform" one into the another. We can do something similar to functions as well. We say that $f : X \to Y$ is *homotopic* to $g : X \to Y$ if there is a "continuous" deformation $\{f_t : t \in I\}$ (here I denotes the unit interval) such that $f_0 = f$ and $f_1 = g$. The problem is that we haven't defined a topology on the space of continuous functions $C(X, Y)$ yet. It would be possible but we take a slightly different approach:

Definition 9.1. *$H : X \times I \to Y$ is a* homotopy *between f and g if $H_0 = f$ and $H_1 = g$ where $H_s = H \circ i_s$ and $i_s : X \to X \times I$ such that $i_s(x) = (x, s)$.*

If there is a homotopy between f and g then we say that f and g are homotopic *or* homotopy equivalent *and we write $f \sim g$.*

To show that $\sim$ is indeed an equivalence relation we need a definiton and a lemma:

Definition 9.2. *Suppose that $F, G : I \times X \to Y$ are maps. Then*

$$F \cdot G(t, x) = \begin{cases} F(2t, x) & \text{if } 0 \le t \le 1/2 \\ G(2t - 1, x) & \text{if } 1/2 \le t \le 1 \end{cases}$$

is the concatenation *of F and G.*

The concatenation of two continuous maps is continuous by the Pasting Lemma:

Lemma 9.3. *Suppose that $X = A \cup B$ and A, $B \subset X$ are closed. If $f : X \to Y$ is continuous restricted to A and restricted to B then f is continuous.*

Proof. Exercise. □

Now you are ready for the following:

Exercise 9.4. Show that $\sim$ is an equivalence relation.

The mapping spaces $C(Z, X)$ and $C(X, Z)$ are fundamental in studying topological invariants of X. But studying only the set of homotopy equivalence classes is an easier task. Homotopy theory studies the sets $[Z, X] = C(Z, X)/\sim$. We will look at the case of Z =circle in details. As we will see it provides us a strong and calculable topological invariant. It also helps us to solve almost all the problems listed in Section 1.

For technical reasons we think a map of the circle to the space X as a map—called loop—$f : I \to X$ such that 0 and 1 are mapped to the same point. In fact we assume that $f(0) = f(1) = x_0$ where x_0 is a distinguished point of X called the *base point*. This seemingly unnecessary restriction becomes important when we want to define a group structure on loops to enter the field of Algebraic Topology:

Definition 9.5. *Suppose that $f, g : I \to X$ are loops. Then*

$$f \cdot g = \begin{cases} 2t & \text{if } 0 \le t \le 1/2 \\ 2t - 1 & \text{if } 1/2 \le t \le 1 \end{cases}$$

is the concatenation *of f and g.*

We can see that $f \cdot g$ is a loop, and by the Pasting Lemma this loop is continuous.

Concatenation resembles to a group multiplication on the set of loops. But it is easy to see that there is not even a unit element in this structure. However, moving to the homotopy level fixes all the problems.

Definition 9.6. *Suppose that $f, g : I \to X$ are loops. Then $H : I \times I \to X$ is a* (based) homotopy *between f and g if $H_0 = f$ and $H_1 = g$ and $H(0, s) = H(1, s) = x_0$ for all $s \in I$.*

Definition 9.7. *Let $\pi_1(X, x_0)$ be the set of homotopy equivalence classes of loops in (X, x_0).*

Theorem 9.8. *Concatenation defines a group structure on $\pi_1(X, x_0)$.*

Proof. This proof is typical in homotopy theory. We work with equivalence classes so with every definition first we have to check whether it makes sense on the level of equivalence classes. In this particular case we have to check the following:

$$\text{If } f_0 \sim f_1 \text{ and } g_0 \sim g_1 \text{ then } f_0 \cdot g_0 \sim f_1 \cdot g_1.$$

So we have a homotopy F between f_0 and f_1 and a homotopy G between g_0 and g_1. Then the concatenation $F \cdot G$ will give a homotopy between $f_0 \cdot g_0$ and $f_1 \cdot g_1$ as soon as we define it:

Remark 9.9. The pasting lemma 9.3 shows that $F \cdot G$ is continuous. Notice that if $F, G : I \times I \to Y$ are maps from the unit square then there are two different ways to concatenate.

You can see that Definition 9.5 is a special case of Definition 9.2. It is also easy to see that if F and G are based homotopies than $F \cdot G$ is a based homotopy, too. So we showed that concatenation defines a binary operation on $\pi_1(X, x_0)$. Now we have to check whether the group axioms

(1) there is a unit e such that $\gamma e = e\gamma = \gamma$ for all $\gamma \in G$,
(2) for all $\gamma \in G$ there is an inverse $\gamma^{-1} \in G$ such that $\gamma\gamma^{-1} = e$,
(3) the associativity $\alpha(\beta\gamma) = (\alpha\beta)\gamma$ holds

are satisfied for $G = \pi_1(X, x_0)$.

(1) It is clear that the unit in this group should be the constant loop $\underline{x}_0$. So we have to show that $f \cdot \underline{x}_0 \sim f$ for all loops f, which can be seen from the following:

Lemma 9.10 (on reparametrization)**.** *Let $h : I \to I$ such that $h(0) = 0$ and $h(1) = 1$. Then for any loop $f : I \to (X, x_0)$ we have $f \sim f \circ h$.*

Proof. We use the trick of convex combination (and this is not the last time ;-). Let $H(t, s) = f\big(sh(t) + (1 - s)t\big)$. □

Exercise 9.11. Show that H is indeed a homotopy between f and $f \circ h$.

(2) If $\gamma = [f]$ (i.e. f is an element of the equivalence class γ), then let $\gamma^{-1} = [\overline{f}]$ where $\overline{f}(t) = f(1 - t)$.

Exercise 9.12. Use the following generalization of the reparametrization lemma 9.10 to show that $\overline{f} \cdot f \sim \underline{x}_0$: Let $h_1, h_2 : I \to I$ such that $h_1(0) = h_2(0)$ and $h_1(1) = h_2(1)$. Then for any map $f : I \to X$ such that $f(h_1(0)) = f(h_1(1) = x_0$ the loops $f \circ h_i$ are homotopic.

(3) Associativity can be checked with an other application of the reparametrization lemma 9.10 to

$$h(t) = \begin{cases} 2t & \text{if } \ 0 \le t \le 1/4 \\ t + 1/4 & \text{if } \ 1/4 \le t \le 1/2 \\ t/2 + 1/2 & \text{if } \ 1/2 \le t \le 1 \end{cases}.$$

□

This group $\pi_1(X, x_0)$ is called the *first homotopy* or *fundamental* group of (X, x_0).

Homework for Section 9.

(1) Prove Exercise 9.4 that homotopy is an equivalence relation.
(2) Prove the Pasting Lemma 9.3.
(3) Prove Exercise 9.11 by proving that the fuctions $x + y$ and $x \cdot y$ are continuous.

10. Induced homomorphisms and Functors

As I mentioned before maps are frequently even more important than the objects. That makes the following simple fact very important:

Proposition 10.1. *Suppose that $f : (X, x_0) \to (Y, y_0)$ is a base point preserving map (i.e. $f(x_0) = y_0$). Then*

(1) f induces a homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ where $f_*[l] = [f \circ l]$.
(2) $(f \circ g)_* = f_* \circ g_*$ for any $g : (Z, z_0) \to (X, x_0)$ base point preserving map.

Proof. Left to the reader. □

An other, less popular notation for f_* is $\pi_1(f)$. The advantage of the latter is that it reminds us that π_1 is a *functor*. In other words it is a map from the category of topological spaces to the category of groups. Though I am not going to use category theory in these notes I give some basic definitions because I believe thinking in these terms helps to get a wider perspective.

Definition 10.2. *A* category *is a pair $\mathcal{C} = (\mathcal{O}, \mathcal{M})$ together with a lot of maps. Elements of $\mathcal{O}$ are called* objects *and elements of $\mathcal{M}$ are called* morphisms. *For every $A, B \in \mathcal{O}$ there is a set* $\mathrm{Mor}_{\mathcal{C}}(A, B) \subset \mathcal{M}$ *called the set of* morphisms from A to B. *We can think of* $\mathrm{Mor}_{\mathcal{C}}(A, B)$ *as the set of maps from A to B with a certain property. We also have maps describing the composition of these morphisms:* $\mathrm{Comp} = \mathrm{Comp}_{A,B,D} : \mathrm{Mor}_{\mathcal{C}}(A, B) \times \mathrm{Mor}_{\mathcal{C}}(B, D) \to \mathrm{Mor}_{\mathcal{C}}(A, D)$. *We can denote* $\mathrm{Comp}(\alpha, \beta)$ *by $\alpha \circ \beta$ or simply $\alpha\beta$. We require two axioms to be satisfied:*

(1) Associativity of composition (try to formalize this using the maps Comp*).*
(2) The existence of identity: For every $A \in \mathcal{O}$ there is an element $\mathrm{Id}_A \in \mathrm{Mor}(A, A)$ such that $\alpha \,\mathrm{Id}_A = \alpha$ and $\mathrm{Id}_A \beta = \beta$ for any morphisms for which the composition is defined.

In most examples Comp is indeed the composition of certain maps, so the axioms are automatically satisfied.

Exercise 10.3. Check that the following examples are categories (i.e. composition of morphisms are morphisms).

(1) Sets and maps.
(2) Metric spaces and isometries.
(3) Topological spaces and continuous maps.

(4) Topological spaces with base point and base point preserving continuous maps.
(5) Groups and homomorphisms.
(6) Vector spaces and linear maps.
(7) Ordered sets and order preserving maps.

Definition 10.4. *A* functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ *is a collection of maps: a map* $F_{\mathcal{O}} : \mathcal{O}_1 \to \mathcal{O}_2$ *and maps* $F_{A,B} : \mathrm{Mor}_{\mathcal{C}_1}(A,B) \to \mathrm{Mor}_{\mathcal{C}_2}(F(A),F(B))$ *such that* F *respects composition. (i.e.* $F(\alpha\beta) = F(\alpha)F(\beta)$*).*

So Proposition 10.1 shows that π_1 is a functor from the category of Topological spaces with base point and base point preserving continuous maps to the category of Groups and homomorphisms.

Exercise 10.5. Find other functors among the categories above (an important example is the forgetful functor, give a definition!).

Category theory helps to apply a construction in one category into an other category. For example you are encouraged to define the product of two objects in an arbitrary category in analogy with the product in topology (hint: use Theorem 4.11). What will be the product in the category of groups and homomorphisms? Warning: not every category has a product.

Proposition 10.6. *If* $f,\ g : (X,x_0) \to (Y,y_0)$ *are (based) homotopic, then* $f_* = g_*$.

Proof. Left to the reader. □

Theorem 10.7 (Brouwer)**.** *Every continuous map* $\phi : D^2 \to D^2$ *has a fixed point* $\phi(x) = x$.

Sketch of Proof. Suppose that there is no fixed point. Then for every $x \in D^2$ there is a unique line l_x through $\phi(x)$ and x. Let $p : D^2 \to S^1$ be defined by the following rule: $p(x)$ is the intersection point of the line l_x and the unit circle S^1 which is closer to x than to $\phi(x)$. p is continuous and $p|_{S^1} = \mathrm{Id}_{S^1}$ (Check these!). In other words $i \circ p = \mathrm{Id}_{S^1}$ where $i : S^1 \to D^2$ is the inclusion of the unit circle into the unit disc. Then by the functorial property of π_1:

$$i_* p_* = \mathrm{Id}_{\pi_1(S^1)}\,.$$

Assuming that $\pi_1(S^1) = \mathbb{Z}$ and knowing that $\pi_1(D^2) = 0$ we get a contradiction. □

Remark 10.8. Of course you can give a more down to earth argument if you dig into the definition of induced homomorphism. Let $f : I \to S^1$ is a loop where the base point of S^1 is $(1,0)$. Then $i \circ f$ is a loop in $(D^2,(1,0))$. So there is a homotopy H showing that $f \sim \underline{(1,0)}$. Since $p \circ i \circ f = f$ then $p \circ H$ is a homotopy connecting f and $\underline{(1,0)}$ *inside* S^1 which would imply that $\pi_1(S^1) = 0$.

Homework for Section 10.

(1) Prove Proposition 10.1.
(2) For all topological space X we define the path category $P(X)$. Objects of $P(X)$ are the points of X and morphisms between $x, y \in X$ are homotopy classes of paths from x to y. Define composition for $P(X)$ and show associativity. This category is also called the *fundamental grupoid* of X.
(3) For any group G we can define a category with only one object and morphisms are the elements of G. Define composition for this category and show associativity.

(4) Give 3 examples of functors.
(5) Show that $\pi_1(X,x_0)=0\ \forall x_0\in X \iff \forall f:S^1\to X$ extends to a continuous map $F:D^2\to X$ (i.e. $F|_{S^1}=f$).
(6) If $f:S^1\to S^1$ is nulhomotopic (i.e. homotopic to a constant map) then f has a fixed point. (Hint: use the Brouwer Fixed Point Theorem 10.7 and Homework (5).)

11. Basic properties of the Fundamental Group

Proposition 11.1. $\pi_1\big(X\times Y,(x_0,y_0)\big)=\pi_1(X,x_0)\times\pi_1(Y,y_0)$

Proof. Enough to show that $f\times g\sim \underline{(x_0,y_0)} \iff f\sim \underline{x_0}$ and $g\sim \underline{y_0}$, which is left to the reader. □

A fancy way to interpret this result is to say that the functor π_1 preserves product.

Now we turn to to a seemingly technical but important issue: the choice of the basepoint.

Proposition 11.2. *If X is path connected then $\pi_1(X,x_0)\cong\pi_1(X,x_1)$ for any $x_1\in X$.*

Proof. Let $p:I\to X$ be a path connecting x_0 and x_1. Then we define a map

$$\hat p:\pi_1(X,x_0)\to\pi_1(X,x_1)\qquad\text{by}\quad \hat p[f]:=[(\overline{p}\cdot f)\cdot p],$$

where $\overline{p}(t)=p(1-t)$. Though p is not a loop, we defined concatenation for homotopies in Definition 9.2. We have to check that this definition is correct i.e. if $f\sim g$ then $(\overline{p}\cdot f)\cdot p\sim(\overline{p}\cdot g)\cdot p$. But if H is a homotopy between f and g, then $(\underline{\overline{p}}\cdot H)\cdot\underline{p}$ is a homotopy between $(\overline{p}\cdot f)\cdot p$ and $(\overline{p}\cdot g)\cdot p$ (Here $\underline{p}$ denotes the constant homotopy $\underline{p}(t,s)=p(t)$). We leave it to the reader to check that $\hat{\overline{p}}$ is an inverse for $\hat p$, therefore $\hat p$ is an isomorphism between $\pi_1(X,x_0)$ and $\pi_1(X,x_1)$. □

So for a path connected space X we can use the notation $\pi_1(X)$ not mentioning the basepoint. However it is an important fact that the isomorphism $\hat p$ depends on the path p:

Proposition 11.3. *Let p, $q:I\to X$ be paths connecting x_0 and x_1. Then*

$$\hat q^{-1}\hat p(\alpha)=[q\cdot\overline{p}]\alpha[q\cdot\overline{p}]^{-1}.$$

Notice that $\hat q^{-1}\hat p$ is an automorphism of $\pi_1(X,x_0)$ and that $q\cdot\overline{p}$ is a loop based at x_0 so $[q\cdot\overline{p}]\in\pi_1(X,x_0)$.

Proof. Let f be a loop representing α. Then

$$\hat q^{-1}\hat p(\alpha)=\hat{\overline{q}}\hat p(\alpha)=\hat{\overline{q}}[(\overline{p}\cdot f)\cdot p]=[(q\cdot((\overline{p}\cdot f)\cdot p))\cdot\overline{q}],$$

but as we have seen in the proof of Theorem 9.8 the bracketing don't matter up to homotopy so:

$$\hat q^{-1}\hat p(\alpha)=[((q\cdot\overline{p})\cdot f)\cdot(p\cdot\overline{q})].$$

We can finish the proof by noticing that $\overline{q\cdot\overline{p}}=p\cdot\overline{q}$. □

Remark 11.4. Proposition 11.3 says that the deviation of the isomorphism $\hat q$ from the isomorphism $\hat p$ is measured by *conjugation* with $[p\cdot\overline{q}]\in\pi_1(X,x_0)$. It is easy to see that a group is commutative iff every conjugation is trivial, which implies that $\hat p$ is independent of the path p iff $\pi_1(X,x_0)$ is a commutative group. So it would be interesting to find a space with non commutative fundamental group:

Exercise 11.5. Try to guess what $\pi_1(\mathbb{R}^2 \setminus \{a,b\})$ is, and give examples of loops such that the corresponding elements in $\pi_1(\mathbb{R}^2 \setminus \{a,b\})$ don't commute.

Homework for Section 11.

(1) Show that $\hat{\bar{p}}$ is an inverse for $\hat{p}$. (One option is to use the path category of X. See Homework (2) in Section 10.)

12. Other constructions for Topological Spaces

Definition 12.1. *Let $f : X \to Y$ be a map and $\mathcal{U}_Y$ be a topology on Y. Then the* pull back *of $\mathcal{U}_Y$ is*

$$\mathcal{U}_{\overleftarrow{f}} = \{f^{-1}(U) \subset X : U \in \mathcal{U}_Y\}.$$

Remarks 12.2. (1) It is easy to see that $\mathcal{U}_{\overleftarrow{f}}$ is a topology on X.

(2) If f is injective we get back the definition of subspace topology. If f is not injective then $\mathcal{U}_{\overleftarrow{f}}$ is not even T_0.

(3) An alternative definition would be that $\mathcal{U}_{\overleftarrow{f}}$ is the smallest topology on X such that $f : X \to Y$ is continuous.

There is a natural dual of this construction.

Definition 12.3. *Let $f : X \to Y$ be a map and $\mathcal{U}_X$ be a topology on X. Then the* push forward *of $\mathcal{U}_X$ is*

$$\mathcal{U}_{\overrightarrow{f}} = \{U \subset Y : f^{-1}(U) \in \mathcal{U}_X\}.$$

In other words $\mathcal{U}_{\overrightarrow{f}}$ is the biggest topology on Y for which f is continuous. $(X, \mathcal{U}_{\overrightarrow{f}})$ restricted to the complement of $f(X)$ has the discrete topology (Show it!) so the interesting case is when f is surjective. In this case $\mathcal{U}_{\overrightarrow{f}}$ is called the *quotient* topology by the following reasons: If $f : X \to Y$ is a surjective map then we can define an equivalence relation on X by taking $x_1 \sim x_2$ if $f(x_1) = f(x_2)$. On the other hand if $\sim$ is an equivalence relation on X then we can take $Y := X/\sim$ (the set of equivalence classes) and we can define $f : X \to Y$ to be the surjective map sending $x \in X$ to its equivalence class $[x] \in Y$. Y is usually called the quotient of X by $\sim$.

Exercise 12.4. Describe the following quotient topologies:

(1) $X := \mathbb{R}^2$ and $(x_1, x_2) \sim (y_1, y_2)$ if $x_1 = y_1$.
(2) $X := \mathbb{R}$ and $x \sim y$ if there is an integer $n \in \mathbb{Z}$ such that $x = y + n$.
(3) $X := [0,1]$ and $0 \sim 1$. All other point are equivalent only to themselves.
(4) $X := \overline{B_1(0)} = D^2 \subset \mathbb{R}^2$ the closed unit disc. Let $(x_1, x_2) \sim (y_1, y_2)$ if $x_1^2 + x_2^2 = y_1^2 + y_2^2 = 1$. All other point are equivalent only to themselves.

At the last three cases you can use the following fact called the Quotient Lemma:

Lemma 12.5. *Suppose that $f : X \to Y$ is a continuous surjection, X is compact and Y is T_2. Then $\mathcal{U}_Y \cong \mathcal{U}_{\overrightarrow{f}}$.*

Proof. f is continuous so $\mathcal{U}_Y \supset \mathcal{U}_{\overrightarrow{f}}$, i.e. $\mathrm{Id}_Y : (Y, \mathcal{U}_{\overrightarrow{f}}) \to (Y, \mathcal{U}_Y)$ is continuous. $(Y, \mathcal{U}_{\overrightarrow{f}})$ is the continuous image of X therefore compact. So we can apply Homework 8 from Section 8 saying that a continuous bijection from a compact space to a T_2 space is a homeomorphism. $\square$

A special case of the quotient topology is the *glueing* of two spaces. The simplest is when we don't glue at all:

Definition 12.6. *The* disjoint union *of the topological spaces* $(X, \mathcal{U}_X)$ *and* $(Y, \mathcal{U}_Y)$ *is* $X \coprod Y$ *(the disjoint union of the two sets) with the topology generated by the basis* $\mathcal{B} = \mathcal{U}_X \cup \mathcal{U}_Y$.

Disjoint union has the following universal property:

Theorem 12.7. *Let X and Y be topological spaces. Then there exists a space T unique up to homeomorphism such that there are continuous maps $t_X : X \to T$ and $t_Y : Y \to T$ such that for any continuous maps $f_X : X \to Z$, $f_Y : Y \to Z$ there is a unique continuous map $h : T \to Z$ with the property: $f_X = h \circ t_X$ and $f_Y = h \circ t_Y$.*

The statement of the Theorem can be summarized with the commutative diagram:

$$\begin{tikzcd} X \arrow[r, "t_X"] \arrow[dr, "f_X"'] & T \arrow[d, dashed, "h"'] & Y \arrow[l, "t_Y"'] \arrow[dl, "f_Y"] \\ & Z & \end{tikzcd}$$

Notice that we simply reversed the arrows in the commutative diagram of the product (Theorem 4.11). In other words this is a "dual" construction. In an arbitrary category this construction is called *sum*. We leave the proof as an exercise that disjoint union is the sum in the category of topological spaces and continuous maps.

Definition 12.8. *Let $(X, \mathcal{U}_X)$ and $(Y, \mathcal{U}_Y)$ be topological spaces, $g_X : A \to X$ and $g_Y : A \to Y$ continuous. Then we can glue X and Y together along g_X and g_Y:*

$$X \cup_{g_X, g_Y} Y := X \coprod Y / \sim$$

where $\sim$ is the equivalence relation generated by $g_X(a) \sim g_Y(a)$ for all $a \in A$. We equip this set with the quotient topology. The maps g_X and g_Y are called the glueing maps.

In most applications $A \subset X$ and g_X is the inclusion map of A into X. Then we use the notation $X \cup_{g_Y} Y$. Of course you can glue a space to itself: If $A \subset X$ and $g : A \to X$ continuous then you can take the quotient topology on $X_g = X/\sim$ where $a \sim g(a)$ for all $a \in A$ and all other point are equivalent only to themselves. Notice that the last two examples in Exercise 12.4 can be obtained via glueing.

Remark 12.9. You can define glueing with a universal property corresponding to the commutative diagram:

$$\begin{tikzcd} & A \arrow[dl, "g_X"'] \arrow[dr, "g_Y"] & \\ X \arrow[r, "t_X"] \arrow[dr, "f_X"'] & T \arrow[d, dashed, "h"'] & Y \arrow[l, "t_Y"'] \arrow[dl, "f_Y"] \\ & Z & \end{tikzcd}$$

so glueing (or push out) can be defined in any category (it may not exist of course). In the category of topological spaces however $T = X \cup_{g_X, g_Y} Y$ is the push out where the maps $t_X : X \to T$ and $t_Y : Y \to T$ are the natural maps.

Exercise 12.10. We noticed that the edge E of the Moebius band Mb is homeomorphic to the unit circle S^1, i.e. we have a homeomorphism $h : E \to S^1$. How does $Mb \cup_f D^2$ look like?

Homework for Section 12.

(1) Solve Exercise 12.4.
(2) Give a glueing function f such that $I^2_f \cong Mb$ the Moebius band.
(3) Show that the quotient topology on $X/\sim$ is T_1 iff the equivalence classes of $\sim$ are closed in X.
(4) Show that in any category the sum is unique.
(5) Show that disjoint union is the sum in the category of topological spaces and continuous maps.
(6) Show that $U \subset X \cup_{g_X, g_Y} Y$ is open iff $t_X^{-1}(U)$ and $t_Y^{-1}(U)$ are open.
(7) Show that if $A \cup B = X$ and $i_A : A \cap B \to A$ and $i_B : A \cap B \to B$ are the corresponding inclusions then $X \cong A \cup_{i_A, i_B} B$ if both A and B are open in X or both A and B are closed in X but not in general. Hint: for A and B closed this is the pasting lemma 9.3.

13. Manifolds

A particularly interesting class of topological spaces is the class of manifolds. They look like a Euclidean space locally. So they have only "global" topological properties.

Definition 13.1. *The topological space M is an n-dimensional manifold if $\forall m \in M \quad \exists U \ni m$ open such that $U \cong \mathbb{R}^n$ and $M \in T_2,\ M_2$.*

Remarks 13.2.

(1) The assumption $M \in T_2$ is important if we want to avoid strange spaces: Let $X = (0, 2)$, $Y = (4, 6)$, $A = (0, 2) \setminus \{1\}$ and $f : A \to Y$, $f(a) = a + 4$. The space $Z = X \cup_f Y$ is locally homeomorphic to $\mathbb{R}$ but not Hausdorff. Z is called the interval with a double point (why?).
(2) The assumption $M \in M_2$ is less important for us. We are mainly interested in compact manifolds and for compact spaces the two other conditions imply M_2.

Theorem 13.3. *The dimension of a manifold is a topological invariant.*

This is a consequence of the deep fact that $\mathbb{R}^m \cong \mathbb{R}^n$ implies $m = n$. We will prove a special case later.

Examples 13.4.

(1) 1-dimensional manifolds are called *curves*. The only connected compact curve is S^1. The only connected non compact curve is $\mathbb{R}$. The proof is not difficult but technical.
(2) 2-dimensional manifolds are called *surfaces*. Examples of surfaces are $\mathbb{R}^2$ and the annulus $(0, 1) \times S^1$. Examples of compact surfaces are the sphere S^2 and the torus (or donut) $T^2 = S^1 \times S^1$. Later we give a complete list of connected compact surfaces.
(3) In higher dimensions no complete list is known. This is a very active area of modern mathematics. Examples of higher dimensional manifolds: $\mathbb{R}^n$, the n-dimensional sphere S^n, the n-dimensional torus T^n. Taking products you can get more examples.

Exercise 13.5. Show that the product of two manifolds is a manifold.

Homework for Section 13.

(1) Show that S^n is an n-dimensional manifold.
(2) Prove Exercise 13.5 that the product of two manifolds is a manifold.

14. Homotopy Equivalences

Definition 14.1. *Two spaces X and Y are* homotopy equivalent *and denoted by $X \simeq Y$ if there are continuous maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \sim \mathrm{Id}_Y$ and $g \circ f \sim \mathrm{Id}_X$.*

Homotopy equivalent spaces can look very different however their homotopic properties are the same.

Exercise 14.2. Find explicit maps and homotopies showing that $* \simeq \mathbb{R}^n$, $S^n \simeq \mathbb{R}^{n+1} \setminus \{0\}$ and $8 \simeq \Theta$,

Definition 14.3. *If a space is homotopy equivalent to the one point space $*$ then it is called* contractible.

Theorem 14.4. *Let X and Y be path connected and $X \simeq Y$. Then $\pi_1(X) \cong \pi_1(Y)$.*

The proof is not difficult but tricky because the homotopies involved don't have to preserve a base point. We prove a weaker statement instead.

Proposition 14.5. *Suppose that (X, x_0) and (Y, y_0) are "based" homotopy equivalent: there are base point preserving maps $f : (X, x_0) \to (Y, y_0)$, $g : (Y, y_0) \to (X, x_0)$ and base point preserving homotopies H_X, H_Y connecting $f \circ g$ to Id_Y and $g \circ f$ to Id_X. Then $\pi_1(X) \cong \pi_1(Y)$.*

Proof. By Proposition 10.6 $g_* \circ f_* = (g \circ f)_* = (\mathrm{Id}_X)_* = \mathrm{Id}_{\pi_1(Y)}$. □

There is a certain type of homotopy equivalence which comes up quite frequently.

Definition 14.6. *Let $A \subset X$. Then a continuous map $r : X \to A$ is called* retraction *if $r(a) = a$ for all $a \in A$. In this case we say that A is a* retract *of X. If in addition $i_A \circ r$ is homotopic to Id_X (where i_A is the inclusion of A into X) then r is called* deformation retraction*. In this case we say that A is a* deformation retract *of X.*

Homework for Section 14.

(1) Prove Exercise 14.2.
(2) * Prove Theorem 14.4.
(3) Give an example of a retraction which is not a deformation retraction.
(4) Show that if A is a deformation retract of X then $A \simeq X$.
(5) Show that if X is path connected and $X \simeq Y$ then Y is path connected, too.

15. Simply Connected Spaces

Definition 15.1. *X is* simply connected *if path connected and $\pi_1(X) = 0$.*

Definition 15.2. *$D \subset \mathbb{R}^n$ is* convex *if for any $x, y \in D$ the line segment $\overline{xy}$ connecting x and y is in D.*

Proposition 15.3. *If $D \subset \mathbb{R}^n$ is convex then simply connected.*

Proof. The trick of the trade is the parametrization of $\overline{xy}$: Let $f(t) = (1-t)x + ty$. The path $f(t)$ shows that D is path connected. The point $(1-t)x + ty$ for $t \in [0,1]$is called a *convex combination of* x and y. Now let $l : I \to D$ is a loop in (D, x_0) (where x_0 is a base point for D). Then $H(s,t) = (1-s)l(t) + sx_0$ is a nulhomotopy. □

Proposition 14.4 implies that a contractible space has trivial fundamental group. So the following exercises give an alternative proof for Proposition 15.3:

Exercise 15.4. Show that a contractible space is path connected. In fact if $X \simeq Y$ and X is path connected then Y is path connected.

Exercise 15.5. Show that if $D \subset \mathbb{R}^n$ is convex then contractible.

Theorem 15.6. *The spheres S^n for $n > 1$ are simply connected.*

Proof. The proof is based on the following

Exercise 15.7. $S^n \setminus \{N\} \cong \mathbb{R}^n$ where $N = (0, \ldots, 0, 1)$ is the North pole.

So we will show that any loop f in S^n can be homotoped to a loop which avoids a point of S^n. Then f is nulhomotopic and we are done. It is difficult even to imagine a loop covering the whole sphere however such monsters exist:

Theorem 15.8 (Peano)**.** *There is a continuous surjective map $p : [0,1] \to \square$.*

Using such a p—called *Peano curve*—you can construct curves covering the cube, or S^n for any n (try!). Our major tool is the following

Lemma 15.9 (Lebesgue lemma for $[0,1]$)**.** *Let $l : I \to X$ be a continuous map and let $\mathcal{C} = \{U_\alpha : \alpha \in A\}$ be an open cover of X. Then there are $t_0 = 0, t_1, \ldots, t_n = 1 \in [0,1]$ such that for every $i \leq n$ there is an $\alpha \in A$ with $l([t_i, t_{i+1}]) \subset U_\alpha$.*

Proof. Let $\mathcal{C}' = \{(a,b) \subset [0,1] : \exists \alpha \in A$ such that $l(a,b) \subset U_\alpha\}$. $\mathcal{C}'$ is an open cover of $[0,1]$. The compactness of the unit interval implies that $\mathcal{C}'$ has a finite subcover

$$\{[0, b_0), (a_1, b_1), \ldots, (a_n, 1]\}.$$

We can assume that $a_1 < b_0 < a_2 < b_1 < \cdots$ so we can pick the t_i from the "overlap" (a_{i+1}, b_i). □

Continuation of the proof of Theorem 15.6: Let $U_S = S^n \setminus \{N\}$ and $U_N = S^n \setminus \{S\}$. Then $\mathcal{C} = \{U_S, U_N\}$ is an open cover of S^n. So if $l : [0,1] \to S^n$ is a loop then by the Lebesgue lemma for $[0,1]$ we have $t_0 = 0, t_1, \ldots, t_n = 1 \in [0,1]$ such that $l([t_i, t_{i+1}]) \subset U_N$ or $l([t_i, t_{i+1}]) \subset U_S$. The paths $l|_{[t_i, t_{i+1}]}$ are homotopic inside U_N or U_S (keeping endpoints fixed) to a path c_i not containing the North pole N. With the usual pasting of these homotopies we can see that l is homotopic to the concatenation $c = c_0 \cdot c_1 \cdots c_n$. So l is nulhomotopic. □

Homework for Section 15.

(1) Prove Exercise 15.4.
(2) Prove Exercise 15.5.

(3) Prove Exercise 15.7. Hint: use stereographic projection $\pi : S^n \setminus \{N\} \to \mathbb{R}^n$ where $\pi(s)$ is the point where the line connecting $N = (0, \dots, 0, 1) \in \mathbb{R}^{n+1}$ and $s = (s_0, \dots, s_n) \in S^n \subset \mathbb{R}^{n+1}$ intersects the horizontal hyperplane $\{x \in \mathbb{R}^{n+1} : x_n = 0\} \cong \mathbb{R}^n$. I. e.

$$\pi(s_0, \dots, s_n) = \frac{1}{1 - s_n}(s_0, \dots, s_{n-1}).$$

Find an inverse for π.

(4) Prove the Lebesgue number lemma: If (X, d) is a compact metric space and $\mathcal{C} \subset \mathcal{U}_d$ is an open cover of X then there is a $\delta > 0$—called the Lebesgue number of the cover $\mathcal{C}$—such that if $A \subset X$ has diameter less then δ then there is a $U \in \mathcal{C}$ with $A \subset U$.

(5) Use the Lebesgue number lemma to give an alternative proof of the Lebesgue lemma for $[0, 1]$.

16. The Fundamental Group of the Circle

Theorem 16.1. *$\pi_1(S^1) \cong \mathbb{Z}$.*

This is a consequence of a general theorem about covering maps we are going to prove in the next section. As a warm up I give a sketch of the proof of $\pi_1(S^1) \cong \mathbb{Z}$:

As we saw $S^1 \cong \mathbb{R}/\sim$ where $a \sim b$ if $a = b + k$ for $k \in \mathbb{Z}$. In other words S^1 can be identified with the modulo 1 real numbers. (From this point of view a natural base point for S^1 is 0=1 ;-) So if we have a map $f : X \to S^1$ we can construct a "lift" $\tilde{f} : X \to \mathbb{R}$ such that $p \circ \tilde{f} = f$ where p is the quotient map. Such a lift is not unique and sometimes it is impossible to find a continuous lift even if f was continuous. However we have the following:

Lemma 16.2 (path lifting). *Let $f : [0, 1] \to S^1$ be a continuous path such that $f(0) = 0$. Then for any $k \in \mathbb{Z}$ there is a unique lift $\tilde{f} : [0, 1] \to \mathbb{R}$ such that $\tilde{f}(0) = k$.*

We will prove this in the next section. An intuitive argument would be the following: Think $\mathbb{R}$ as an infinite helix above S^1 and think f as the driving instructions for our car standing at 0. We don't see whether our car is on S^1 or on the helix (locally they look the same). So the directions work in both case giving f and $\tilde{f}$ respectively.

Lemma 16.2 allows us to assign an integer to any loop $l : [0, 1] \to S^1$ measuring the number of times l "goes around" S^1:

Definition 16.3. *$\Phi(l) := \tilde{l}(1) \in \mathbb{Z}$ where $\tilde{l} : [0, 1] \to \mathbb{R}$ is the unique lift of l such that $\tilde{l}(0) = 0$.*

Proposition 16.4. *Φ induces a map $\phi : \pi_1(S^1, 0) \to \mathbb{Z}$.*

Proof. We have to show that if $l_0 \sim l_1$ then $\tilde{l}_0(1) = \tilde{l}_1(1)$. For this we need the following

Lemma 16.5 (homotopy lifting). *If $H : \square \to S^1$ is a based homotopy (i.e. $H(s, 0) = H(s, 1) = 0$ for all $s \in [0, 1]$) then there is a unique lift $\tilde{H} : \square \to \mathbb{R}$ such that $\tilde{H}(0, 0) = 0$.*

Notice that the unicity follows from the path lifting lemma applied to the paths H_s. We will prove the continuity part in the next section.

So if H is a homotopy connecting l_0 and l_1, then $\tilde{H}$ connects $\tilde{l}_0$ and $\tilde{l}_1$. In particular $\tilde{H}(s, 1)$ is a path connecting $\tilde{l}_0(1)$ and $\tilde{l}_1(1)$. Since $\tilde{H}$ is a lift $\tilde{H}(s, 1)$ has to be entirely in $p^{-1}(0) = \mathbb{Z} \subset \mathbb{R}$ so it is the constant path: $\tilde{l}_0(1) = \tilde{l}_1(1)$. $\square$

Exercise 16.6. Show that ϕ is a homomorphism.

Proposition 16.7. *$\phi : \pi_1(S^1, 0) \to \mathbb{Z}$ is an isomorphism.*

Proof.

(1) ϕ is surjective: Let $l_k(t) = [kt] \in S^1$ for $k \in \mathbb{Z}$ then $\tilde{l}_k(t) = kt$ so $\phi[l_k] = \tilde{l}_k(1) = k$.
(2) ϕ is injective: Since ϕ is a homomorphism it is enough to show that if $\phi[l] = 0$ then $l \sim \underline{0}$. But $\phi[l] = 0$ means that $\tilde{l}(1) = 0$. Let $\tilde{H} : \square \to \mathbb{R}$ be defined by $\tilde{H}(s,t) = s\tilde{l}(t)$. $\tilde{H}$ is nulhomotopy for $\tilde{l}$ so $H := p \circ \tilde{H}$ is a nulhomotopy for l.

$\square$

17. Covering Spaces and Maps

We used some special properties of the quotient map $p : \mathbb{R} \to S^1$. The proper generalization is the notion of *covering map*. We follow Munkres: Topology (p. 336) for a while:

Definition 17.1. *Let $p : \tilde{X} \to X$ be a continuous surjective map. An open set $U \in X$ is* evenly covered *by p if $p^{-1}(U)$ is the disjoint union of open subsets V_α of $\tilde{X}$ such that for each α the restriction $p|_{V_\alpha} : V_\alpha \to U$ is a homeomorphism.*

Definition 17.2. *Let $p : \tilde{X} \to X$ be a continuous surjective map. If X is covered by open subsets evenly covered by P then p is called a* covering map *and $\tilde{X}$ is said to be a* covering space *of X.*

Exercise 17.3.

(1) Show that the quotient map $p : \mathbb{R} \to S^1$ is a covering map.
(2) Show that if $p : \tilde{X} \to X$ is a covering map then for all $x \in X$ the preimage $p^{-1}(x)$ has the discrete topology.
(3) Show that if $p : \tilde{X} \to X$ is a covering map and X is connected then $p^{-1}(x)$ has the same number of points for all $x \in X$. This number is called the number of *sheets* of p.
(4) Find a 3-sheeted covering of S^1. OK, find an other one ;-)
(5) Find a covering map $p : \mathbb{R}^2 \to S^1 \times S^1$.

Definition 17.4. *Let $p : \tilde{X} \to X$ be a map. If $f : Z \to X$ is a continuous map then a lift of f is a continuous map $\tilde{f} : Z \to \tilde{X}$ such that $p \circ \tilde{f} = f$.*

Lemma 17.5 (path lifting)**.** *Let $p : \tilde{X} \to X$ be a covering map such that $p(\tilde{x}_0) = x_0$. Then any path $f : [0,1] \to X$ beginning at x_0 has a unique lift $\tilde{f} : [0,1] \to \tilde{X}$ beginning at $\tilde{x}_0$.*

Proof. Existence of a lift: Apply the Lebesgue lemma for $[0,1]$ to the open cover of X provided by the definition of covering space. Let $t_0 = 0, t_1, \ldots, t_n = 1 \in [0,1]$ such that $f([t_i, t_{i+1}]) \subset U_i$ for an evenly covered open set $U_i \subset X$. Let $\tilde{f}(0) = \tilde{x}_0$. Then suppose that $\tilde{f}$ is defined for $0 \le t \le t_i$. By assumption $p^{-1}(U_i)$ is the disjoint union of open sets V_α such that for each α the restriction $p|_{V_\alpha} : V_\alpha \to U$ is a homeomorphism. $\tilde{f}(t_i)$ lies in one of these, say in V_0. Define $\tilde{f}(t) := p|_{V_0}^{-1}(f(t))$ for $t \in [t_i, t_{i+1}]$. Continuing this way we define $\tilde{f}$ on the whole interval. By the pasting lemma $\tilde{f}$ is continuous.

Unicity: Suppose we have an other lift $\tilde{g}$ of f. Then let t_i be the largest of the division points such that $\tilde{g}(t) = \tilde{f}(t)$ for $t \in [0, t_i]$. But then $\tilde{g}$ and $\tilde{f}$ has to agree on $[t_i, t_{i+1}]$ as well since the image should stay in the same connected component of $p^{-1}(U_i)$. $\square$

Lemma 17.6 (homotopy lifting). *If $H : \square \to X$ is a based homotopy (i.e. $H(s,0) = H(s,1) = x_0$ for all $s \in [0,1]$) then there is a unique lift $\tilde{H} : \square \to \tilde{X}$ such that $\tilde{H}(0,0) = \tilde{x}_0$.*

Notice that the unicity follows from the path lifting lemma applied to the paths H_s. We will prove the continuity part using a version of the Lebesgue lemma for $[0,1]$:

Lemma 17.7 (Lebesgue lemma for $\square$). *Let $H : I \times I \to X$ be a continuous map and let $\mathcal{C} = \{U_\alpha : \alpha \in A\}$ be an open cover of X. Then there are $s_0 = 0, s_1, \dots, s_m = 1 \in [0,1]$ and $t_0 = 0, t_1, \dots, t_n = 1 \in [0,1]$ such that for every $i \leq m$, $j \leq n$ there is an $\alpha \in A$ with $H([s_i, s_{i+1}] \times [t_j, t_{j+1}]) \subset U_\alpha$.*

Proof. Let

$$\mathcal{C}' = \{(a_1,b_1) \times (a_2,b_2) \subset [0,1] : \exists \alpha \in A \text{ such that } H\big((a_1,b_1) \times (a_2,b_2)\big) \subset U_\alpha\}.$$

$\mathcal{C}'$ is an open cover of $[0,1]$. The compactness of the unit square implies that $\mathcal{C}'$ has a finite subcover. Slightly decrease the size of the rectangles occuring in the finite subcover. Let $s_0 = 0, s_1, \dots, s_m = 1 \in [0,1]$ be the possible s-coordinates for the corners of these rectangles and $t_0 = 0, t_1, \dots, t_n = 1 \in [0,1]$ be the possible t-coordinates for the corners of these rectangles. $\square$

Exercise 17.8. Explain what "slightly" means exactly in the previous proof.

Alternatively you can use the Lebesgue number Lemma (Homework (4) in Section 15) to prove the Lebesgue lemma for $\square$.

Proof of Lemma 17.6. Apply the Lebesgue lemma for $\square$ to the open cover of X provided by the definition of covering space. Let R_i the i^{th} rectangle, where we ordered the rectangles column by column. We can define the lift $\tilde{H}$ by induction if we notice that $L_{n+1} = R_{n+1} \cap \cup_{i=1}^n R_i$ is connected (one or two sides of R_{n+1}) so if $U \supset H(R_{n+1})$ is evenly covered i.e. $p^{-1}(U) = \coprod V_\alpha$ then $\tilde{H}(L_{n+1}) \subset V_\alpha$ for some α so we can define $\tilde{H}|_{R_{n+1}}$ by $(p|_{V_\alpha})^{-1} \circ H|_{R_{n+1}}$. Continuity follows from the pasting lemma. $\square$

Now we actually finished the proof of that $\pi_1(S^1) \cong \mathbb{Z}$, but we can get a more general statement. For this we generalize the map ϕ of the previous section:

Proposition 17.9. *Let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering map and $l : [0,1] \to X$ be a loop. Let $\Phi(l) = \tilde{l}(1) \in p^{-1}(x_0)$ where $\tilde{l} : [0,1] \to \tilde{X}$ is the unique lift of l such that $\tilde{l}(0) = \tilde{x}_0$. Then Φ induces a map $\phi : \pi_1(X, x_0) \to p^{-1}(x_0)$ called the* lifting correspondence.

Proof is the same as of Proposition 16.4.

Theorem 17.10. *Let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering map and $\phi : \pi_1(X, x_0) \to p^{-1}(x_0)$ be the lifting correspondence. If $\tilde{X}$ is path connected then ϕ is surjective. If $\tilde{X}$ is simply connected then ϕ is bijective.*

Proof is the same as of Proposition 16.7.

18. Compact surfaces

Definition 18.1. *A surface is* non-oriented *if it has a subspace homeomorphic to a Moebius band and* oriented *otherwise.*

The projective plane is the first example of a compact non oriented surface. Difficult to imagine since it cannot be embedded into $\mathbb{R}^3$.

Definition 18.2.

$$\mathbb{RP}^2 := \{\textit{lines in } \mathbb{R}^3 \textit{ through origin}\}.$$

$\mathbb{RP}^2$ is a metric space called projective plane *with the metric $d(l_1, l_2) = \sphericalangle(l_1, l_2)$ where $\sphericalangle(l_1, l_2)$ is measured between 0 and $\pi/2$.*

Exercise 18.3. Show that $(\mathbb{RP}^2, \mathcal{U}_\sphericalangle)$ is homeomorphic to

(1) $S^2/\sim$ where $a \sim -a$ (think $S^2 \subset \mathbb{R}^3$ the unit sphere).
(2) $D^2/\sim$ where $a \sim -a$ if $|A| = 1$.
(3) $Mb \cup_\phi D^2$ where $\phi : \partial Mb \to S^1 \subset D^2$ is a homeomorphism.
(4) $S^1 \cup_\psi D^2$ where $\psi : S^1 \to S^1 \subset D^2$, $\psi(t) = [2t]$.

Proposition 18.4. $\pi_1(\mathbb{RP}^2) \cong \mathbb{Z}_2$.

Proof. The quotient map $p : S^2 \to \mathbb{RP}^2$ is a 2-sheeted covering map, S^2 is simply connected so Theorem 17.10 implies that $\pi_1(\mathbb{RP}^2)$ has two elements. The only group with two elements is $\mathbb{Z}_2$. □

All compact connected surfaces can be obtained from the sphere S^2, the torus T^2 and the projective plane $\mathbb{RP}^2$ using *connected sum*:

Definition 18.5. *If M and N are n-dimensional manifolds then their connected sum*

$$M\#N = M \setminus B^n \cup_\phi N \setminus B^n,$$

where B^n is a small open disc and ϕ is a homeomorphism between the boundaries of the two discs (they are both homeomorphic to S^{n-1}).

Theorem 18.6.

(1) A compact connected oriented surface is homeomorphic to the sphere S^2 or to $\Sigma_g = \overbrace{T^2\#\cdots\#T^2}^{g}$ and these are all non-homeomorphic surfaces.
(2) A compact connected non-oriented surface is homeomorphic to $K_g = \overbrace{\mathbb{RP}^2\#\cdots\#\mathbb{RP}^2}^{g}$ and these are all non-homeomorphic surfaces.

Remark 18.7. We will show later that these surfaces all have different fundamental groups, so they are not even homotopy equivalent. We will not prove however that this is a complete list.

Example 18.8. $K_2 = \mathbb{RP}^2\#\mathbb{RP}^2$ is called the *Klein bottle*. In other words $K_2 = Mb \cup_\phi Mb$ where ϕ glues the two Moebius bands together along their boundary.

Homework for Section 18.

(1) Prove Exercise 18.3.
(2) What is $\square/\sim$ if
   (a) $(s, 0) \sim (s, 1)$ and $(0, t) \sim (1, t)$,
   (b) $(s, 0) \sim (s, 1)$ and $(0, t) \sim (1, 1-t)$,

(c) $(s,0) \sim (1-s,1)$ and $(0,t) \sim (1,1-t)$.
(3) What is $S\#S^2$ if S is any surface?
(4) Where is $\mathbb{RP}^2\#T^2$ on the list?

19. Group Actions

We say that a group acts on the set X if for any $g \in G$ we have a map $\rho(g) : X \to X$ such that $\rho(e) = \mathrm{Id}_X$ and $\rho(g_1 g_2) = \rho(g_1)\rho(g_2)$. Usually the action ρ is not mentioned and we think g as a self map of X. If there is a topology on X then we also require the maps $\rho(g)$ to be continuous (therefore homeomorphism). So we can say that ρ is a homomorphism from the group G to $\mathrm{Homeo}(X)$ the group of self homeomorphisms of X.

If G also admits a topology (for example S^1 the group of rotations) then you want ρ to be continuous. Since we haven't define topology on $\mathrm{Homeo}(X)$ we can use the following trick, similarly to the definition of homotopy:

Definition 19.1. *Let G be a group with a topological structure and X be a topological space. An* action *of G on X is a continuous map $\mu : G \times X \to X$ such that $\mu(e,x) = x$ and $\mu(g_1, \mu(g_2, x)) = \mu(g_1 g_2, x)$.*

We are mainly concerned about the case when the topology on G is discrete. In this case the two definitions are equivalent.

Definition 19.2. *Suppose that G acts on X. Then we can define an equivalence relation on X by $x \sim y$ if $\exists g \in G$ such that $y = g(x)$. (Check that this is indeed an equivalence relation.) The equivalence classes of $\sim$ are called* orbits *and $X/\sim$ is called the* orbit space *and denoted by X/G. If X was a topological space then we have the quotient topology on X/G .*

Exercise 19.3.
(1) Find a $\mathbb{Z}$-action and a $\mathbb{Z}_2$-action on $\mathbb{R}$.
(2) Find a $\mathbb{Z}_n$-action on S^1 and on S^2.

We look at closely to two examples:
(1) $\mathbb{R}/\mathbb{Z} \cong S^1$, the quotient map is exactly the covering map we studied to calculate $\pi_1(S^1) \cong \mathbb{Z}$.
(2) $S^2/\mathbb{Z}_2 \cong \mathbb{RP}^2$, the quotient map is exactly the covering map we studied to calculate $\pi_1(\mathbb{RP}^2) \cong \mathbb{Z}_2$.

You may wonder whether this is a coincidence that the group acting is the same as the the fundamental group. Of course not, but we need that the quotient map has to be a covering map as well.

Definition 19.4. *The action of G on X is called a* covering action *(also called properly discontinuous) if for all $x \in X$ there is an open set $U \ni x$ such that $g(U) \cap U = \emptyset$ if $g \neq e$.*

Proposition 19.5. *The quotient map $q : X \to X/G$ is a covering map for any covering action.*

Proof. Let $y \in X/G$ and $x \in X$ such that $q(x) = y$. Then take $V = q(U)$ for the $U \ni x$ in the definition of covering action. We claim that V is evenly covered. It follows from the equality

$$q^{-1}(V) = q^{-1}(q(U)) = \coprod_{g \in G} g(U)$$

and the definition of quotient topology since all the sets $g(U)$ are open (g^{-1} is continuous!). □

Remark 19.6. It is easy to see that a covering action is always a free action, i.e. $g(x) = x$ implies $g = e$. So a covering action defines a $|G|$-sheeted covering. Applying Theorem 17.10 we get that if X is simply connected then there is a bijection between the groups $\pi_1(X/G)$ and G.

Theorem 19.7. *Suppose that the action of the group G is a covering action on the simply connected space X. Then $\pi_1(X/G) \cong G$.*

Proof. Pick base points $x_0 \in X$ and $y_0 = q(x_0) \in X/G$. Since the action is free we have a bijection $m : q^{-1}(y_0) \to G$ such that $m(x) = g$ if $x = g(x_0)$. We claim that $h = m \circ \phi : \pi_1(X/G, y_0) \to G$ is a homomorphism therefore an isomorphism where $\phi : \pi_1(X/G, y_0) \to q^{-1}(y_0)$ is the lifting correspondence. To see that it is enough to show that if $a, b : [0,1] \to X/G$ are loops with lifts $\tilde{a}, \tilde{b} : [0,1] \to X$, then $\tilde{a} \cdot h[a](\tilde{b})$ is a lift for $\tilde{a} \cdot \tilde{b}$. Here $h[a](\tilde{b})$ denotes the path $t \mapsto h[a](\tilde{b}(t))$. By the definition of h we have that $h[a](x_0) = \tilde{a}(1)$ which shows that the concatenation $\tilde{a} \cdot h[a](\tilde{b})$ is continuous. It is a routine to check that $\tilde{a} \cdot h[a](\tilde{b})$ is a lift of $a \cdot b$, so $\phi([a][b]) = (\tilde{a} \cdot h[a](\tilde{b}))(1) = h[a](\tilde{b}(1)) = h[a](\phi([b])) = h[a](h[b](x_0))$ which means that h is a homomorphism. □

Every reasonable path connected space X has a simply connected cover $\tilde{X}$ called the *universal cover*. Moreover there is a covering action of $\pi_1(X)$ on $\tilde{X}$. This gives a way to determine the fundamental group of any reasonable space. However the construction of the universal cover is not too constructive so we need an other method to calculate fundamental groups.

Homework for Section 19

(1) Show that the formula $(k,l)(x,y) = (k+x, l+y)$ defines a covering action of $\mathbb{Z} \times \mathbb{Z}$ on $\mathbb{R}^2$. Use this result to calculate the fundamental group of something.
(2) Let $X = D^2 \coprod D^2 \coprod D^2 / \sim$ where all three boundary circles are identified. You can call X the 3-layered disc or the pita bread with two pockets. Define a covering action of $\mathbb{Z}_3$ on X. (Later we will see that X is simply connected, so $X/\mathbb{Z}_3$ has fundamental group $\mathbb{Z}_3$.)
(3) Show that the universal cover is unique: if $\tilde{X}_1$ and $\tilde{X}_2$ are simply connected coverings of X then $\tilde{X}_1 \cong \tilde{X}_2$.
(4) What is the universal cover of the figure eight?

20. Seifert-van Kampen Theorem

Goal: Calculate $\pi_1(A \cup B)$ where A and B are open in $A \cup B$ if we know $\pi_1(A)$, $\pi_1(B)$ and $\pi_1(A \cap B)$. This way you can calculate the fundamental group of any space "built up" from simpler pieces.

Idea: By the Lebesgue lemma for $[0,1]$ any loop γ in $A \cup B$ is homotopy equivalent to a "factorized" loop $\gamma_1 \cdot \gamma_2 \cdots \gamma_n$, where $\gamma_i : [0,1] \to A$ or $\gamma_i : [0,1] \to B$. So $\pi_1(A \cup B)$ is going to be a group generated by elements of $\pi_1(A)$ and $\pi_1(B)$ subject to the condition that loops in $A \cap B$ are considered to be the same no matter if we think them as elements of $\pi_1(A)$ or $\pi_1(B)$.

The algebraic construction: By Homework 7 in Section 12 we can consider $A \cup B$ as A and B glued together along $A \cap B$. So we can expect that $\pi_1(A \cup B)$ is going to be the glueing of $\pi_1(A)$ and $\pi_1(B)$ in the category of groups and homomorphisms called *amalgamation* or amalgamated product. The Seifert-van Kampen theorem says that under some conditions this is indeed the case:

Definition 20.1. *Suppose we have groups G, H, S and homomorphisms $g : S \to G$, $h : S \to H$. Then we denote by $G *_S H$ the amalgamation i.e. the unique group satisfying the universal property corresponding to the commutative diagram:*

$$\begin{tikzcd}
 & S \arrow[dl, "g"'] \arrow[dr, "h"] & \\
G \arrow[r, "t_g"] \arrow[dr, "f_g"'] & G *_S H \arrow[d, dashed, "\alpha"'] & H \arrow[l, "t_h"'] \arrow[dl, "f_h"] \\
 & Z &
\end{tikzcd}$$

It would be more precise to have the maps g and h in the notation but this is the traditional notation.

Theorem 20.2 (Seifert-van Kampen)**.** *Let $X = A \cup B$ s.t. A, B are open in X and A, B, $A \cap B$ are all path connected. Let $x \in A \cap B$ and suppose that x is the base point for all fundamental groups appearing. Then*

$$\pi_1(A \cup B) \cong \pi_1(A) *_{\pi_1(A\cap B)} \pi_1(B),$$

where we use the maps $a_* : \pi_1(A \cap B) \to \pi_1(A)$ and $b_* : \pi_1(A \cap B) \to \pi_1(B)$ induced by the inclusions $a : A \cap B \to A$ and $b : A \cap B \to B$.

We defined amalgamation using the universal property of glueing. It is not easy to see how this group looks like. Let me list some group theoretic facts—mainly without proof—to understand the structure of amalgamation better. The main tool will be presenting a group with generators and relations.

Generators and relations:

Examples 20.3.

(1) $\mathbb{Z}_2 = \langle g : g^2 = 1\rangle$,
(2) $\mathbb{Z} \times \mathbb{Z} = \langle g, h : ghg^{-1}h^{-1} = 1\rangle$,
(3) $D_3 = \langle g, h : g^3 = 1, h^2 = 1, hgh^{-1}g = 1\rangle$, where D_3 is the symmetry group of the equilateral triangle.

A formal definition would be:

Definition 20.4. *Let $g_1, \dots, g_n$ be different letters and $r_1, \dots, r_k$ are words using the letters $g_1, \dots, g_n, g_1^{-1}, \dots, g_n^{-1}$. The group $G = \langle g_1, \dots, g_n : r_1 = 1, \dots, r_k = 1\rangle$ is the set of equivalence classes of words using the letters $g_1, \dots, g_n, g_1^{-1}, \dots, g_n^{-1}$. Two such words w_1, w_2 are equivalent if w_1 can be transformed into w_2 using finitely many steps of the following types:*

(1) a "syllable" of the form $g_i g_i^{-1}$ can be deleted or inserted anywhere.
(2) a "syllable" of the form r_i can be deleted or inserted anywhere.

Remark 20.5. The number of generators or relations can be infinite as well.

Theorem 20.6. *Concatenation induces a group structure on G.*

Remark 20.7. If there are no relations, the group $F_n = \langle f_1, \dots, f_n \rangle$ is called the *free group* with n generators. $F_1 \cong \mathbb{Z}$, the other free groups are not Abelian.

The most important property of the free groups is the following:

Theorem 20.8. *Suppose that $F_n = \langle f_1, \dots, f_n \rangle$ is the free group with n generators and G is an arbitrary group with $g_1, \dots, g_n \in G$. Then there is a unique homomorphism $\phi : F_n \to G$ such that $\phi(f_i) = g_i$ for $i = 1, \dots, n$.*

Every group can be described with generators and relations. A not so economic way is to list all elements as generators and all products which happen to be 1 in this group as relations.

Theorem 20.9. $G = \langle g_1, \dots, g_n : r_1 = 1, \dots, r_k = 1 \rangle \cong F_n/N$*, where N is the normal subgroup generated by the relations $r_1, \dots, r_k$.*

This follows from Theorem 20.8.

Corollary 20.10. *Every group is a factor of a free group.*

Suppose now that we have homomorphisms g, h from a group S to $G = \langle g_1, \dots, g_n : r_1 = 1, \dots, r_k = 1 \rangle$ and $H = \langle h_1, \dots, h_m : q_1 = 1, \dots, q_l = 1 \rangle$.

Theorem 20.11. *The amalgamation $G *_S H$ of G and H over S can be presented as*

$$G *_S H = \langle g_1, \dots, g_n, h_1, \dots, h_m | r_1 = 1, \dots, r_k = 1, q_1 = 1, \dots, q_l = 1, g(s)h(s)^{-1} = 1 : s \in S \rangle.$$

Sum in the category of groups and homomorphisms is called *free product.*

Corollary 20.12. *The free product $G * H$ of $G = \langle g_1, \dots, g_n : r_1 = 1, \dots, r_k = 1 \rangle$ and $H = \langle h_1, \dots, h_m : q_1 = 1, \dots, q_l = 1 \rangle$ is*

$$G * H = \langle g_1, \dots, g_n, h_1, \dots, h_m : r_1 = 1, \dots, r_k = 1, q_1 = 1, \dots, q_l = 1 \rangle.$$

So you can see that free groups are free products of copies of the group $\mathbb{Z}$. This corollary also shows that any element of $G * H$ can be written as a product $g_1 h_1 \cdots g_n h_n$.

Definition 20.13. *A word using letters which are non unit elements of G or H is called* reduced word *if no consecutive letters are from the same group.*

Theorem 20.14. *If $a \in G * H$ then there is a unique reduced word representing a.*

The theorem generalizes to the free product of more that two groups.

Corollary 20.15. *The free group F_2 is not commutative.*

Proof. xy and yx are different reduced words in $F_2 = \langle x \rangle * \langle y \rangle$. □

Exercise 20.16. Show that $\mathbb{Z}_2 * \mathbb{Z}_2$ is the semidirect product of $\mathbb{Z}_2$ and $\mathbb{Z}$.

Example 20.17. Let $G \cong \mathbb{Z}$, $H \cong 1$ (the trivial group), $S \cong \mathbb{Z}$, $g(n) := 7n$ and h is the trivial map. Then $G *_S H \cong \mathbb{Z}_7$.

Similarly you can show that if H is trivial then $G *_S H \cong G/\langle \operatorname{Im} g \rangle$, where $\langle \operatorname{Im} g \rangle$ denotes the normal subgroup generated by the image of g.

Remark 20.18. $\operatorname{Im} g$ is usually not a normal subgroup.

Exercise 20.19.

(1) $G \cong F_2$, $H \cong 1$, $S \cong \mathbb{Z}$ and $g(1) = aba^{-1}b^{-1}$. $G *_S H \cong ?$
(2) $G \cong H \cong S \cong \mathbb{Z}$, $g(1) = m$ and $h(1) = n$. $G *_S H \cong ?$

After this detour in algebra we turn to the proof of the Seifert-van Kampen theorem:

Proof of Theorem 20.2. We want use the universal property of amalgamation, so suppose we have homomorphisms $f_A : \pi_1(A) \to Z$ and $f_B : \pi_1(B) \to Z$ to an arbitrary group Z. Then we have to show that there is a unique homomorphism $\alpha : \pi_1(A \cup B) \to Z$ such that the following diagram commutes:

$$\begin{tikzcd}
 & \pi_1(A \cap B) \arrow[dl, "a_*"'] \arrow[dr, "b_*"] & \\
\pi_1(A) \arrow[r, "t_A"] \arrow[dr, "f_A"'] & \pi_1(A \cup B) \arrow[d, dashed, "\alpha"'] & Y \arrow[l, "t_B"'] \arrow[dl, "f_B"] \\
 & Z &
\end{tikzcd}$$

where t_A and t_B are induced by the inclusions $A \to A \cup B$ and $B \to A \cup B$ respectively.

First we show the uniqueness of α:

Lemma 20.20. *The image of t_A and t_B generates $\pi_1(A \cup B)$.*

Proof. The Lebesgue lemma for $[0, 1]$ implies that any loop γ in $A \cup B$ has a factorization $\gamma = p_1 \cdot p_2 \cdots p_n$ where p_i are paths in A or B. Using the path connectedness of $A \cap B$ we can choose paths q_i connecting the base point x with the division points. Inserting "round trips" back to x into γ along the paths q_i we get a factorization of $[\gamma] \in \pi_1(A \cup B)$: We get that $[\gamma] = [\gamma_1][\gamma_2] \cdots [\gamma_n]$ where γ_i are loops in A or B. □

So the only possible choice is $\alpha([\gamma]) = f_A([\gamma_1])f_B([\gamma_2]) \cdots f_B([\gamma_n])$. Here for simplicity we assumed that γ_1 is a loop in A, γ_2 is a loop in B and so on.

The difficult part of the proof is to show that this is a correct definition of α, i.e. if we choose an other factorization of γ or an other representant of $[\gamma]$ then we get the same result.

Independence of factorization: There are two things to check. One is what happens if γ_i is a loop in $A \cap B$ and we switch from $f_A([\gamma_i])$ to $f_B([\gamma_i])$ in the product. But the diagram implies that $f_A([\gamma_i]) = f_B([\gamma_i])$. The other is what happens if we add an extra division point. I leave this case as an (easy) exercise.

So we have now a well defined map $\alpha : \{\text{loops in } A \cup B\} \to Z$, and we have to check that α depends only the homotopy class of the loops:

Independence of the choice of representant: Suppose that $[\delta] = [\gamma] \in \pi_1(A \cup B)$ i.e. $\delta \sim_H \gamma$. Apply the Lebesgue lemma for □ to the homotopy $H : [0, 1] \times [0, 1] \to A \cup B$. So we get a subdivision of $[0, 1] \times [0, 1]$ to smaller rectangles $\{R_i : i = 1, \dots, k\}$. Using this subdivision we can factorize H, i.e. we construct a sequence of loops $\gamma(0) = \underline{x} \cdot \gamma, \gamma(1), \dots, \gamma(k) = \delta \cdot \underline{x}$ (I denoted the constant loop by $\underline{x}$) such that $\gamma(i)$ is the restriction of H to the piecewise linear curve "cutting down the first i rectangles", see figure:

It is easy to see that $\alpha(\gamma) = \alpha(\underline{x} \cdot \gamma)$ and $\alpha(\delta) = \alpha(\delta \cdot \underline{x})$. What left is to show that $\alpha(\gamma(i)) = \alpha(\gamma(i + 1))$. We can choose a factorization of $\gamma(i)$ and $\gamma(i + 1)$ such that all factor

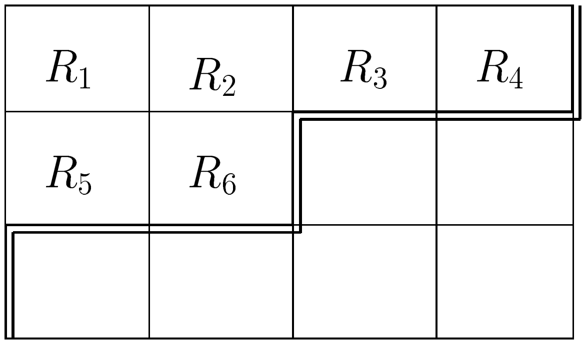

FIGURE 1. $\gamma(6)$

are the same except one, the different factor is homotopic in A (or B) so the value of f_A (or f_B) is the same. And we finished the proof of the Seifert-van Kampen theorem. $\square$

21. The Dirac experiment and the no-wind theorem.

The Dirac experiment. Moving an object like the coat hanger in space defines a path of isometries of the 3-dimensional space $\mathbb{R}^3$. If we keep one point of the object fixed (think of it to be the origin of a 3-dimensional vector space) then the path is in the space of linear isometries of $\mathbb{R}^3$: This is the group usually denoted by $O(3)$ (Check that this is a group indeed). Elements of $O(3)$ can be represented by 3×3 matrices, so $O(3) \subset \mathbb{R}^9$ and admit a natural topology. In fact by moving an object we can only realize *orientation preserving* isometries: your left shoe never becomes a right shoe though they are isometric. It means that our path is in the group of orientation preserving linear isometries denoted by $SO(3)$. If at the end the coat hanger arrives to the same position, then this path is a loop. So the Dirac experiment suggests that $\pi_1(SO(3))$ has an order two element.

Theorem 21.1. *Any isometry $X \in SO(3)$ is a rotation along an axis through origin.*

The proof can be found in most geometry or linear algebra textbook.

So we can describe an element of $SO(3)$ by giving an axis and an angle. Let $c : \bar{B}_\pi(0) \cong D^3 \to SO(3)$ defined by sending $v \in \bar{B}_\pi(0)$ to the rotation around the axis spanned by the vector v rotating with angle $|v|$. It is not difficult but technical to see that c is continuous so we omit the proof.

Exercise 21.2. Show that c is one to one in the interior $B_\pi(0)$ but sends antipodal points on the boundary sphere to the same element in $SO(3)$.

Since $\bar{B}_\pi(0)$ is compact and $SO(3)$ is T_2 we can apply the quotient lemma 8 to show that $SO(3) \cong D^3/\sim$ where $x \sim -x$ on the boundary sphere.

Corollary 21.3. *$SO(3) \cong \mathbb{RP}^3$.*

The proof is left to the reader. Since $\pi_1(\mathbb{RP}^3) \cong \mathbb{Z}_2$, the next exercise gives the explanation for the Dirac experiment.

Exercise 21.4. Show that making a full rotation along a fixed axis defines a loop in $SO(3)$ which represents the non-trivial element in $\pi_1(SO(3)) \cong \mathbb{Z}_2$.

The no-wind theorem. Suppose there is a wind blowing everywhere on Earth. It means that we can assign a nonzero tangent vector to every point of the sphere S^2 continuously. By normalizing these vectors we can assume that these are unit vectors. Such an object is usually called a unit tangent vector field.

Theorem 21.5 (No-wind theorem)**.** *There is no continuous unit tangent vector field on the sphere.*

Let T_1S^2 denote the space of unit tangent vectors (define a topology on this set!). An element of T_1S^2 can be described by a point p in S^2 and a direction in the tangent plane of S^2 at p. If a unit tangent vector field is given then this direction can be described via the angle between the direction and the vector given by the vector field. In other words we get a map $T_1S^2 \to S^2 \times S^1$. It is not difficult to see that this map would be a homeomorphism. We know that $\pi_1(S^2 \times S^1) \cong \mathbb{Z}$ so to prove the no-wind theorem it is enough to show that $\pi_1(T_1S^2) \ncong \mathbb{Z}$.

Theorem 21.6. $SO(3) \cong T_1S^2$.

Proof. The first observation is that the image of the standard basis of $\mathbb{R}^3$ uniquely determines a linear transformation. Since the image of the standard basis under an orientation preserving isometry is a right handed orthonormal basis ($RHONB$) we get that $SO(3) \cong \{RHONB\text{'s}\}$. Now we define a map $h : T_1S^2 \to \{RHONB\text{'s}\}$: If $(p, v) \in T_1S^2$ then let $b_1 = p$, $b_2 = v$ and $b_3 = p \times v$ the unique unit vector forming a $RHONB$ with p and v. It is not difficult to check that h is a homeomorphism. So we showed that $\pi_1(T_1S^2) \cong \mathbb{Z}_2$ and proved the no-wind theorem. □