CALCULATION OF THOM POLYNOMIALS FOR GROUP ACTIONS

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1. Introduction

In this paper the authors' intention is to connect two effective theories of global singularities: the Vassiliev theory of global singularities (as extended by Kazarian [Kaz97]) on one hand, and the Szűcs theory of generalized Pontryagin-Thom construction (started at [Szű79], see also references in [RS98]) on the other. It turns out that the idea for calculating Thom polynomials of singularities using the generalized Pontryagin-Thom construction in [Rimb] can be adapted to the case of generalized Thom polynomials.

The authors are grateful to William Dwyer for helpful discussions and in particular to the paper [Kaz97] in which essential steps are already taken to unify the two theories mentioned above.

As Kazarian points out in [Kaz97] several problems in global singularity theory can be formulated in a very general framework: Let $\rho : G \to \mathrm{Diff}(V)$ be a smooth action of the Lie group G on a smooth contractible manifold V. Then for any principal G-bundle $P \longrightarrow M$ we can associate a bundle $\xi : E = P \times_\rho V \longrightarrow M$. We call such a bundle a ρ-bundle for short. We can define a map $\omega : E \to V/G$—where V/G denotes the orbit space—by $\omega[p, v] := [v]$. The fact that ω is well defined says that though G doesn't act on E we can still define for any element $e \in E$ to which G-orbit it belongs. If τ is a ρ-invariant subset of V—i.e. a union of orbits—then we use the notation $\tau(\xi) := \omega^{-1}(\tau)$. If a section $s : M \to E$ is given then we can define $\tau(s) := s^{-1}\tau(\xi) = (\omega \circ s)^{-1}(\tau)$. In certain situations the closure of $\tau(s)$ is a singular submanifold of M carrying a fundamental cycle and by Poincaré duality it defines a cohomology class in M. It turns out that this class can be expressed as a polynomial $\mathrm{Tp}(\tau)$—depending

Supported by OTKA D29234 (first author), OTKA T029759, FKFP 0226/99 (second author)

Keywords: singularities, Kazarian spectral sequence, Thom polynomials, generalized Pontryagin-Thom construction, global singularity theory

Mathematical Subject Classification 2000: 14N10, 57R45, 32s20, 58K30.

only on τ—of G-characteristic classes of ξ. We shall call these polynomials (generalized) Thom polynomials. Our approach is based on the simple observation that it suffices to calculate these cohomology classes in a single case—for the *universal ρ-bundle.*

In Section 2 we explain this in detail, recall the Vassiliev conditions for the existence of $\mathrm{Tp}(\tau)$ and give a method to calculate it.

In Sections 3 and 4 we give very simple examples of the theory. In Section 5 we continue with a case which provides a very short proof of the Porteous formulas. In Section 6 we describe our main example: singularities of smooth maps. We also present some applications of the Kazarian spectral sequence: In Remark 5.4 we derive an interesting combinatorial identity and in Section 6 we show how to get bounds on the number of singularities in a given codimension.

2. The construction—generalized Thom polynomials and the spectral sequence

In this section the case of the universal ρ-bundle will be presented as well as its relation to the Kazarian spectral sequence and the Thom polynomials.

Let $\xi : E \longrightarrow M$ be a ρ-bundle and $k : M \to BG$ its classifying map. Which means that we have a commutative diagram

$$\begin{tikzcd} E \arrow[r, "\tilde{k}"] \arrow[d] & E_\rho \arrow[d] \\ M \arrow[r, "k"] & BG \end{tikzcd}$$

where $\xi_\rho : E_\rho = E \times_\rho V \longrightarrow BG$ is the universal ρ-bundle over BG. Then for every ρ-invariant subset τ in V we have $\tau(E) = \tilde{k}^{-1}(\tau(E_\rho))$. Moreover if $\overline{\tau(E_\rho)}$ carries a cohomology class $[\overline{\tau(E_\rho)}] \in H^*(E_\rho)$ then

$$[\overline{\tau(E)}] = \tilde{k}^*[\overline{\tau(E_\rho)}] \in H^*(E).$$

Since V is contractible then for any section $s : M \to E$ we have $s^* : H^*(E) \cong H^*(M)$ and this isomorphism doesn't depend on s. Using this isomorphism we have the following

Proposition 2.1. *If $\overline{\tau(s)}$ carries a cohomology class $[\overline{\tau(s)}]$ then*

$$[\overline{\tau(s)}] = \tilde{k}^*[\overline{\tau(E_\rho)}].$$

This motivates the following heuristic definition of the Thom polynomial of τ:

Definition 2.2. *If $\overline{\tau(E_\rho)}$ carries a cohomology class $[\overline{\tau(E_\rho)}]$ then*

$$\operatorname{Tp}(\tau) := [\overline{\tau(E_\rho)}] \in H^*(BG).$$

So we can see that indeed the cohomology class defined by $\tau(s)$ can be expressed in terms of G-characteristic classes of ξ.

For $\tau(E_\rho)$ we also use the notation B_τ because of the following simple observation:

Proposition 2.3. *Every ρ-bundle $\xi : E \longrightarrow M$ which admits a section s such that $\tau(s) = M$ can be classified by the bundle $(\xi_\rho^* E_\rho)\Big|_{B_\tau} \longrightarrow B_\tau$.*

We don't use this result in the paper so we omit the simple proof.

Remark 2.4. If V is not contractible we can still define a Thom polynomial $\operatorname{Tp}(\tau) := [\overline{\tau(E_\rho)}] \in H^*(E_\rho)$ and we have $[\overline{\tau(s)}] = s^*\tilde{k}^*[\overline{\tau(E_\rho)}]$ but in this case s^* depends on the section s.

Now we give a precise definition of the Thom polynomial by recalling the Vassiliev-Kazarian theory:

Existence of Thom polynomials

Let us consider a stratification Ξ of V into strata satisfying the Vassiliev conditions (see cellular G-classification in [Vas88, \$8.6.5] and also [AVGL91], [Kaz97]). Roughly speaking these conditions say that the stratification is regular, locally finite, points in one orbit belong to the same stratum, points in one stratum have isomorphic stabilizer subgroups, and that the moduli spaces (stratum/G) are contractible. Associated to this stratification we can naturally assign a stratification $\Xi(E)$ of the total space E of any ρ-bundle – in particular of E_ρ: we get $\Xi(E_\rho) = \{B_\eta : \eta \in \Xi\}$. Then one can consider the filtration $\emptyset = F_{-1} \subset F_0 \subset F_1 \subset \dots$ of this space E_ρ induced by the codimension filtration of V:

$$S_i := \bigcup_{\operatorname{codim}_V(\eta)=i} B_\eta \qquad\qquad F_i := \bigcup_{j\le i} S_j.$$

The E_1 term of the spectral sequence corresponding to this filtration contains the following relative cohomology groups: $E_1^{p,q} = H^{p+q}(F_p, F_{p-1})$. Let us now suppose that η is cooriented in

V relative to the cohomology theory we are using. In this case B_η will have a cooriented normal bundle in E_ρ, too. Then the relative cohomology groups can be identified—using excision and the Thom isomorphism—as absolute cohomology groups: $E_1^{p,q} = H^q(S_p)$. (If the coorientation condition does not hold we can still use Thom isomorphism with twisted coefficients.)

To have more information on the cohomologies of S_p's we need a finer analysis of the stratification $\Xi(E_\rho)$.

Theorem 2.5. *Let η be a stratum in the stratification of V. The maximal compact subgroup of the stabilizer (isotropy) group* $\mathrm{Stab}(\eta)$ *of a point in η will be denoted by G_η (its inclusion into G by i) and its action on an invariant normal slice N_η to η by ρ_η.*

1. $B_\eta \simeq BG_\eta$.
2. *The diagram*

$$\begin{array}{ccc} B_\eta & \xrightarrow{\simeq} & BG_\eta \\ \Big\downarrow{\scriptstyle j_\eta} & & \Big\downarrow{\scriptstyle Bi} \\ E_\rho & \xrightarrow{\simeq} & BG \end{array}$$

   is homotopy commutative.
3. *The normal bundle ν_η of B_η in E_ρ is isomorphic to $EG_\eta \times_{\rho_\eta} N_\eta$.*

Proof. The first two statements can be read off the following commutative diagram.

$$\begin{array}{ccccccccccc} B_\eta & = & EG\times_\rho \eta & \xrightarrow[(A)]{\simeq} & EG\times_\rho \eta_0 & = & EG\times_\rho (G/\mathrm{Stab}(\eta)) & \underset{(B)}{=} & EG/\mathrm{Stab}(\eta) & \xrightarrow{\simeq} & BG_\eta \\ \Big\downarrow{\scriptstyle j_\eta} & & \Big\downarrow & & \Big\downarrow & & & & \Big\downarrow & & \Big\downarrow{\scriptstyle Bi} \\ E_\rho & = & EG\times_\rho V & = & EG\times_\rho V & & \xrightarrow{\qquad\simeq\qquad} & & EG/G & = & BG \end{array}$$

Here η_0 is an orbit in η, at (A) we have homotopy equivalence since η/G is contractible and the map (B) is $[e,[g]] \mapsto [eg]$.

To prove the third statement observe that an appropriate tubular neighbourhood of $\eta(E_\rho)$ in E_ρ restricted to $\eta_0(E_\rho)$ is $EG\times_\rho \mathrm{Tube}(\eta\subset V)|_{\eta_0}$ and this latter is equal to $EG\times_{\rho_\eta} N_\eta$ (by the map $[e,t]\mapsto[eg,g^{-1}t]$ where $g^{-1}t\in N_\eta$). These identifications, in fact, commute with the identifications in the first row of the above diagram.

We used the fact that for every finite dimensional Lie group G we have $BG \simeq B(\mathrm{Mc}(G))$ where $\mathrm{Mc}(G)$ denotes the maximal compact subgroup of G. It can be a subtle problem for infinite dimensional Lie groups, see Theorem 6.2. □

Remark 2.6. If η/G is not contractible then B_η is a fiber bundle with fiber BG_η and base η/G.

Now (using only the first assertion of the theorem) the terms of the spectral sequence $E_1^{p,q} \cong H^q(S_p)$ can be identified as direct sums of computable groups:

$$E_1^{p,q} \cong \bigoplus_{\text{codim } \eta=p} H^q(BG_\eta), \tag{1}$$

and the equivalence with the Kazarian spectral sequence is transparent: the stratification of E_ρ is the underlying topology of the Kazarian spectral sequence originally defined algebraically in [Kaz97]. Following [Kaz97] the 0^{th} row $(E_1^{p,0}, d_1)$ is a cochain complex – the Vassiliev complex, see also [Vas88] as universal complex. Now we can give the precise definition for the Thom polynomial:

Definition 2.7 ([Kaz97])**.** *The edge homomorphism*

$$\mathrm{Tp} : H^*(E_1^{p,0}, d_1) \to H^*(BG)$$

is called the Thom polynomial corresponding to the action ρ.

Remark 2.8. To see the connection with our heuristic definition 2.2 first notice that by the isomorphism (1) we have $E_1^{p,0} \cong \bigoplus H^0(BG_\eta)$ so we can think of the elements of $E_1^{p,0}$ as linear combinations $\sum a_i\eta_i$ of p-codimensional strata. Such a linear combination admits a Thom polynomial if $d_1(\sum a_i\eta_i) = 0$.

Now to compare the two definitions we dwell deeper into the definition of this spectral sequence: Since the strata B_η have a normal bundle they define a Thom class (with twisted coefficients if necessary) in a tubular neighborhood. Whether this class extends to a unique cohomology class of the union of strata is measured by a cohomology exact sequence of a pair and the coboundary map can be identified with d_1.

Computation of Thom polynomials

Let us turn to our second goal: the computation of (general) Thom polynomials. For simplicity let us assume that a stratum alone defines a Thom polynomial—the general case (linear sum) is not more complicated either.

We need to define the natural partial ordering of the strata in V as follows. The stratum η will be called *more complicated* than θ (notation $\eta > \theta$) if η is in the closure of θ. It follows from the Vassiliev conditions that if codim $\eta \leq$ codim θ then $\eta > \theta$ can not hold unless $\eta = \theta$. The following simple observation (a direct analogue of the main observation in [Rimb]) will be quite fruitful when computing Thom polynomials.

Theorem 2.9. *Let η and θ be strata and $j_\theta \simeq Bi_\theta : BG_\theta \to BG$ as in Theorem 2.5. Then*

$$j_\theta^* \operatorname{Tp}(\eta) = \begin{cases} \text{Euler class of } \nu_\theta & \text{if } \theta = \eta \qquad \textit{`principal equation'} \\ 0 & \text{if } \theta \not> \eta \qquad \textit{`homogeneous equations'}. \end{cases}$$

Proof. The homogeneous equations are consequences of the definition of $\theta \not> \eta$. The principal equation is due to the fact that a cohomology class represented by a submanifold restricted to the submanifold itself is the Euler class of its normal bundle. □

3. First application: the usual representation of $GL_n(\mathbb{C})$

As in [Kaz97] the easiest application of the theory is to the usual representation of $GL(n) = GL(n, \mathbb{C})$ on $\mathbb{C}^n$. Let us consider the stratification given by the two orbits $\theta := \{0\}$, $\eta := \mathbb{C}^n \setminus \{0\}$. Associated with this we have a stratification of $E_\rho \simeq BU(n)$ to $B_\theta \cup B_\eta$. Since $G_\theta = U(n)$, $G_\eta = U(n-1)$ (and so we have the familiar partition of $BU(n)$ into $BU(n) \cup BU(n-1)$), the E_1 term of the Kazarian spectral sequence is $E_1^{0,q} = H^q(BU(n-1))$, $E_1^{n,q} = H^q(BU(n))$ and everything else is 0. By parity reasons there are no non-trivial differentials. So we can define Thom polynomials of the two strata. Theorem 2.9 gives the (otherwise trivial) results: $\operatorname{Tp}(\theta) = c_n \in H^*(BU(n))$ and $\operatorname{Tp}(\eta) = 1$.

4. The action of $GL(2)$ on degree two polynomials

$G = GL(2)$ has an irreducible representation ρ on a three dimensional vector space, which can be identified with the space of degree two polynomials in two variables or the space of symmetric 2×2 matrices: $\rho(g)M = g^T M g$. By Sylvester's Theorem we have three orbits η_0, η_1

and η_2 which can be represented by $0 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $X = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. $\operatorname{codim}_{\mathbb{C}}(\eta_1) = 1$ since $M \in \eta_2$ if and only if $\det(M) \neq 0$. Therefore d_1 is trivial, every orbit has a Thom polynomial. $\operatorname{Tp}(\eta_2) = 1$ since η_2 is the open orbit. $\operatorname{Tp}(\eta_0) = c_3(S^2\gamma_2) = 4c_1c_2$ where γ_2 is the universal 2-bundle over $BGL(2)$ and S^2 denotes the second symmetric power. The only case when we really have to use Theorem 2.9 is η_1:

$$N_X = \left\langle \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \right\rangle, \qquad G_X = \left\{ \left(\begin{smallmatrix} \pm 1 & 0 \\ 0 & \delta \end{smallmatrix}\right) \right\} \cong U(1) \times \mathbb{Z}_2, \qquad \rho_X \left(\begin{smallmatrix} \pm 1 & 0 \\ 0 & \delta \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & \delta^2 \end{smallmatrix}\right).$$

Since the $\mathbb{Z}_2$-action is trivial we use only the $U(1)$ factor a in the inclusion $i = a \times b : U(1) \times \mathbb{Z}_2 \to GL(2)$. The map $j = Ba : BU(1) \to BGL(2)$ induces isomorphism on H^2: $j^*c_1(\gamma_2) = c_1(\gamma_1)$. On the other hand $e(\nu_{\eta_1}) = 2c_1$ so $\operatorname{Tp}(\eta_1) = 2c_1$.

In the real case when $GL(2, \mathbb{R})$ acts on $\mathbb{R}^3$ there are six orbits by the real Sylvester's Theorem. The equation $\det(M) = 0$ defines a cone, which cuts $\mathbb{R}^3$ into three open regions represented by $(+) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $(\pm) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $(-) = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, these are the 0-codimensional orbits. The two parts of the cone and 0 give the other three orbits. In this case d_1 of the Kazarian spectral sequence is nontrivial, and the 0-codimensional orbits together define a d_1-cocycle: $d_1((+) + (-) + (\pm)) = 0$. There are no nontrivial Thom polynomials however so we omit these calculations. We only mention that the orbit of 0 is not cooriented, so it doesn't define a Thom polynomial in cohomology with $\mathbb{Z}$-coefficients. It has a Thom polynomial with $\mathbb{Z}_2$-coefficients but it is 0.

5. The classical case: Giambelli-Thom-Porteous formula

In this section we show how to recover the classical Thom polynomial formula (the so called Giambelli-Thom-Porteous formula, see [Tho56], [Por71]) in our theory. We choose our field to be $\mathbb{C}$ and our cohomology theory to be $H^*(\ , \mathbb{Z})$.

Suppose that $f : N \to P$ is a smooth map of manifolds. The Giambelli-Thom-Porteous formula describes the cohomology class defined by $\Sigma_s(f)$, the subset of N where df has corank s.

In terms of the theory described above we calculate the Thom polynomials of the representation $\rho = \hom\big(\rho(n), \rho(n+k)\big)$ of the group $G := GL(n) \times GL(n+k)$ on the linear space $\mathbb{C}^{(n+k)\times n}$ where $\rho(n)$ is the standard representation of $GL(n)$. So $(R, L) \in G$ acts on an $(n+k) \times n$ matrix X by: $(R, L) \cdot X := LXR^{-1}$. We can assume that $k \geq 0$.

As it is well known the orbits Σ_s of this action are characterized by corank. A representative from Σ_s is $X_s := \begin{pmatrix} 0 & 0 \\ 0 & I_{n-s} \end{pmatrix}_{(n+k)\times n}$. The maximal compact stabilizer subgroup of X_s is

$$G_s := G_{X_s} = \left\{ \left(\left(\begin{smallmatrix} A & 0 \\ 0 & C \end{smallmatrix}\right), \left(\begin{smallmatrix} B & 0 \\ 0 & C \end{smallmatrix}\right) \right) \;\middle|\; (A,B,C) \in U(s) \times U(s+k) \times U(n-s) \right\}.$$

An invariant normal slice Σ_s at X_s is $N_s = \left\{ \left(\begin{smallmatrix} M^{(s+k)\times s} & 0 \\ 0 & 0 \end{smallmatrix} \right) \right\}$. It implies that the—complex—codimension of Σ_s is $s(s+k)$. Therefore d_1 is trivial, every orbit has a Thom polynomial. To determine the principal equation for the Thom polynomial associated to Σ_s we need two data: the Euler class of the normal bundle ν_s of Σ_s in E_ρ and the map $H^*(Bi_s): H^*(BG) \to H^*(BG_s)$ where i_s is the inclusion of G_s into G. We will use the notation

$$\begin{aligned} H^*(G) &= \mathbb{Z}[R_1,\dots,R_n,L_1,\dots,L_{n+k}] \\ &= \mathbb{Z}^{S_n\times S_{n+k}}[r_1,\dots,r_n,l_1,\dots,l_{n+k}] \\ H^*(G_s) &= \mathbb{Z}[A_1,\dots,A_s,B_1,\dots,B_{s+k},C_1,\dots,C_{n-s}] \\ &= \mathbb{Z}^{S_s\times S_{s+k}\times S_{n-s}}[a_{n-s+1},\dots,a_n,b_{n-s+1},\dots,b_{n+k},c_1,\dots,c_{n-s}], \end{aligned}$$

where the capitals mean universal Chern classes while the lower letters mean Chern roots, and e.g. $\mathbb{Z}^{S_n\times S_{n+k}}[\ ]$ means the part of the polynomial ring $\mathbb{Z}[\ ]$ invariant under the action of $S_n \times S_{n+k}$, i. e. the permutations of the a_i's and b_j's.

The action of $(A,B,C) \in G_s$ on the invariant normal slice N_s is given by changing M to BMA^{-1}. So the Euler class of the normal bundle ν_s is (written in terms of Chern roots):

$$\prod_{i=n-s+1}^{n} \prod_{j=n-s+1}^{n+k} (b_j - a_i) \in H^{s(s+k)}(BG_s).$$

The map $H^*(Bi_s)$ is given by (again in terms of Chern roots):

$$r_i \mapsto \begin{cases} c_i & \text{if} \quad i \le n-s \\ a_i & \text{if} \quad i > n-s \end{cases}$$

$$l_i \mapsto \begin{cases} c_i & \text{if} \quad i \le n-s \\ b_i & \text{if} \quad i > n-s \end{cases}$$

Lemma 5.1. *$h := H^{s(s+k)}(Bi_s)$ is injective.*

Proof. Let $p(r_1, \dots, r_n, l_1, \dots, l_{n+k})$ be a polynomial representing a nonzero element in $H^*(BG)$. Let us define the following polynomials:

$$p_\sigma(c_1, \dots, c_\sigma, r_{\sigma+1}, \dots, r_n, l_{\sigma+1}, \dots, l_{n+k}) := p(c_1, \dots, c_\sigma, a_{\sigma+1}, \dots, a_n, c_1, \dots, c_\sigma, b_{\sigma+1}, \dots, b_{n+k}).$$

We have $p = p_0$ and $p_{n-s} = h(p)$. Suppose now that $h(p) = 0$. Then there is a smallest σ such that $0 < \sigma \le n-s$ and $p_\sigma = 0$. Then $p_{\sigma-1}$ is divisible by $r_\sigma - l_\sigma$. On the other hand $p_{\sigma-1}$ is invariant under the permutations of the variables $r_\sigma, \dots, r_n$ and that of $l_\sigma, \dots, l_{n+k}$. So $p_{\sigma-1}$ is in fact divisible by $\prod_{i=\sigma}^n \prod_{j=\sigma}^{n+k}(r_j - l_i)$ which implies that:

$$\deg p_0 \ge \deg p_{\sigma-1} \ge (n-\sigma+1)(n+k-\sigma+1) \ge (s+1)(s+k+1),$$

which is a contradiction. □

Lemma 5.2. *Using the notation*

$$\frac{1 + L_1 t + L_2 t^2 + \ldots + L_{n+k} t^{n+k}}{1 + R_1 t + R_2 t^2 + \ldots + R_n t^n} = 1 + H_1 t + H_2 t^2 + \ldots$$

$\det(H_{s+i-j})_{(s+k)\times(s+k)}$ *maps to* $\prod_{i=n-s+1}^n \prod_{j=n-s+1}^{n+k}(b_j - a_i)$ *under the map* $H^{s(s+k)}(Bi_s)$.

Proof. The image H_i' of H_i are the coefficients of the Taylor series

$$\frac{\prod(1+a_i t)\prod(1+c_k t)}{\prod(1+b_j t)\prod(1+c_k t)} = \frac{\prod(1+a_i t)}{\prod(1+b_j t)} = \frac{1 + B_1 t + B_2 t^2 + \ldots + B_{s+k} t^{s+k}}{1 + A_1 t + A_2 t^2 + \ldots + A_s t^s}.$$

But $\det(H'_{s+i-j})_{(s+k)\times(s+k)}$ equals to the resultant of the two polynomials $1 + B_1 t + B_2 t^2 + \ldots + B_{s+k} t^{s+k}$ and $1 + A_1 t + A_2 t^2 + \ldots + A_s t^s$. (This is a less known form of the resultant $R(p,q)$ which can be obtained by multiplying the Sylvester-matrix by a matrix obtained from the coefficients of the Taylor series of $1/p$ or as in [Ful98, p.420].) On the other hand the resultant is equal to the product of differences of the roots of the two polynomials. $\square$

The two lemmas together proves that there is only one polynomial that satisfies the principal equation, i.e.

$$\mathrm{Tp}(\Sigma_s) = \det(H_{s+i-j})_{(s+k)\times(s+k)}.$$

Notice that a ρ-bundle in this case is a pair of vector bundles E, F of rank n and $n+k$. In the classical situation of a map $f : N \to P$ these bundles are TN and f^*TP, and H_i can be interpreted as the i^{th} Chern class of the virtual bundle $F \ominus E$. Also notice that the formula doesn't depend on n.

Remark 5.3. The fact that we used only the principal equations and got a unique polynomial is highly atypical. Usually we need lots (all) of homogeneous equations as well, and there are examples where even all the equations together do not determine the Thom polynomial. In the present case the homogeneous equations are explicit consequences of the principal one.

Remark 5.4. We can also calculate the Kazarian spectral sequence of this representation. It collapses at E_1 since the odd rows and columns are zero and gives some combinatorial identities. Particularly interesting is the limiting case $n = \infty$. We assume now that $k = 0$. Here the cohomology groups of $U(s)^2 \times U(\infty)$ are listed in the s^{th} column. Summing up the ranks in the skew diagonals one gets the ranks of the cohomologies of $U(\infty)^2$. Some combinatorics shows that "one can drop a $\times U(\infty)$ term" from everywhere, i.e. one can write the cohomology groups of $U(s)^2$ in the s^{th} column and get the ranks of the cohomologies of $U(\infty)$ (i.e. $1, 1, 2, 3, 5, 7, 11, 15, \dots$) in the skew diagonals, as follows:

	0.	2.	4.	6.	8.	10.	12.	14.	16.	18.
8.	.	5	.	.	14	.	.	.	.	16
6.	.	4	.	.	8	.	.	.	.	10
4.	.	3	.	.	5	.	.	.	.	5
2.	.	2	.	.	2	.	.	.	.	2
0.	1	1	.	.	1	.	.	.	.	1

In this table only the ranks of the free abelian groups are written (i.e. a is written instead of $\mathbb{Z}^a$) and only the terms with two even coordinates are indicated, since everything else is 0. This leads to the combinatorical identity

$$\pi(n,[1,1,\dots]) = \pi(n-1,[2,0,0,\dots]) + \pi(n-4,[2,2,0,0,\dots]) + \pi(n-9,[2,2,2,0,0,\dots]) + \cdots,$$

where $\pi(n,[a_1,a_2,\dots])$ denotes the numbers of degree n monomials in terms of a_i copies of variables of degree i. This identity—already known to Euler— can be directly proved by using Ferrer diagrams of partitions (thanks to Aart Blokhuis for these informations).

6. Singularities

In this section we show how our theory applies to the case of singularities of maps between manifolds—the case where Thom polynomials were originally defined by Thom in [Tho56]. We will work over the complex field, so manifolds and maps are assumed to be complex analytic. What we really show is that the equations we get by Theorem 2.9 for the Thom polynomials of simple singularities are the same as were studied and solved in [Rimb], so we will not repeat their solution here. On the other hand, when we apply the other feature—the Kazarian spectral sequence—of the theory to this case we get interesting relations (e.g. bounds) for the number of different strata—i.e. in some sense, the number of different singularity types—in a fixed codimension.

Now we recall some standard definitions of singularity theory (see e.g. [AVGL91]): $\mathcal{E}^0(n,n+k)$ will be the vector space of smooth germs $(\mathbb{C}^n,0)\to(\mathbb{C}^{n+k},0)$. We will think of $\mathcal{E}^0(n,n+k)$ as a subset of $\mathcal{E}^0(n+a,n+k+a)$ by trivial unfolding. Fixing k let $\mathcal{E}^0(\infty,\infty+k)$ be the union (or formally the direct limit): $\cup_{n=0}^{\infty}\mathcal{E}^0(n,n+k)$. This space will play the role of V of the general theory. Also we have maps $u_\infty:\mathcal{E}^0(n,n+k)\to\mathcal{E}^0(\infty,\infty+k)$ (u_∞ stands for infinite *unfolding*). Let $\mathrm{Hol}(\mathbb{C}^n,0)$ denote the group of biholomorphism germs of $(\mathbb{C}^n,0)$. The group

$$\mathcal{A}(n,n+k) := \mathrm{Hol}(\mathbb{C}^n,0)\times\mathrm{Hol}(\mathbb{C}^{n+k},0)$$

acts on $\mathcal{E}^0(n,n+k)$ by $(\varphi,\psi)\cdot f := \psi\circ f\circ\varphi^{-1}$. Similarly the limit group

$$\mathcal{A}(\infty,\infty+k) := \cup_{n=0}^{\infty}\mathrm{Hol}(\mathbb{C}^n,0)\times\mathrm{Hol}(\mathbb{C}^{n+k},0)$$

acts on $\mathcal{E}^0(\infty,\infty+k)$ by the same formula.

We will mainly be concerned with the bigger—*contact*—groups

$$\mathcal{K}(n,n+k) := \{(\varphi, M) : \varphi \in \mathrm{Hol}(\mathbb{C}^n,0), M \text{ is a germ } (\mathbb{C}^n,0) \to \mathrm{Hol}(\mathbb{C}^{n+k},0)\},$$

acting on $\mathcal{E}^0(n,n+k)$ by $\rho^{\mathcal{K}(n,n+k)}(f) = M(x) \circ f \circ \varphi^{-1}(x)$. The limit group $\mathcal{K}(\infty,\infty+k)$ acts on $\mathcal{E}^0(\infty,\infty+k)$ by the same formula. This group will play the role of G in the general theory. We use the notations $\mathcal{E}^0, \mathcal{A}$ and $\mathcal{K}$ if the value of n ($n=\infty$ allowed) is clear from the context.

So, consider the action of $\mathcal{K}(\infty,\infty+k)$ on $\mathcal{E}^0(\infty,\infty+k)$. The nicest orbits are the so called *simple* ones: an orbit (or a representative) is simple, if a neighbourhood intersects only finitely many different orbits. Simple orbits will be strata in an appropriate Vassiliev stratification. Let η be a simple orbit, and let us choose a representative $f \in \mathcal{E}^0(n,n+k)$ with minimal n. In other words we choose a minimal dimensional representative with $\eta =$ orbit of $u_\infty(f)$. Such an f (defined up to $\mathcal{K}(n,n+k)$-equivalence) is called a *genotype* for η in [AVGL91, p. 157]. The contact automorphism group $\mathrm{Stab}^{\mathcal{K}}(f) = \{(\varphi,M) \in \mathcal{K}(n,n+k) | (\varphi,M)\cdot f = f\}$ and the analogously defined $\mathrm{Stab}^{\mathcal{A}}(f)$ are not finite dimensional (moreover they do not possess convenient topologies) so we need the following definition—inspired by the classical Bochner theorem—using that $GL(n) \times GL(n+k) \subset \mathcal{A}(n,n+k) \subset \mathcal{K}(n,n+k)$.

Definition 6.1 ([Jän78]). *If M is a subgroup of $\mathcal{A}(n,n+k)$ or $\mathcal{K}(n,n+k)$ then M is* compact *if M is conjugate to a compact subgroup $N \subset GL(n) \times GL(n+k)$.*

Luckily enough the groups $\mathrm{Stab}^{\mathcal{K}}(f)$ and $\mathrm{Stab}^{\mathcal{A}}(f)$ share many properties with finite dimensional groups, as follows.

Theorem 6.2.

1. $\mathrm{Stab}^{\mathcal{K}}(f)$ *(* $\mathrm{Stab}^{\mathcal{A}}(f)$*) has a maximal compact subgroup $G_f = G_f^{\mathcal{K}}$ ($G_f^{\mathcal{A}}$).*
2. *Any two maximal compact subgroups are conjugate.*
3. $B\,\mathrm{Stab}^{\mathcal{K}}(f) \simeq BG_f^{\mathcal{K}}$ *and* $B\,\mathrm{Stab}^{\mathcal{A}}(f) \simeq BG_f^{\mathcal{A}}$.

The proof of 1 and 2 can be found in [Jän78], [Wal80]. As we mentioned $\mathrm{Stab}(f)$ does not possess convenient topology, so strictly speaking $B\,\mathrm{Stab}(f)$ is not defined. However, it is possible to define the notion of $\mathrm{Stab}(f)$-principal bundle over a *smooth manifold* and BG_f classifies those bundles ([Rim96, Thm 1.3.6] or [Rima]). So from our point of view we can replace $B\,\mathrm{Stab}(f)$ by BG_f. In particular we have $B\mathcal{K}(n,n+k) \simeq B\mathcal{A}(n,n+k) \simeq BGL(n) \times BGL(n+k)$.

Remark 6.3. Theorem 6.2 allows us to use the same algorithm to calculate the Thom polynomials as in the finite dimensional case.

Definition 6.1 and Theorem 6.2 shows that by choosing f carefully from its $\mathcal{K}$-orbit, we can assume that $G_f^{\mathcal{K}} \subset GL(n) \times GL(n+k)$, so we have representations μ_0, μ_1 of $G_f^{\mathcal{K}}$ on the source and target spaces respectively. By part 2 of Theorem 6.2 the isomorphism classes of these representations are uniquely defined. The groups $G_f^{\mathcal{K}}$ and representations μ_0, μ_1 were calculated for low codimensional singularities in [Rima].

For the identification of G_η for $G = \mathcal{K}(\infty, \infty + k)$ we cannot directly use Theorem 6.2, but it is not difficult to get around:

Definition 6.4.

$$G_\eta := G_f \times U(\infty) \subset \mathrm{Stab}(\eta)$$

where the inclusion of $U(\infty)$ *into* $\mathrm{Stab}(\eta)$ *corresponds to the diagonal action on the unfolding dimensions.*

Though G_η is not compact in any reasonable sense, it is still true that $B\,\mathrm{Stab}(\eta) \simeq BG_\eta$ (in the sense of our remark after Theorem 6.2) and as we will see the $U(\infty)$ summand acts trivially on N_η anyway.

Below we explain how to calculate the two inputs of the algorithm for computing the Thom polynomials for an orbit η, i. e. the map $H^*(Bi) : H^*(BG) \to H^*(BG_\eta)$ and the representation $\rho_\eta : G_\eta \to GL(N_\eta)$.

Proposition 6.5. *The homomorphism* $H^*(B\mathcal{K}) = H^*(BU(\infty) \times U(\infty)) \to H^*(BG_\eta)$ *induced by the inclusion* $G_\eta \to \mathcal{A} \subset \mathcal{K}$ *is given by*

$$\begin{array}{ccc} \mathbb{Z}[\underline{a}, \underline{b}] & \longrightarrow & H^*(BG_f) \otimes \mathbb{Z}[\underline{d}] \\ a & \mapsto & c(\mu_0) \cdot d \\ b & \mapsto & c(\mu_1) \cdot d, \end{array}$$

where $\underline{a} = a_1, a_2, \dots$ *and* $\underline{b} = b_1, b_2, \dots$ *are the universal Chern classes of the two factors of* $U(\infty) \times U(\infty)$*;* $a = 1 + a_1 + a_2 + \dots$*,* $b = 1 + b_1 + b_2 + \dots$ *are the total Chern classes, and the definitions for* $\underline{d}$*,* d *are similar. The class* $c(\mu_i)$ *is the total Chern class of the vector bundle* E_{μ_i} *over* BG_f*.* ∎

Let us turn to our second goal. It is enough to calculate the $G_f^{\mathcal{K}}$-action ρ_f on the normal space $N_f^{\mathcal{K}}$ to the $\mathcal{K}(n,n+k)$-orbit in $\mathcal{E}^0(n,n+k)$ since $u_\infty(N_f^{\mathcal{K}})$ is normal to η in $\mathcal{E}^0(\infty,\infty+k)$ too, and once the action is G_f-invariant, it is also G_η-invariant with the trivial action of the $U(\infty)$ factor.

The representation ρ_f is also explicitly computable for low codimensional singularities: The miniversal unfolding F of f is a stable germ $(\mathbb{C}^n\oplus U,0)\to(\mathbb{C}^{n+k}\oplus U,0)$ where U is the unfolding space. The group $G_f^{\mathcal{K}}$ ats linearly on U: let us denote this representation by μ_U. Then we have $\rho_f\cong\mu_0\oplus\mu_U$. We can also see that the source space of F is naturally isomorphic to $N_f^{\mathcal{K}}$. (For more details see [Wal80] or [Rima].)

Remark 6.6. Calculation of the Thom polynomials for the stable orbits of the $\mathcal{A}(\infty,\infty+k)$-action doesn't give anything new: every such orbit is a dense open subset of a $\mathcal{K}(\infty,\infty+k)$-orbit —we usually use the same notation for the two orbits—so their Thom polynomials are the same.

Now, we might as well write down the equations of Theorem 2.9 for the Thom polynomials for $\mathcal{K}$, but it is possible to simplify these equations as we will see in Theorem 6.12.

We can also calculate Thom polynomials for $\mathcal{K}(n,n+k)$. These results are not independent: If η is an orbit of $\mathcal{K}(n,n+k)$ then its d-dimensional unfolding $u_d(\eta)$ is an orbit of $\mathcal{K}(n+d,n+k+d)$ with the same codimension. This is a consequence of the fact that the codimension of the $\mathcal{K}$-orbit can be read off from its local algebra, which doesn't change by trivial unfolding. To understand the connection between the Thom polynomial of η and of $u_d(\eta)$ we look at the unfolding map $u_d:\mathcal{K}(n,n+k)\to\mathcal{K}(n+d,n+k+d)$. It induces a map

$$H^*(Bu_d):H^*(B\mathcal{K}(n+d,n+k+d)\cong\mathbb{Z}[a_1,\dots,a_{n+d},b_1,\dots,b_{n+k+d}]\to$$
$$H^*(B\mathcal{K}(n,n+k))\cong\mathbb{Z}[a_1,\dots,a_n,b_1,\dots,b_{n+k}]$$

($d=\infty$ is allowed) such that

$$H^*(Bu_d)(a_i)=\begin{cases}a_i & \text{if} \quad i\le n\\ 0 & \text{if} \quad i>n\end{cases}\qquad H^*(Bu_d)(b_i)=\begin{cases}b_i & \text{if} \quad i\le n+k\\ 0 & \text{if} \quad i>n+k\end{cases}$$

Proposition 6.7. $\mathrm{Tp}(\eta)=H^*(Bu_d)\big(\mathrm{Tp}(u_d(\eta))\big)$ *where u_d denotes the d-fold trivial unfolding ($d=\infty$ is allowed).*

Proof. The bundle map $\tilde{u}_d : E_{\rho^{\mathcal{K}(n,n+k)}} \to E_{\rho^{\mathcal{K}(n+d,n+k+d)}}$ keeps the codimension filtration (at least after some refinement of the Vassiliev stratification of $E_{\rho^{\mathcal{K}(n+d,n+k+d)}}$) so it induces a map between the Kazarian spectral sequences. □

So using the generators a_i, b_i for all of these groups we can say that for d large enough ($n + d \geq \operatorname{codim} \eta$) unfoldings don't change the Thom polynomial. Pursueing these arguments further, we find another important property of these Thom polynomials:

Proposition 6.8 (folklore, see [AVGL91])**.** $\mathrm{Tp}(\eta) \in Q$ *where* Q *is the subring of* $H^*(B\mathcal{K})$ *generated by* $1, h_1, h_2, \dots,$ *where* $1 + h_1 + h_2 + \dots = \frac{1+b_1+b_2+\dots}{1+a_1+a_2+\dots}$.

Before getting into the proof we need some definitions:

Definition 6.9. *If* A^n *and* B^{n+k} *are vector bundles over a manifold* M*, then let* $\mathcal{E}^0(A, B)$ *denote the* $\mathcal{E}^0(n, n+k)$*-bundle over* M *such that*

$$\mathcal{E}^0_m(A, B) = \{\text{germs of smooth maps } (A_m, 0) \to (B_m, 0)\}.$$

Using Theorem 6.2 it is not difficult to see that

$$a_i(\mathcal{E}^0(A, B)) = c_i(A) \quad \text{and} \quad b_i(\mathcal{E}^0(A, B)) = c_i(B)$$

through the obvious identifications. (In fact it also follows from Theorem 6.2 that every $\mathcal{E}^0(n, n+k)$-bundle is isomorphic to a bundle of the form $\mathcal{E}^0(A, B)$ but we don't use this in this paper.)

Definition 6.10. *Let* C^d *be a* d*-dimensional vector bundle. Then*

$$u_C : \mathcal{E}^0(A, B) \to \mathcal{E}^0(A \oplus C, B \oplus C)$$

denotes the twisted unfolding map*:*

$$u_C(\varphi) := \varphi \oplus \mathrm{Id}_C\,.$$

Proof of Proposition 6.8. We have a commutative diagram

$$\begin{array}{ccc} E_{\rho^{\mathcal{K}(n,n+k)}} & \xrightarrow{\tilde{u}_d} & E_{\rho^{\mathcal{K}(n+d,n+k+d)}} \\ \tilde{k}\uparrow & & \uparrow\tilde{k}_C \\ \mathcal{E}^0(A, B) & \xrightarrow{u_c} & \mathcal{E}^0(A \oplus C, B \oplus C) \end{array}$$

where $\tilde{k}$ and $\tilde{k}_C$ are the bundle maps induced by the corresponding classifying maps. It shows that

$$k^* \operatorname{Tp}(\eta) = k_C^* \operatorname{Tp}(u_d(\eta))$$

Choose C to be the 'inverse' of A. Then using that we can think of Tp as a polynomial of $\underline{a}$ and $\underline{b}$ we get (with some abuse of notation):

$$\operatorname{Tp}(\eta)\big(c(A), c(B)\big) = \operatorname{Tp}(u_d(\eta))\big(1, c(B)/c(A)\big)$$

□

Definition 6.11. *Let* $\operatorname{tp}(\eta)$ *be the unique polynomial with the property*

$$\operatorname{tp}(\eta)(1, h_1, \dots) = \operatorname{Tp}(\eta)(\underline{a}, \underline{b}),$$

where $1 + h_1 + h_2 + \dots = \frac{1+b_1+b_2+\dots}{1+a_1+a_2+\dots}$.

Proposition 6.5 is enough to write down the equations for the Thom polynomials of simple singularities in the cohomology ring of $B\mathcal{A}$ or $B\mathcal{K}$, namely in $\mathbb{Z}[\underline{a}, \underline{b}]$. In the light of Proposition 6.8 we write it in terms of the 'quotient' variables h_i:

Theorem 6.12.

$$\operatorname{tp}(\eta)(c(\theta)) = \begin{cases} \text{Euler class of } E_{\rho_\eta} & \text{if } \theta = \eta \qquad \textit{`principal equation'} \\ 0 & \text{if } \theta \not> \eta \qquad \textit{`homogeneous equations'}, \end{cases}$$

where $c(\theta) = c(E_{\mu_1(\theta)})/c(E_{\mu_0(\theta)})$.

These are exactly the equations that were solved for many cases in [Rimb].

Remark 6.13. At this point we would like to comment on the history of these ideas. The systematic study of classifying spaces of the symmetry groups of singularities and a powerful construction out of these spaces was pioneered by A. Szűcs (see e.g. [Szű79]). In the language of the present paper he calculated G_η for several singularities, and described a way how to glue these spaces together to get a space whose algebraic topological properties can be translated to differential topological theorems. He applied his construction to various differential topological questions, such as e.g. the cobordism groups of maps with given singularities (e.g. [Szű80], [Szű91], [Szű94]). A general method of calculating more of the symmetry groups was given in [Rim96], [RS98], [Rima]. In the present paper we explored the fact that roughly speaking (the 'source space' in) Szűcs' construction is a union of strata in the stratification of $B\mathcal{A}$ defined by the $\mathcal{A}$-action, and that the ideas fruitful there (e.g. Thom polynomial calculations [Rimb]) turn out to be fruitful viewing any G-action.

Example 6.14. Let us start with $V = \mathcal{E}^0(1,1)$. The group $\mathcal{K}(1,1)$ acts on it as usual. Here the finite codimensional strata will be the contact orbits A_k represented by the germ $x \mapsto x^{k+1}$. Just like above one can write up the equations for the Thom polynomials of these strata. Carrying out the computation one finds that here the 'principal' and 'homogeneous' equations are *enough* to determine the Thom polynomials:

$$\mathrm{Tp}(A_n) = \prod_{i=1}^{n}(b - ia) \in H^*\big(BU(1) \times BU(1)\big) = \mathbb{Z}[a,b].$$

This is interesting in the light of the fact that when we want to compute the Thom polynomials of A_n in the original way, then the equations dealing with only A_i singularities are *not* enough: e.g. to compute the Thom polynomial of A_4 one has to consider $I_{2,2}$ maps, too (this is clear in any approach, see [Gaf83], [Rimb]). This apparent contradiction is due to the fact that

$$H^*(Bu) : H^*(BU(\infty) \times BU(\infty)) = \mathbb{Z}[\underline{a}, \underline{b}] \to H^*(BU(1) \times BU(1))$$

is not injective, not even on the subring Q generated by $1, h_1, h_2, \ldots$, where $1 + h_1 + h_2 + \ldots = \frac{1+b_1+b_2+\ldots}{1+a_1+a_2+\ldots}$. (We know from Proposition 6.8 that the image of $H^*(Bu)$ is in Q.)

Let us now turn to the Kazarian spectral sequence of the theory. For simplicity let $k = 0$. The list of simple singularities (for codimension ≤ 8) is given in the following table.

$\mathrm{codim}_{\mathbb{C}}$	0	1	2	3	4	5	6	7	8
	A_0	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8
					$I_{2,2}$	$I_{2,3}$	$I_{2,4}$	$I_{2,5}$	$I_{2,6}$
							$I_{3,3}$	$I_{3,4}$	$I_{3,5}$
									$I_{4,4}$
								(x^2, y^3)	$(x^2 + y^3, xy^2)$

Suppose for the moment that we know this classification only up to codimension 7. The maximal compact symmetry groups of these singularities can be computed as in [Rima], which gives us the E_1 term for the Kazarian spectral sequence. Here we show this with the '$U(\infty)$ term dropped' everywhere (as in remark 5.4), and writing only the ranks of the occuring groups:

22.	0											
20.	0	1										
18.	0	1	1									
16.	0	1	1	1								
14.	0	1	1	1	5							
12.	0	1	1	1	5	2						
10.	0	1	1	1	4	2	3					
8.	0	1	1	1	4	2	3	8				
6.	0	1	1	1	3	2	3	7	a_6			
4.	0	1	1	1	3	2	3	6	a_4	b_4		
2.	0	1	1	1	2	2	3	5	a_2	b_2	c_2	
0.	1	1	1	1	2	2	3	4	a_0	b_0	c_0	d_0
	0.	2.	4.	6.	8.	10.	12.	14.	16.	18.	20.	22.

Up to codimension 8 all the singularities are simple, so up to the 16^{th} column we will not have non-zero ranks in entries with an odd coordinate. Hence, after considering the directions of the differentials we have that up to $p+q \leq 16$ this table 'degenerates', i.e. in that region $E_2 = E_\infty$. In E_∞ the sums of skew diagonals should be equal to the ranks of the cohomologies of $BU(\infty)$: $1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 55, \ldots$ (the number of partitions). This gives us that a_0 must be 5, i.e. there must be 5 Vassiliev strata in codimension 8, which is consistent with our table above. In fact we can easily compute the maximal compact symmetry groups of the codimension 8 singularities: they are all $U(1)$, so not only a_0 but every a_{2i} is equal to 5.

Now, blindly adding up the 9^{th} skew diagonal we find that b_0 should be 7.

Recall that there are 6 simple singularities in codimension 9: A_9, $I_{2,7}$, $I_{3,6}$, $I_{4,5}$, (x^2+y^3, y^4), (x^2+y^4, xy^2) (see e.g. [PW95] or [DG83]). So we are looking for one more. An obvious guess is the one-parameter family of simplest Σ^3-singularities (of codimension 10)—corresponding to the ideals:

$$(x^2+2vyz, y^2+2vxz, z^2+2vxy) \qquad\qquad v\in\mathbb{C}\setminus\{\text{roots of } v(v^3-1)(8v^3+1)\}\Big/\mathbb{Z}_3 .$$

(Here the $\mathbb{Z}_3$-action is the multiplication by 3^{rd} roots of unity.) This shows that the moduli space of this stratum is a sphere minus 4 points, which is not contractible, so it is not allowed by the Vassiliev conditions. We have two ways to overcome this difficulty.

(1) We slice the moduli space by 3 segments connecting the missing points from the sphere. What remains can be a Vassiliev stratum in codimension 18. Calculation shows that its symmetry group is $U(1)\times S_3$, which then gives us $b_0=b_2=b_4=\ldots=7$. However the slicing gives us 3 new strata in codimension 19 (!), so the number between b_0 and c_0 (not shown on the table) is 3. The coordinate 19 being odd, this 3 has to be killed before E_∞. The first chance where it can happen is the horizontal differential mapping *from* $E^{19,0}$, i.e. the differential in the Vassiliev complex. Let the rank of this differential be r $(0\le r\le 3)$. Then the remaining rank $3-r$ is to be killed by a differential mapping *to* $E^{19,0}$. But this latter would decrease the $p+q=18$ skew diagonal sum, so this would imply that $b_0>7$, which is not the case, see [PW95]. So $r=3$, which by similar argument shows that $c_0\ge 13$.

(2) Another way, which is an advantage of the geometric approach, is that we do not slice the moduli space, but consider it as a whole. Then, as remarked earlier, the only change in the theory is that B_η will not be BG_η, but a BG_η-bundle over the moduli space. In our case this bundle is cohomologically trivial, which gives us: $b_0=b_2=b_4=\ldots=7$ and $b_1=3$. Again, this 3 must be killed before E_∞, which happens at the map 'pointing' to c_0; and we arrive at the same consequences.

Remark 6.15. A more detailed analysis of the mentioned and the coming region of the Kazarian spectral sequence is worthwhile. For example, a geometric understanding of the rank 3 differential (in both approaches) mapping to $E^{19,0}$ gives us 3 independent linear equations among Thom polynomials of singularities of codimension 10. This and similar study of the spectral table, as well as more detailed proofs of the statements above will be given in a subsequent paper.

Example 6.16. Now we look at the case $V=\mathcal{E}^0(2,2)$ and write up the corresponding Kazarian spectral sequence (now there is nothing like '$U(\infty)$ term dropped'). We get the following E_1

table:

22.	6											
20.	6	11										
18.	5	10	10									
16.	5	9	9	9								
14.	4	8	8	8	12							
12.	4	7	7	7	11	8						
10.	3	6	6	6	9	7	8					
8.	3	5	5	5	8	6	7	12				
6.	2	4	4	4	6	5	6	10	8			
4.	2	3	3	3	5	4	5	8	7	b_4		
2.	1	2	2	2	3	3	4	6	6	b_2	c_2	
0.	1	1	1	1	2	2	3	4	5	b_0	c_0	d_0
	0.	2.	4.	6.	8.	10.	12.	14.	16.	18.	20.	22.

Here, in E_∞ the skew diagonal sums are equal to the ranks of the cohomologies of $BU(2) \times BU(2)$, i.e. $1, 2, 5, 8, 14, 20, 30, 40, 55, 70, 91, 112, \ldots$ This (and some arguments about differentials in the bottom left region) shows that $b_0 = 6$ which is consistent with the above, since the Σ^3 stratum does not occur here. We can go on with similar study and get numerical results on the strata not worse than Σ^2. The comparison of the original and this kind of $\mathcal{E}^0$(finite, finite) tables helps us in filling them. These investigations are also subject to future study.

References

[AVGL91] V. I. Arnold, V. A. Vassiliev, V. V. Goryunov, and O. V. Lyashko. *Singularities. Local and global theory.* Enc. Math. Sci. Dynamical Systems VI. Springer, 1991.

[DG83] A. Dimca and C. G. Gibson. Contact unimodular germs from the plane to the plane. *Quart. J. Math. Oxford (2)*, 34:281–295, 1983.

[Ful98] W. Fulton. *Intersection Theory.* Springer, 1984, 1998.

[Gaf83] T. Gaffney. The Thom polynomial of $\overline{\Sigma^{1111}}$. In *Singularities*, Proc. Symp. Pure Math. 40. Part I, pages 399–408. AMS, 1983.

[Jän78] K. Jänich. Symmetry properties of singularities of C^∞ functions. *Math. Ann.*, 238:147–156, 1978.

[Kaz97] M. É. Kazarian. Characteristic classes of singularity theory. In V. I. Arnold et al., editors, *The Arnold-Gelfand mathematical seminars: geometry and singularity theory*, pages 325–340, 1997.

[Por71] I. Porteous. Simple singularities of maps. In *Liverpool Singularities — Symposium I*, number 192 in SLNM, pages 286–307, 1971.

[PW95] A. du Plessis and C. T. C. Wall. *The geometry of topological stability*. Oxford University Press, 1995.

[Rima] R. Rimányi. On right-left symmetries of stable singularities. www.cs.elte.hu/anal/ricsi/cikkek.

[Rimb] R. Rimányi. Thom polynomials, symmetries and incidences of singularities. www.cs.elte.hu/anal/ricsi/cikkek.

[Rim96] R. Rimányi. *Generalized Pontrjagin-Thom construction for singular maps*. PhD thesis, Eötvös University, 1996.

[RS98] R. Rimányi and A. Szűcs. Pontrjagin-Thom type construction for maps with singularities. *Topology*, Vol. 37.(No 6):1177–1191, 1998.

[Szű79] A. Szűcs. Analogue of the Thom space for mapping with singularity of type Σ^1. *Math. Sb. (N. S.)*, 108 (150)(3):438–456, 1979.

[Szű80] A. Szűcs. Cobordism of maps with simplest singularities. In *Topology Symposium, Siegen*, SLNM 788, pages 223–244, 1980.

[Szű91] A. Szűcs. On the cobordism groups of immersions and embeddings. *Math. Proc. Camb. Phil. Soc.*, 109:343–349, 1991.

[Szű94] A. Szűcs. Cobordism groups of immersions of oriented manifolds. *Acta Math. Hungar.*, 64 (2):191–230, 1994.

[Tho56] R. Thom. Les singularités des applications différentiables. *Ann. Inst. Fourier 6*, pages 43–87, 1956.

[Vas88] V. A. Vassiliev. *Lagrange and Legendre Characteristic Classes*. Gordon and Breach, 1988.

[Wal80] C. T. C. Wall. A second note on symmetry of singularities. *Bull. London Math. Soc.*, 12:347–354, 1980.

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