SCHUR AND SCHUBERT POLYNOMIALS AS THOM POLYNOMIALS—COHOMOLOGY OF MODULI SPACES

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1. Introduction

In this paper we approach the theory of Schur and Schubert polynomials from the the point of view of generalized Thom polynomials. The starting point is that we realize Schur and Schubert polynomials—of Chern classes—as first obstructions of certain fibre bundles (in the spirit of Stiefel, Whitney and Steenrod who defined Chern classes as first obstructions). This realization makes Schur and Schubert polynomials a special case of the generalized Thom polynomials. Applying a general method to compute them as Thom polynomials we obtain new definitions for these polynomials. Along the way we also redefine the double Schubert polynomials and define "double Schur polynomials" which are also first obstructions. We show how these results are related to the structure of the cohomology ring of the Grassmannian and the flag manifold. The authors hope that this Thom polynomial technique may turn to be useful in the study of the cohomology ring structure of various moduli spaces, see section 6 for a discussion.

We will define Schur and Schubert polynomials as unique solutions of linear equations. This might seem somewhat implicit however it allows us to give closed formulas for the double Schur polynomials (Theorem 5.1) and to deduce the recursion formula of Lascoux and Schützenberger [11] for double Schubert polynomials (Theorem 4.2).

For geometrically defined fibre bundles—on which we mean fibre bundles associated to a principal G-bundle where G is a Lie group—first obstructions are usually called Thom polynomials (see [3] on this subject). The word polynomial is justified since the first obstruction of the universal bundle is an element in $H^*(BG)$ which is a polynomial ring, at least rationally.

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Calculating Thom polynomials has a long history. It was René Thom who initiated their study in the case of singularities of smooth maps. The major tool for calculating them was the method of resolution (see [1] for an account of the method and results).

Works of V. Vassiliev [15] and M. Kazarian [7], [9] clarified the connection of Thom polynomials with the underlying symmetry groups. Their works also show that the so called *degeneracy loci formulas* in algebraic geometry are also Thom polynomials for group actions (see [5] for many examples). In particular in [2] double Schubert polynomials are described as degeneracy loci formulas for certain quiver representations. In [4] we showed how to describe and calculate quiver type formulas as Thom polynomials for group actions. In this paper however we use a more "economic" group action. In Remark 6.9 we compare the two methods.

Based on works of A. Szűcs ([14]) the second author introduced a different method (called the method of restriction equations) to calculate Thom polynomials for singularities of smooth maps ([12]). In [3] we generalized the method to the case of Thom polynomials for group actions. It turned out that though the idea is quite simple the method of restriction equations is very powerful. We reproduced and improved earlier results in several directions ([3]). In all these applications we start from a representation of G with the property that G has only finitely many orbits (at least up to a given codimension). In the final section of this paper we make an attempt to generalize the definitions and methods to the case where continuous families of orbits occur—the case of "moduli". We also propose that via Thom polynomials we can calculate the cohomology rings of certain moduli spaces.

The idea of the paper was born from a question of Tamás Hausel who drew our attention to the problem of calculating cohomology rings of moduli spaces.

2. Thom polynomials for group actions

In this section we give a short introduction to the theory of Thom polynomials for group actions. They were defined by M. Kazarian in [9]. The approach given here is somewhat different since we concentrate on complex representations, see [3] for a detailed introduction along these lines.

Given a complex representation $\rho : G \to GL(V)$ and a principal G-bundle $P \to M$ we can look for obstruction of having a section of the V-bundle $E = P \times_G V$ associated to this representation *avoiding* a certain orbit η (or more generally a G-invariant subset of V). Of course the zero section avoids any orbit different from the zero orbit but this is pathological: we want obstructions for a *generic* section. In effect we want to avoid the *closure* of η. The obstruction we will deal with is the cohomology class represented by $\bar{\eta}(s) \subset M$ for a generic section s. Thus this class is an obstruction for having a section in the complement $V \setminus \bar{\eta}$. Let us call this class the Thom polynomial $\mathrm{Tp}(\eta)(E)$.

In [3] we explain that $\mathrm{Tp}(\eta)(E)$ is equal to the first obstruction class (in the homotopy theoretic sense) of the fibre bundle $P \times_G (V \setminus \bar{\eta})$. By naturality it is enough to look at the universal V-bundle $BV = EG \times_G V$ (where B refers to the Borel construction), and the value of the Thom polynomial here: let $\mathrm{Tp}(\eta)$ denote $\mathrm{Tp}(\eta)(BV) \in H^*(BG) = H^*_G(pt)$. This characteristic class $\mathrm{Tp}(\eta)$ can be thought as the G-equivariant Poincaré dual of $\bar{\eta}$ (see [3]).

One of the early examples is the case what Stiefel, Whitney and Steenrod studied (see e.g. [13]) to define Chern classes:

Example 2.1. Let $V = \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^{n+k})$ and let $G = GL(n+k)$ act on V by composition. We define a G-invariant subset of V:

$$\Sigma_1(k) := \{v \in V : \dim_{\mathbb{C}}(\ker v) = 1\}.$$

Then $\mathrm{Tp}(\Sigma_1(k))$ is the first obstruction of the bundle with fibre $V \setminus \bar{\Sigma}_1(k)$. This set can be identified with the set of n-frames in $\mathbb{C}^{n+k}$ i.e. the complex Stiefel manifold $St(n, n+k)$.

So given a vector bundle $E = P \times_{GL(n+k)} \mathbb{C}^{n+k}$, the cohomology class $\mathrm{Tp}(\Sigma_1(k))$ is the first obstruction for having n linearly independent sections of E. In other words if we have n sections $s_1, \dots, s_n$ in generic position then $\mathrm{Tp}(\Sigma_1(k))$ is the cohomology class represented by the subset of the base where $s_1, \dots, s_n$ are not linearly independent (notice that $s_1, \dots, s_n$ defines a section of $\mathrm{Hom}(\mathbb{C}^n, E) = P \times_{GL(n+k)} V$).

Theorem 2.2 (Stiefel, Whitney and Steenrod)**.** $\mathrm{Tp}(\Sigma_1(k)) = c_{k+1}$.

3. Calculation of Thom polynomials via the method of restriction equations

The G-equivariant Poincaré dual has similar properties as the ordinary Poincaré dual. Since $\mathrm{Tp}(\eta)$ is the G-equivariant Poincaré dual of $\bar{\eta}$ we get:

Proposition 3.1 ([3])**.** *Suppose that θ and η are orbits of ρ and $j_\theta^* : H_G^*(V) \to H_G^*(\theta)$ is induced by the inclusion $\theta \subset V$.*

Then

$$j_\theta^* \operatorname{Tp}(\eta) = \begin{cases} e(\nu_\eta) & \text{if } \theta = \eta \\ 0 & \text{if } \eta \not\subset \bar{\theta}, \end{cases}$$

where ν_η is the normal bundle of η in V and e denotes the equivariant Euler class.

Remark 3.2. ν_η is a G-equivariant bundle so it has an equivariant Euler class in $H_G^*(\eta)$. A simple calculation shows (see e.g. [3]) that $H_G^*(\eta) \cong H_{G_\eta}^*(pt)$ and $e(\nu_\eta)$ is equal to the Euler class of the representation of G_η on a normal space of η. The Euler class of a representation—using the description of $H_{G_\eta}^*(pt)$ as symmetric polynomials—is simply the product of all weights.

In certain cases $\operatorname{Tp}(\eta)$ is the unique solution of these equations. In fact even less equations are enough to determine $\operatorname{Tp}(\eta)$:

Theorem 3.3 ([3])**.** *Let $\rho : G \to GL(V)$ be a linear representation on a complex vector space V with finitely many orbits. Suppose that for every orbit η we have $e(\nu_\eta) \neq 0$. Then the restriction equations*

$$j_\theta^* \operatorname{Tp}(\eta) = \begin{cases} e(\nu_\eta) & \text{if } \theta = \eta & \textit{‘principal equation’} \\ 0 & \text{if } \theta \neq \eta,\ \operatorname{codim}\theta \leq \operatorname{codim}\eta & \textit{‘homogeneous equations’} \end{cases}$$

have a unique solution.

4. Schubert polynomials

Let the group $G := ◹(n) \times ◺(n)$—where $◹(n) = \{$upper triangular n by n matrices$\}$ and $◺(n) = \{$lower triangular n by n matrices$\}$—act on the vector space $V := \operatorname{Hom}(\mathbb{C}^n, \mathbb{C}^n)$ by $(Y, X) \cdot M := XMY^{-1}$. The orbits of this action are in a one-to-one correspondence with the rook-arrangements on an n by n chessboard (c.f. Bruhat-form of matrices). A rook-arrangement is a number of rooks placed on a chessboard with the property that no two are in one row or column. Associated to such a rook-arrangement we consider the $n \times n$ matrix where 1's stand for rooks and 0's otherwise. We will encode such a rook-arrangement by a permutation $\pi \in S_m$ for some $m \geq n$ as follows (and as is in [6, p.9.]). Expand the matrix (to the right and down) to an $m \times m$ matrix and add 1's to rows which do not have any, starting from the top down, putting a 1 in the left-most column outside the $n \times n$ matrix that does not yet have a 1 in it. Then set $\pi(i) = j$ if in the i^{th} row the 1 is in the j^{th} column. For example for the rook-arrangement $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ the permutation $\pi \in S_3$ is $2, 3, 1$ or $\pi = 2, 3, 1, 4, 5, 6$ in S_6. The ambiguity of which S_m is considered π in will not effect our reasoning or formulas.

Let M_π denote the $n \times n$ rook-arrangement matrix encoded by the permutation $\pi \in S_{m \geq n}$ and let Σ_π be its orbit. Easy computation (matrix multiplication) shows that the tangent space to Σ_π at M_π can be obtained as follows. Let T_π be the set of boxes in the matrix which are either directly below or right of any 1 in M_π (including where the 1's are). The set of the remaining boxes will be denoted by N_π. (Observe that N_π does not depend on n only on $\pi \in S_m$.) Then the mentioned tangent space is $\mathbb{C}^{T_\pi}$, so for a normal slice to its orbit at M_π we can take $\mathbb{C}^{N_\pi}$. This also gives us the codimension of the orbit Σ_π:

$$\operatorname{codim} \Sigma_\pi = |N_\pi| = |\{(i,j) | \text{there are 0's directly above and to the left of } (i,j) \text{ in } M_\pi\}| =$$

$$= |\{(i,j) | \pi(i) > j, \pi(l) \neq j \text{ for all } l = 1, 2, \ldots, i-1\}| =$$

$$= |\{(i,j) | \pi(i) > j, \pi^{-1}(j) > i\}| = l(\pi),$$

where the length $l(\pi)$ of π is the number of inversions in π, i.e. the number of $i < j$ such that $\pi(i) > \pi(j)$.

According to the general theory now we need to determine the maximal compact symmetry group of M_π together with its representation on an invariant normal slice at M_π to Σ_π. This is also an easy computation. For the sake of simplicity in our formulas, instead of considering the actual maximal compact symmetry group we will *map* a group onto the maximal compact symmetry group. Here $\operatorname{diag}(a, b, \ldots)$ means the diagonal matrix with $a, b, \ldots$ in the diagonal.

Proposition 4.1. *The homomorphism*

$$G_\pi := \{\operatorname{diag}(\alpha_1, \alpha_2, \ldots, \alpha_m) : \ |\alpha_i| = 1\} \ \to \ \big\{\big(\operatorname{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n), \operatorname{diag}(\alpha_{\pi(1)}, \alpha_{\pi(2)}, \ldots, \alpha_{\pi(n)})\big)\big\}$$

maps $G_\pi \cong U(1)^m$ onto the maximal compact symmetry group of M_π.

Fortunately enough the normal slice $\mathbb{C}^{N_\pi}$ is invariant under the action of G_π, so now we have the two ingredients of the restriction equations:

- the map
$$\begin{array}{ccc} H^*(BG) & \xrightarrow{j_\pi^*} & H^*(BG_\pi) \\ \downarrow \cong & & \downarrow \cong \\ \mathbb{Z}[y_1,\dots,y_n,x_1,\dots,x_n] & & \mathbb{Z}[a_1,\dots,a_m] \end{array}$$
is given by $y_i \mapsto a_i,\ x_i \mapsto a_{\pi(i)}$.
- the Euler class of the representation of G_π on $\mathbb{C}^{N_\pi}$ is

$$e_\pi = \prod_{(i,j)\in N_\pi} \big(j_\pi^*(x_i) - j_\pi^*(y_j)\big) = \prod_{(i,j)\in N_\pi} (a_{\pi(i)} - a_j). \tag{1}$$

Observe that this latter Euler class is never zero (since $(i,j) \in N_\pi$ implies $\pi(i) > j$), so according to Theorem 3.3 the restriction equations are uniquely solvable, giving the Thom polynomial $\mathrm{Tp}(\pi) := \mathrm{Tp}(\Sigma_\pi)$ associated to the orbit Σ_π. If we consider a permutation $\pi \in S_n$ in a larger permutation group S_m (where the permutation acts by identity on the extra $m-n$ elements) then the equations do not change, so Tp does not change either. That is, the map Tp is really defined on $S_\infty := \cup S_i$ (here $S_i \subset S_j$ is meant if $i < j$).

These polynomials have already been studied in geometry and named *double Schubert polynomials*. So our approach can be regarded as a new way to compute double Schubert polynomials as solutions of (large) linear equation systems.

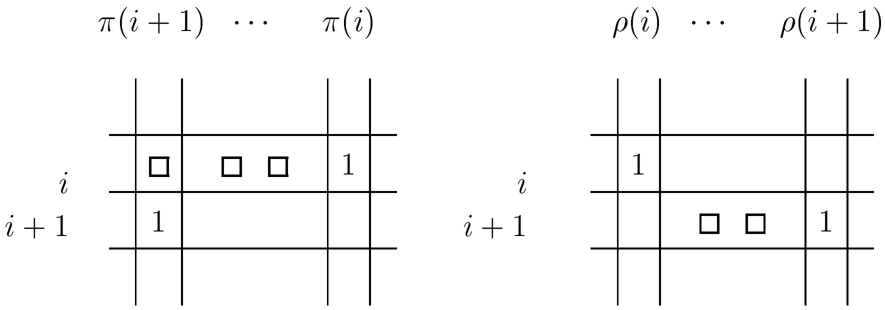
In enumerative geometry the standard procedure is to compute double Schubert polynomials by recursion. In the rest of this section we will show that these recursion formulas occur naturally in our approach.

Theorem 4.2. *The recursion formulas of Lascoux and Schützenberger ([11]) hold for* $\mathrm{Tp}(\pi)$*'s, i.e.*

1. $\mathrm{Tp}(n, n-1, \dots, 2, 1) = \prod_{i+j<n} (x_i - y_j)$*;*
2. *if for some i the inequality $\pi(i) > \pi(i+1)$ holds and $\rho = \pi \cdot s_i$, where s_i is the transposition $(i, i+1)$, then*
$$\mathrm{Tp}(\rho) = \partial_i \,\mathrm{Tp}(\pi) := \frac{\mathrm{Tp}(\pi) - s_i(\mathrm{Tp}(\pi))}{x_i - x_{i+1}},$$
where $s_i(\mathrm{Tp}(\pi))$ is obtained from $\mathrm{Tp}(\pi)$ by interchanging x_i and x_{i+1}.

Proof. Let π_n be the permutation $n, n-1, \dots, 2, 1$, and let $E_{i,j}$ be the matrix which is 1 at the (i,j) position and 0 otherwise. Matrix multiplication shows that any matrix in the orbit of M_{π_n} has zeros in the (i,j) entries if $i+j<n$. It means that for such (i,j)'s $E_{i,j}$ does not belong to the closure of Σ_{π_n}, so the homogeneous equation $j^*_{\text{orbit of } E_{i,j}}(\mathrm{Tp}(\pi_n))=0$ holds. This j^* assigns different indeterminates to x_k's and y_l's $(k,l=1,\ldots,n)$ except x_i and y_j are both mapped to a_j. This means that $\mathrm{Tp}(\pi_n)$ must be divisible by (x_i-y_j), and so in effect it must be a constant times $\prod_{i+j<n}(x_i-y_j)$. The constant is calculated to be 1 using the principal equation $j^*_{\pi_n}(\mathrm{Tp}(\pi_n))=\prod_{i+j<n}(a_{n+1-i}-a_j)$.

Now we turn to the proof of (2). For this let us study the matrices M_π and M_ρ. On the figure below we show the parts in which they differ:



M_π $\qquad$ M_ρ

Here the boxes correspond to N_π and N_ρ respectively. This shows that $\mathrm{codim}(\Sigma_\rho)$ is one less than $\mathrm{codim}(\Sigma_\pi)$. Also we can keep track of how the corresponding Euler classes change: $e_\pi=e_\rho\cdot(a_{\pi(i)}-a_{\pi(i+1)})$ (c.f. formula (1)).

To prove the statement we need to prove the principal and the homogeneous equations (corresponding to ρ) for $\partial_i\,\mathrm{Tp}(\pi)$. The principal one is easily verified as follows

$$j^*_\rho(\partial_i\,\mathrm{Tp}(\pi))=\frac{j^*_\rho(\mathrm{Tp}(\pi))-j^*_\rho(s_i\,\mathrm{Tp}(\pi))}{a_{\rho(i)}-a_{\rho(i+1)}}=\frac{0-j^*_\pi(\mathrm{Tp}(\pi))}{a_{\pi(i+1)}-a_{\pi(i)}}=\frac{-e(\pi)}{a_{\pi(i+1)}-a_{\pi(i)}}=e(\rho).$$

For the homogeneous ones let $\tau\neq\rho$ be a permutation with length not greater than that of ρ. Then we have that $\mathrm{codim}(\Sigma_\tau)\le\mathrm{codim}(\Sigma_\rho)=\mathrm{codim}(\Sigma_\pi)-1$, so $\mathrm{codim}(\Sigma_{\tau s_i})=\mathrm{codim}(\Sigma_\tau)-1\le\mathrm{codim}(\Sigma_\pi)$ and $\tau s_i\neq\pi$. Then

$$j^*_\tau(\partial_i\,\mathrm{Tp}(\pi))=\frac{j^*_\tau(\mathrm{Tp}(\pi))-j^*_\tau(s_i\,\mathrm{Tp}(\pi))}{a_{\rho(i)}-a_{\rho(i+1)}}=\frac{0-j^*_{\tau s_i}(\mathrm{Tp}(\pi))}{a_{\rho(i)}-a_{\rho(i+1)}}=\frac{0-0}{a_{\rho(i)}-a_{\rho(i+1)}}=0.$$

This proves the theorem. ∎

Remark 4.3. There were some awkward points in our reasonings—e.g. a matrix of size $n \times n$ was encoded by a permutation in S_m greater than n (and m was not even fixed) or that we did not describe the maximal compact symmetry group of M_π explicitly but as an image of a homomorphism. These could have been avoided by considering the 'infinite' case from the very beginning: infinitely big triangular matrices act on the space of infinitely big matrices. Then the orbits correspond exactly to elements in S_∞. And the described homomorphism to the symmetry group is an isomorphism. However doing that we would face other technical obstacles, such as: the described symmetry group would not be compact and there would be infinitely many equations for infinitely many indeterminates, etc. Although these problems can be overcome they would make the reasoning even more technical.

Remark 4.4. To obtain the orbits Σ_π we could have started with a smaller group action. If we act (e.g.) from the left by *unipotent* triangular matrices—i.e. ones having 1 in the diagonal—then the orbits are the same, and we expect the 'single' Schubert polynomials as solutions. However carrying out the calculation one finds that in this case the Euler classes are 0, so our method does not work: the solutions of the restriction equations are not unique. This phenomenon, i.e. 'the addition of extra symmetries makes the method work' was noticed by M. Kazarian too in [8].

5. Schur polynomials

Let $d \leq n$, and let the group $G := GL(d) \times \text{◺}(n)$ act on the vector space $V := \mathrm{Hom}(\mathbb{C}^d, \mathbb{C}^n)$ by $(Y, X) \cdot M := XMY^{-1}$. The action of $GL(d)$ in the source geometrically means that two linear maps are in one orbit if their image is the same. The action of the triangular matrices (= the automorphism group of a complete flag) geometrically means that two linear subspaces are equivalent if they have the same intersection structure with the complete flag. So the orbits of this action are exactly the so called Schubert cells of the Grassmannian $\mathrm{Gr}_r(\mathbb{C}^n)$ for $r \leq d$. Encoding a Schubert cell by a partition as usual we have that orbits are in a 1-to-1 correspondence with the set

$$\{(r, \lambda') \mid r = 0, \dots, d; \lambda' \subset (n-r)^r \text{ a partition}\}$$

It will be convenient to index the orbits not with such a pair (r, λ') but with a single partition, so we embed the above set in

$$\{\lambda \subset n^d\}$$

by attaching a $d-r$ by $n-r$ rectangle above λ', i.e.

$$\lambda_i(r, \lambda') = \begin{cases} n-r & \text{if } \ i \leq d-r \\ \lambda'_{i-(d-r)} & \text{if } \ i > d-r. \end{cases}$$

We will also associate a 'strict partition' μ to λ by $\mu_i := \lambda_{d+1-i} + i$. Then

$$1 \leq \mu_1 < \mu_2 < \ldots < \mu_d \leq n+d.$$

Let the orbit corresponding to λ be called Σ_λ. A representative of this is the n by d matrix M_λ having 1 at (μ_i, i) (if $\mu_i \leq n$) and 0 everywhere else. We should not forget about the (μ_i, i) entries when this does not fit into the n by d matrix. We imagine them outside M_λ. Observe that they form a $d-r$ by $d-r$ identity matrix below M_λ as in Figure 1.

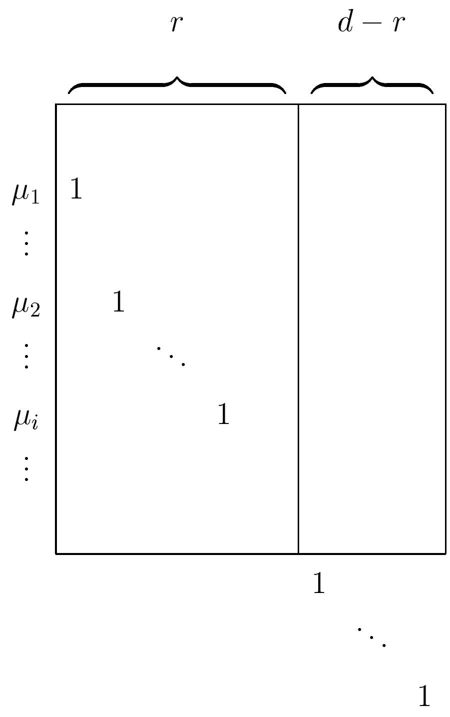


FIGURE 1. The matrix M_λ

Computing the image of the Lie algebra of G in the tangent space of V at M_λ (i.e. carrying out matrix multiplications) we find that the tangent space of Σ_λ at M_λ is $\mathbb{C}^{T_\lambda}$ where T_λ is the set of boxes which are directly below or in the row of an entry 1 in M_λ. Denoting the set of the remaining boxes by N_λ it is clear that $\mathbb{C}^{N_\lambda}$ is a normal slice to Σ_λ at M_λ. As a byproduct we have that

$$\operatorname{codim}\Sigma_\lambda = |N_\lambda| = \sum |N_\lambda \cap i^{\text{th}} \text{ column}| = \lambda_d + \ldots + \lambda_2 + \lambda_1 = |\lambda|.$$

Now we need to determine the maximal compact symmetry group of M_λ together with its representation on an invariant normal slice at M_λ to Σ_λ. Easy computation (e.g. using the symmetry group calculations for $GL(n) \times GL(n)$ as in [3] and restricting to G) shows that

$$G_\lambda := \Big\{ \big(\operatorname{diag}(\alpha_{\mu_1}, \alpha_{\mu_2}, \ldots, \alpha_{\mu_r}) \oplus A, \operatorname{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n) \big) : |\alpha_i| = 1, A \in U(d-r) \Big\}$$

$$\cong U(1)^n \times U(d-r)$$

is the maximal compact symmetry group. It keeps $\mathbb{C}^{N_\lambda}$ invariant and its maximal torus acts on $\mathbb{C}^{N_\lambda}$ with the representation

$$\bigoplus_{(i,j)\in N_\lambda} \varepsilon_i \otimes \overline{\varepsilon_{\mu_j}},$$

where ε_k is the standard 1-dimensional representation of the k^{th} $U(1)$ and ε_{μ_j} for $j > r$ correspond to the maximal torus of the $U(d-r)$ part. From these we can compute

- the map
$$\begin{array}{ccc} H^*(BG) & \xrightarrow{\quad j_\lambda^* \quad} & H^*(BG_\lambda) \\ \downarrow \cong & & \downarrow \cong \\ \mathbb{Z}[x_1, \ldots, x_n, y_1, \ldots, y_d]^{S_d} & & \mathbb{Z}[a_1, \ldots, a_n, a_{n+1}, \ldots, a_{n+d-r}]^{S_{d-r}} \end{array}$$
  is given by $x_i \mapsto a_i,\ y_i \mapsto a_{\mu_i}$.
- the Euler class of the representation of G_λ on $\mathbb{C}^{N_\lambda}$ is
$$e_\lambda = \prod_{(i,j)\in N_\lambda} (a_i - a_{\mu_j}).$$

Observe that this latter is never zero, so according to Theorem 3.3 the restriction equations are uniquely solvable, giving the Thom polynomial $\operatorname{Tp}(\lambda) := \operatorname{Tp}(\Sigma_\lambda)$ associated to the orbit Σ_λ. Variants of these polynomials have been extensively studied in geometry, and by analogy with them let us call $\mathrm{Tp}(\lambda)$ the "double Schur polynomial" associated to λ. Our approach (the solution of the restriction equations) is thus a way of computing double Schur polynomials.

In the rest of this section we show a closed form for $\mathrm{Tp}(\lambda)$.

Theorem 5.1.

$$\mathrm{Tp}(\lambda) = \Delta_\lambda(c^{(d)}, c^{(d-1)}, \dots, c^{(1)}),$$

where $c^{(j)} = c_0^{(j)} + c_1^{(j)}t + c_2^{(j)}t^2 + \dots$ *is the Taylor expansion of*

$$\frac{\prod_{i<\mu_j}(1+x_i t)}{\prod_{i=1}^d (1+y_i t)},$$

and Δ_λ *is the determinant of* $\left(c_{\lambda_i+j-i}^{(d+1-i)}\right)_{d\times d}$.

Proof. First we show that the principal equation holds, i.e. the j_λ^* image of $\Delta_\lambda(c^{(d)}, c^{(d-1)}, \dots, c^{(1)})$ is equal to the Euler class $e(\lambda)$. We have

$$j_\lambda^*\big(\Delta_\lambda(c^{(d)}, c^{(d-1)}, \dots, c^{(1)})\big) = \Delta_\lambda\Big(\frac{\prod_{N_d}(1+a_j t)}{(1+a_{\mu_d}t)}, \frac{\prod_{N_{d-1}}(1+a_j t)}{(1+a_{\mu_d}t)(1+a_{\mu d-1}t)}, \dots, \frac{\prod_{N_1}(1+a_j t)}{\prod_{j=1}^d (1+a_{\mu_j}t)}\Big),$$

where $N_i = \{j < \mu_i, j \neq \mu_l \forall l\}$. (Let us remark that $N_\lambda = \bigcup N_i \times \{i\}$.) In this latter determinant carry out the following column transformations: add a_{μ_d} times the $j-1^{\text{st}}$ column to the j^{th} one for $j = d, d-1, \dots, 2$ (in this order). Then—keeping in mind that

$$\frac{\prod_{N_i}(1+a_j t)}{\prod_{j=i}^d (1+a_{\mu_j}t)}\Big|_v + a_{\mu_d}\frac{\prod_{N_i}(1+a_j t)}{\prod_{j=i}^d (1+a_{\mu_j}t)}\Big|_{v-1} = \frac{(1+a_{\mu_d}t)\prod_{N_i}(1+a_j t)}{\prod_{j=i}^d (1+a_{\mu_j}t)}\Big|_v = \frac{\prod_{N_i}(1+a_j t)}{\prod_{j=i}^{d-1} (1+a_{\mu_j}t)}\Big|_v,$$

we get the matrix

$$\begin{pmatrix} \frac{\prod_{N_d}(1+a_j t)}{(1+a_{\mu_d}t)}\Big|_{\lambda_1} & 0 \\ * & \square \end{pmatrix},$$

where the bottom right $d-1$ by $d-1$ matrix is exactly the one whose determinant is

$$\Delta_{\lambda\setminus\{\lambda_1\}}(c^{(d-1)}, \dots, c^{(1)}).$$

The top left entry is exactly the resultant of the enumerator and the denominator, so it is $\prod_{N_d}(a_j - a_{\mu_d})$. So either going on with similar column transformations or by induction we get

$$\prod_{i=1}^d \prod_{j\in N_i}(a_j - a_{\mu_i}),$$

which is $e(\lambda)$, what we wanted to prove.

Now we prove the homogeneous equations, that is if $\operatorname{codim}\Sigma_\lambda = |\lambda| \geq \operatorname{codim}\Sigma_{\eta'} = |\lambda'|$, $\lambda \neq \lambda'$ then applying $j^*_{\lambda'}$ to the formula in the theorem yields 0. Let the strict partition assigned to λ and λ' be μ and μ' respectively. We will use the following index sets

$$\begin{array}{rlrl} T_0 := & \{\mu'_l \mid \mu'_l < \mu_1\} & S_0 := & \{j < \mu_1\} \setminus T_0 \\ T_1 := & \{\mu'_l \mid \mu_1 \leq \mu'_l < \mu_2\} & S_1 := & \{\mu_1 \leq j < \mu_2\} \setminus T_1 \\ & \vdots & & \vdots \\ T_d := & \{\mu'_l \mid \mu_d \leq \mu'_l\} & S_d := & \{\mu_d \leq j\} \setminus T_d \end{array}$$

Applying $j^*_{\lambda'}$ to the formula in the theorem we get

$$\Delta_\lambda\Big(\frac{S_0T_0S_1T_1\cdots S_{d-1}T_{d-1}}{T_0T_1\cdots T_d}, \frac{S_0T_0S_1T_1\cdots S_{d-2}T_{d-2}}{T_0T_1\cdots T_d}, \dots, \frac{S_0T_0}{T_0T_1\cdots T_d}\Big) =$$

$$\Delta_\lambda\Big(\frac{S_0S_1\cdots S_{d-1}}{T_d}, \frac{S_0S_1\cdots S_{d-2}}{T_{d-1}T_d}, \dots, \frac{S_0}{T_1T_2\cdots T_d}\Big),$$

where for simplicity instead of $\prod_{j\in J}(1+a_jt)$ we only wrote J in the formulas.

Now we are going to apply column operations as earlier: adding a_i times the $v-1^{\text{st}}$ column to the v^{th} one. First do it with the greatest index element of T_d and $v = d, d-1, \dots, 2$. Then with the next a_i in T_d and $v = n, n-1, \dots, 3$, and so on. Keeping our eyes on how the entries change (see in the first part of the proof) wee see that in the first row there will be no denominators after the $|T_d|^{\text{th}}$ element. The degree of the enumerator will be $\sum_{i=0}^{d-1}|S_i| = \lambda_1 - 1 + |T_d|$. After the $|T_d|^{\text{th}}$ element we have to take the (at least) $\lambda_1 + |T_d|^{\text{th}}$ coefficient. So, we see that after the $|T_d|^{\text{th}}$ element we actually get 0.

Now forget the first $|T_d|$ column, and consider only the rest. Here the a_i's from T_d already disappeared. Now do the same column operations as before, but now with the elements of $|T_{d-1}|$. As above we get that after the $|T_d| + |T_{d-1}|^{\text{th}}$ element (in the original matrix) in the second row we get polynomials with degree $\sum_{i=0}^{d-2}|S_i| = \lambda_2 - 2 + |T_d| + |T_{d-1}|$. After the $|T_d| + |T_{d-1}|^{\text{th}}$ element we have to take at least the $\lambda_2 - 1 + |T_d| + |T_{d-1}|^{\text{th}}$ coefficients, which are thus all zero.

Carrying out this procedure with all rows, we arrive at a matrix which is zero in the i^{th} row after the $\sum_{j=d-i+1}^{d}|T_j|^{\text{th}}$ element. In other words, a (reflected) Young diagram is covered from a matrix (i.e. 0's stand there), and we would like to conclude that the determinant of the matrix is 0. We can certainly conclude so—in the case (and only in the case) when this Young diagram covers at least one box of the diagonal. That is, the determinant is 0 unless $|T_d| > 0$ and $|T_d| + |T_{d-1}| > 1, \dots$. But if these all hold then the definitions of the index sets T_i gives that either $\lambda = \lambda'$ or $|\lambda'| > |\lambda|$, which we wanted to prove. $\square$

Remark 5.2. The ordinary Schur polynomials can be obtained from our double Schur polynomials by substituting 0 for all x_i. This can be seen either from the definition (only *unipotent* triangular matrices act in the target space) or from the concrete form of these polynomials. Interestingly enough the ordinary Schur polynomials can not be obtained directly by our method (c.f. Remark 4.4) only via the double ones.

6. Cohomology of moduli spaces

The computation of the cohomology ring of various quotients spaces is a hot area in numerous branches of geometry and topology, see e.g. [10]. In this section we show how the theory of Thom polynomials for group actions can be applied to this problem; and we also show how this approach works in the case of the Grassmannians.

6.1. **Obstructions as relations.** The idea is very simple: Suppose that X is a G-space. Then we have a map $j_X^* : H_G^*(pt) \to H_G^*(X)$ induced by the G-equivariant map $X \to pt$. In [3] we called the kernel $\mathcal{O}_X$ of this map the *obstruction ideal* of X since the elements of $\mathcal{O}_X$ are the G-characteristic classes which are obstructions for having a section of an X-bundle associated to a principal G-bundle. If in addition the map j_X^* is surjective, then the generators of $H_G^*(pt)$ are generators for the ring $H_G^*(X)$, too and the elements of $\mathcal{O}_X$ are the relations for $H_G^*(X)$. If the G-action is free, then $H_G^*(X) \cong H^*(X/G)$ i.e. we calculate the ordinary cohomology ring of the space X/G. (In this case X itself is a principal G-bundle and elements of $\mathcal{O}_X$ are the vanishing G-characteristic classes of X.)

6.2. **Using Thom polynomials.** If there is a geometric realization of the map $X \to pt$ i.e. a G-equivariant embedding $i_X : X \to V$ into a contractible G-space V—in other words X is an invariant subset of a representation of G—then we can use Thom polynomial methods to generate $\mathcal{O}_X$: It is easy to see that Thom polynomials of orbits *outside* of X are in $\mathcal{O}_X$. However we have to extend the definition for the case where continuous families of orbits occur.

6.3. **Thom polynomials for moduli.**

Definition 6.1. *A G-invariant stratification Ξ of V is called a* Vassiliev stratification *if for every $S \in \Xi$ the projection $p : S \to S/G = M$ is a fibre bundle over the manifold M.*

If M has positive dimension we call it *moduli space*. Suppose now that $p : S \to S/G = M$ is a complex algebraic map and the manifold M admits a complex algebraic cell stratification $C = \{C_i\}$ (i.e. the cells C_i are complex submanifolds and their closures are complex algebraic varieties). Since in this case the G-invariant subsets $p^{-1}(\bar{C}_i) \subset V$ are also complex algebraic varieties they have Thom polynomials ([3]) and we can define a homomorphism:

Definition 6.2. *The* Thom polynomial map $\mathrm{Tp}_M : H^*(M) \to H_G^{*+d}(pt)$ *is defined by*

$$\mathrm{Tp}_M([C_i]) := \mathrm{Tp}\left(p^{-1}(C_i)\right),$$

where $[C_i]$ denotes the cohomology class defined by the cell C_i in the cellular cohomology of the cell decomposition C.

Remark 6.3. In general M doesn't have a complex cell decomposition. Then take any cell decomposition $C = \{C_i\}$ which is a stratification of M at the same time. Extend this stratification into a strict Vassiliev stratification Ξ of V (i.e. we additionally assume that for every $S \in \Xi$ the manifold S/G is contractible). Then we have a map from the group of cellular cochains $\mathcal{C}^*(C)$ to $E_1^{0,*+d}$ of the Kazarian spectral sequence of Ξ (i.e. the cohomology spectral sequence of the induced filtration, see [9]). If this map is a cochain map (see Example 6.5 for a counterexample) then it induces a map $H^*(M) \to E_2^{0,*+d}$. Composing with the edge homomorphism we can define Tp_M.

Examples 6.4. In the following cases the Thom polynomial map can be defined:

(i) Let $V = \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n)$ as in Section 4 but restrict the action to the subgroup $\nabla(n)$. The orbit of a map $\varphi \in V$ is determined by the image of the standard flag. So a natural stratification of V is $V = \bigcup S_\lambda$ where $S_\lambda = \{\varphi \in V : \dim \varphi(\mathbb{C}^i) = \lambda_i\}$. The moduli space $S_\lambda/\nabla(n)$ is a partial flag manifold $\mathrm{Fl}_\lambda(\mathbb{C}^n)$. In particular the moduli space corresponding to the open stratum is the full flag manifold $\mathrm{Fl}(\mathbb{C}^n)$.

We can see that the orbits of $\text{◹}(n) \times \text{◺}(n)$ correspond to the Schubert cells of $\mathrm{Fl}_\lambda(\mathbb{C}^n)$ so not only they give a complex algebraic cell stratification but we can calculate their Thom polynomials by restricting the $\text{◹}(n) \times \text{◺}(n)$ Thom polynomials to $H^*_{\text{◹}(n)}(pt)$. In other words we substitute $y_i = 0$ into the $\text{◹}(n) \times \text{◺}(n)$ Thom polynomials.

This method can be generalized: Whenever we can enlarge the the symmetry group such that the moduli space becomes the union of finitely many orbits of the larger group we have a chance to calculate Tp_M. In an ideal situation the orbit decomposition defines a complex algebraic cell stratification of M.

(ii) Let $V = \mathrm{Hom}(\mathbb{C}^d, \mathbb{C}^n)$ as in Section 5 but restrict the action to the subgroup $GL(d)$. The orbit of a map $\varphi \in V$ is determined by the image $\mathrm{Im}(\varphi)$. So a natural stratification of V is $V = \bigcup \Sigma_i$ where $\Sigma_i = \{\varphi \in V : \dim\ker(\varphi) = i\}$. The moduli space $\Sigma_i/GL(d) \cong \mathrm{Gr}_{d-i}(\mathbb{C}^n)$. Again the Schubert cells give a complex algebraic cell stratification and the Thom polynomial maps can be calculated by restricting the $GL(d) \times \text{◺}(n)$ Thom polynomials calculated in Section 4.

(iii) Generalizing case (i) let $V = V(Q)$ be a quiver representation of A_n-type (see [4]) and forget the action on the last vertex space $\mathbb{C}^{d_n}$. Then the moduli spaces are partial flag manifolds as in (i) but the moduli space corresponding to the open stratum is also a partial flag manifold. Now if we enlarge the symmetry group by letting $\text{◺}(d_n)$ act on the last vertex space $\mathbb{C}^{d_n}$ then we get a situation analogous to case (i). (Notice that if we forget the action on the last vertex space of a representation finite quiver different from A_n then the moduli spaces are not compact and don't admit complex algebraic cell stratifications.)

Example 6.5. This example shows how bad the situation can be even in a very simple case. Let $G = GL(1)$ acting on $V = \mathbb{C}^2$ by $\rho(\alpha) = \begin{pmatrix} \alpha & \\ & \alpha^{-1} \end{pmatrix}$. There are three exceptional orbits: the zero orbit, $\eta_x = \{(x, y) : x \neq 0,\ y = 0\}$ and $\eta_y = \{(x, y) : y \neq 0,\ x = 0\}$. And there are the orbits $\eta_c = \{(x, y) : xy = c\}$ for $c \neq 0$. So a Vassiliev stratification can be $0, \eta_x, \eta_y, S = \cup\eta_c$. The moduli space $S/G \cong \mathbb{C}^\times$ has a cell stratification $C_0 = \mathbb{C} \setminus [0, \infty),\ C_1 = (0, \infty)$. Though C_1 is a cocycle in the cellular cochain complex and $[C_1]$ generates $H^1(M) \cong \mathbb{Z}$ but $\delta_1 p^{-1}(C_1) = \eta_x - \eta_y$ in the Vassiliev complex $(E_1^{0,*}, \delta_1)$. So it cannot have a Thom polynomial. (It is easy to construct a transversal section s of the trivial $\mathbb{C}^2$-bundle over the two-torus T^2 such that $[\overline{s^{-1}(C_1)}]$ is not zero in $H^1(T^2)$. On the other hand for a trivial bundle any Thom polynomial has to be zero.)

It might be tempting to ask what is the Thom polynomial of a circle around 0 in M, but it represents $0 \in H^1(M)$ so its Thom polynomial has to be zero by trivial reasons.

6.4. **Surjectivity.** As we mentioned at the beginning of the section, surjectivity of the map $j_X^* : H_G^*(pt) \to H_G^*(X)$ is an essential step in calculating $H_G^*(X)$. Using the Thom polynomial map we can prove surjectivity for the zero codimensional moduli space in cases of Examples 6.4:

Proposition 6.6. *Suppose that for an open and G-invariant $X \subset V$ the G-action on X is free and $M = X/G$ admits a complex algebraic cell stratification. Then the map $j_X^* : H_G^*(pt) \to H_G^*(X) \cong H^*(M)$ is surjective.*

Proof. $E = X \times_G V$ has a tautological section which is transversal since X is an open subset of V (notice that transversality will fail for higher codimensional moduli). It shows that $\mathrm{Tp}_M(\alpha)(E) = j_X^*(\mathrm{Tp}_M(\alpha)) = \alpha$, i. e. Tp_M is a left inverse to j_X^*. □

Remark 6.7. The proof of Proposition 6.6 also shows that in the Grassmannian $\mathrm{Gr}_n(\mathbb{C}^{n+k})$ the Poincaré dual $[\sigma_\lambda]$ of the Schubert variety σ_λ is equal to $\Delta_\lambda(c_1, \dots, c_n)$ where c_i are the Chern classes of the universal $\mathbb{C}^n$-bundle over $\mathrm{Gr}_n(\mathbb{C}^{n+k})$, so we gave an independent proof of the classical Schubert calculus. Similarly the single Schubert polynomials $\mathrm{Schubert}_\pi(x_1, \dots, x_n)$—when we substitute $y_i = 0$ into the double Schubert polynomials—expresses the Poincaré duals of the Schubert varieties σ_π of the flag manifold $\mathrm{Fl}(\mathbb{C}^n)$ in terms of $x_i = c_1(L_i)$ where L_i is the i^{th} tautological line bundle over the flag manifold.

6.5. **Cohomology of the Grassmannian.** According to Section 5 we can calculate the Thom polynomial maps for Example 6.4.(ii):

$$\mathrm{Tp}_{M_r}(\sigma_{\lambda'}) = \Delta_{\lambda(r,\lambda')}|_{\{x_i=0\}}.$$

As we mentioned, all Thom polynomials coming from the higher codimensional moduli are relations for $H^*(M_0)$. In fact, Thom polynomials of M_1 already generate the ideal of relations $\mathcal{O}(M_0)$:

Theorem 6.8. *For any $0 < d \leq n$*

$$H^*(M_0) \cong \operatorname{Im} \mathrm{Tp}_{M_0} / \operatorname{Im} \mathrm{Tp}_{M_1}.$$

Proof. There are well known descriptions of the ideal I for which $H^*(M_0) = \mathbb{C}[c_1, \dots, c_d]/I$, see e.g. [6, p.27.]. One such is $I = \oplus_{\lambda \not\subset (n-d)^d} \mathbb{Z}\Delta_\lambda(1/c)$. Another is $I = (\Delta_{(n-d+i)}(1/c) | i = 1, \dots, d)$ (ideal generators). What we have to prove is that I is also equal to $J := (\Delta_\lambda(1/c) | \lambda \subset (n-d+1)^d, \lambda_1 = n-d+1)$. According to the first description of I clearly $J \subset I$. To prove the converse observe that $\Delta_{(n-d+i)}(1/c) = \Delta_{1^{n-d+i}}(c)$ and in the latter determinant carry out the Lagrange expansion corresponding to the first $i-1$ columns. In each term we get a polynomial times an $(n-d+1) \times (n-d+1)$ determinant, and this latter determinant is verified to be of the form $\Delta_{\lambda^T}(c) = \Delta_\lambda(1/c)$, with $\lambda \subset (n-d+1)$, $\lambda_1 = n-d+1$. This proves $J \supset I$, so our theorem. [Remark that an even finer analysis of this proof gives an even more economical ideal generator system for I, namely $\Delta_{(n-d+1,1^i)}(1/c)$, $i = 0, \dots, d-1$.] $\square$

Remark 6.9. In [2] there is a different description of the double Schubert polynomials. They use the quiver representation $G(Q) = \mathsf{X}_{i=1}^n GL(i) \times \mathsf{X}_{i=n}^1 GL(i)$ acting on

$$V(Q) = \bigoplus_{i=1}^{n-1} \mathrm{Hom}(\mathbb{C}^i, \mathbb{C}^{i+1}) \oplus \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n) \oplus \bigoplus_{i=n-1}^{1} \mathrm{Hom}(\mathbb{C}^{i+1}, \mathbb{C}^i).$$

Looking at the map $V = \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n) \to V(Q)$ we can see that the the two methods should give the same result.

Remark 6.10. It would be interesting to find analogues of Theorem 6.8 for the flag manifold (Example 6.4 (i)), for partial flags (Example 6.4(iii)), for analogues where $GL(n)$ is replaced by different complex simple groups and for real Grassmannians and flag manifolds.

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