Eta invariant and concordance classes of positive scalar curvature metrics

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It turned out that the proof of Proposition 3.26 wasn't correct. I am working on a proof now. The consequence of this missing proof is that I can prove only a weaker version of the main Theorem 3.14. Surjectivity should be replaced by p-local surjectivity. (I inserted the necessary changes in the text in boldface.) However p-local surjectivity can be proved by different methods like the techniques developed by Peter Gilkey and Boris Botvinnik. See e.g. [BG95]. So my excuses for using deeper results are that on one hand I establish a language which is useful in the field and on the other hand this result on the topological description of the mod $\mathbb{Z}$ reduced eta invariant might be of independent interest.

Abstract

Suppose that M is a compact n-1-dimensional ($n \geq 6$) spin manifold admitting positive scalar curvature metric with finite fundamental group $\pi = \pi_1(M)$. We give a "lower bound" on the size of $\tilde{\pi}_0(\mathrm{Riem}^+(M))$, the set of concordance classes of positive scalar curvature metrics on M. Concordance is a (possibly) weaker equivalence relation than homotopy through positive scalar curvature metrics. S. Stolz showed in [Sto] that there is a certain relative bordism theory such that the corresponding bordism groups $R_n(B\pi)$ acts on $\tilde{\pi}_0(\mathrm{Riem}^+(M))$ freely and transitively. The groups $R_n(B\pi)$ are not known for any π. But there is an index type map: $\theta : R_n(B\pi) \to KO_n(\mathbb{R}\pi)$ which is usually non-trivial. (This map can be defined for infinite groups, too. Then the group algebra $\mathbb{R}\pi$ should be replaced by the reduced C^*-algebra of π.) We show that for a finite p-group π the map θ is surjective if n is large enough. As a main ingredient of the proof we use a real version of the Atiyah-Patodi-Singer index theorem for flat bundles, which compares the η-invariant with the the assembly map with $\mathbb{Z}_{p^\infty}$-coefficients.

1 Introduction

A well known result of Lichnerowicz shows that the index of the Dirac operator on a spin manifold M is an obstruction for the existence of a positive scalar curvature metric on M. Nigel Hitchin [Hit74] was the first to observe that the Dirac operator method can be used for classification problems, too: An appropriate index can distinguish different components of $\mathrm{Riem}^+(M)$ – the space of positive scalar curvature metrics on M. The eta invariant of Atiyah, Patodi and Singer [APS75a] gave a powerful tool to distinguish plenty of components of $\mathrm{Riem}^+(M)$ provided that M has a non-trivial fundamental group (see e.g. [BG95]). In fact our method can be considered as a systematic way to exhaust the η-invariant method. The novelty is that instead of constructing examples, we use abstract homotopy theoretic arguments.

Instead of $\pi_0(\mathrm{Riem}^+(M))$ we look at $\tilde{\pi}_0(\mathrm{Riem}^+(M))$, the set of concordance classes of positive scalar curvature metrics on M. Concordance is a (possibly) weaker equivalence relation than homotopy through positive scalar curvature metrics. S. Stolz showed in [Sto] that there is a certain relative bordism theory such that the corresponding bordism groups $R_n(B\pi)$ have the following properties: Suppose that M is a compact n-1-dimensional ($n \geq 6$) connected spin manifold with fundamental group $\pi = \pi_1(M)$. Then:

1. M admits a positive scalar curvature metric if and only if $R_{n-1}(B\pi) \ni [M] = 0$.

2. If M admits a positive scalar curvature metric then $R_n(B\pi)$ acts on $\tilde{\pi}_0(\mathrm{Riem}^+(M))$ freely and transitively.

This result shows that these groups are of fundamental importance in this field. The group $R_n(B\pi)$ can be defined as follows:

$$R_n(B\pi) := \left\{ (W, f, g) \;\middle|\; \begin{array}{c} W \text{ is an } n\text{-dimensional spin manifold with boundary} \\ f : W \to B\pi \ \text{ and } g \in \mathrm{Riem}^+(\partial W) \end{array} \right\} \Big/ \text{ bordism},$$

Unfortunately the groups $R_n(B\pi)$ $(n \geq 6)$ are not known for any group π. However, we can define an index type map: $\theta : R_n(B\pi) \to KO_n(\mathbb{R}\pi)$ where $\mathbb{R}\pi$ is the real group algebra of π. (This map can be defined for infinite groups, too. Then the group algebra $\mathbb{R}\pi$ should be replaced by the reduced C^*-algebra of π.) The groups

$$KO_n(\mathbb{R}\pi) = \{\mathbb{Z}_2\text{-graded } \mathbb{R}\pi \otimes \mathrm{Cl}_n\text{-modules}\}/\sim,$$

where Cl_n denotes the n-th real Clifford algebra, are very easy to calculate and all the previous statements about different positive scalar curvature metrics on a manifold can be translated to statements about the image of θ. We will define a certain $\mathbb{R}\pi \otimes \mathrm{Cl}_n$-equivariant Dirac operator $D(W, f, g)$ for any triple (W, f, g) representing an element in $R_n(B\pi)$. The equivariance implies that $\mathrm{Ker}(D(W))$ has an $\mathbb{R}\pi \otimes \mathrm{Cl}_n$-module structure. So $\theta(W, f, g) := [\mathrm{Ker}(D(W))]$ defines an element in $KO_n(\mathbb{R}\pi)$. By a theorem of U. Bunke $\theta(W, f, g)$ depends only on the bordism class of (W, f, g) in $R_n(B\pi)$.

Closed manifolds represent elements in $KO_n(B\pi)$ (see 2.29) and the restriction of θ to these elements defines a map $A : KO_n(B\pi) \to KO_n(\mathbb{R}\pi)$. The map A is called the assembly map. This map is induced by a map of spectra $\mathbf{A} : KO \wedge B\pi_+ \to KO(\mathbb{R}\pi)$ where $KO(\mathbb{R}\pi)$ is the KO-spectrum of $\mathbb{R}\pi$. This enables us to define the assembly map with any coefficients.

$KO_n(\mathbb{R}\pi)$ – the range of θ – is 8-periodic. It has the consequence that θ can only distinguish stable concordance classes of positive scalar curvature metrics: Two positive scalar curvature metrics g_1, g_2 on M are stably concordant if for sufficiently many copies of B the induced positive scalar curvature metrics on $M \times B \times \cdots \times B$ are concordant. B denotes the Bott manifold, a certain 8-dimensional spin manifold which represents the Bott periodicity in KO-theory. So we can define a stable version of θ:

$$\theta_{\mathrm{st}} : R_n(B\pi)[B^{-1}] \to KO_n(\mathbb{R}\pi),$$

where $R_n(B\pi)[B^{-1}]$ denotes $R_n(B\pi)$ with $[B] \in R_8(pt)$ inverted. A recent example of T. Schick (see Remark 2.17) shows that stable concordance is different from concordance, i.e. there is a group and an n such that the map $R_n(B\pi) \to R_n(B\pi)[B^{-1}]$ is not injective. The most general conjecture was formulated by Stephan Stolz:

Conjecture 1.1 *θ_{st} is an isomorphism for any finitely presented group π.*

Proving the injectivity of θ_{st} is unlikely with the existing methods. For torsion free groups the Baum-Connes conjecture implies the surjectivity of θ_{st}. In this paper we prove the **(p-local)** surjectivity for finite p-groups:

Theorem 3.14 For every finite p-group $\theta_{\text{st}} : R_n(B\pi)[B^{-1}] \to KO_n(\mathbb{R}\pi)$ is **(p-locally)** surjective.

(In other words the cokernel of θ_{st} is a finite group with order coprime to p.)

To prove the Theorem we define a map $\boldsymbol{\eta}$ on bordism classes of spin manifolds admitting positive scalar curvature metric with finite fundamental group $\pi = \pi_1(M)$, which can be considered as the collection of all the η_α's where α runs through the irreducible representations of π. These η_α's are the spectral invariants defined by Atiyah, Patodi and Singer [APS75a]. The target of $\boldsymbol{\eta}$ is $KO_n(\tilde{\mathbb{R}}\pi) \otimes \mathbb{Z}[\frac{1}{p}]$, where $\tilde{\mathbb{R}}\pi$ is the augmentation ideal of the real group algebra of π. The key step in the proof is to find a close relationship between $\boldsymbol{\eta}$ and $A(\mathbb{Z}_{p^\infty})$ – the assembly map with $\mathbb{Z}_{p^\infty}$-coefficients. This relation is a real version of the Atiyah-Patodi-Singer index theorem for flat bundles.

We need the finite p-group assumption since a major step in the proof is that $A(\mathbb{Z}_{p^\infty})$ is an isomorphism for finite p-groups and the analogous statement is false for non-p-groups. However there are some promising results in this direction. Joint work in progress with Stephan Stolz indicates that θ_{st} is surjective for finite groups which are products of an odd order group and a 2-group (in particular for nilpotent groups).

This paper is based on my PhD. thesis at the University of Notre Dame, Indiana. The help of my advisor, Stephan Stolz, was essential at every stage of this work. I also owe a lot to the faculty and fellow graduate students of the Mathematics Department for stimulating discussions. Part of this research was done while enjoying the hospitality of the Max Plack Institute in Bonn.

1.1 Outline

In Chapter 2 we introduce the assembly map A and review the history of the Existence or Gromov-Lawson-Rosenberg Conjecture. Then we describe the Stable Existence Theorem which says that a spin manifold M with finite fundamental group admits stably a positive scalar curvature metric if and only if $A(M) = 0$.

Also we describe the various KO-groups involved, and describe a spectrum map $\mathbf{A}$ which induces the assembly map. That allows us to define the assembly map with coefficients.

In Chapter 3 we introduce the notion of concordance. We define the R_n-groups and the map $\theta : R_n(B\pi) \to KO_n(\mathbb{R}\pi)$. In 3.3 we define a diagram (Diagram 1) involving θ, the analytically defined map $\boldsymbol{\eta}$ and a topologically defined map $\overline{\boldsymbol{\eta}}$. Diagram chasing shows that the commutativity of Diagram 1 together with the surjectivity of $\boldsymbol{\eta}$, $\overline{\boldsymbol{\eta}}$ and a third map f implies the surjectivity of θ. In 3.3.1 we prove most of the commutativity of Diagram 1. In 3.3.2 we show the surjectivity of $\boldsymbol{\eta}$ and $\overline{\boldsymbol{\eta}}$.

In Chapter 4 we explain the connection between the classical eta invariants and $\boldsymbol{\eta}$. This result is needed to prove the the surjectivity of f.

In Chapter 5 – using results from Chapter 4 – we complete the proof of the commutativity of Diagram 1, in other words we compare $\boldsymbol{\eta}$ and $\overline{\boldsymbol{\eta}}$. This proof is based on the Atiyah-Patodi-Singer index theorem for flat bundles [APS76, (5.3)] which describes the classical eta invariants in topological terms.

2 The assembly map and existence of positive scalar curvature metrics

The results of this section can be found in several place in the literature, e. g. in [RS95].

2.1 The equivariant Dirac operator $D(W)$

The history of this area starts with the result of Lichnerowicz [Lic63] who showed that if a spin manifold W admits a positive scalar curvature metric then the index of a certain Dirac operator vanishes. For certain manifolds with infinite fundamental group [SY79] and recently for four dimensional manifolds – the Seiberg-Witten theory – different obstructions were developed later. But for spin manifolds W with $\pi_1(W)$ finite and $\dim(W) > 4$ still all the known obstructions are indices of certain Dirac operators. There is a way to combine all these Dirac operators into a single one:

Let Cl_n denote the n-th real Clifford algebra:

$$\mathrm{Cl}_n := \langle e_1, \cdots, e_n : e_i e_j = -e_j e_i \text{ for } i \neq j, e_i^2 = -1 \rangle.$$

Cl_n comes with a $\mathbb{Z}_2$-grading: $\mathrm{Cl}_n = Cl_{\mathrm{n}}^0 \oplus Cl_{\mathrm{n}}^1$ where $Cl_{\mathrm{n}}^0 = \langle e_i e_j : 1 \leq i, j \leq n \rangle$.

There is an invariant way to define the Clifford algebras [ML89, I.§1]:

Definition 2.1 For any vector space V equipped with a quadratic form q

$$\mathrm{Cl}(V, q) = \mathcal{T}(V)/\mathcal{I}_q(V)$$

where $\mathcal{T}(V) = \sum_{i=0}^{\infty} \otimes^i V$ is the tensor algebra of V and the ideal $\mathcal{I}_q(V)$ is generated by the elements of the form

$$v \otimes v + q(v)1.$$

In this terminology $\mathrm{Cl}_n = \mathrm{Cl}(\mathbb{R}^n, q)$ where $q(v) = \langle v, v \rangle$ is the standard positive definite quadratic form on $\mathbb{R}^n$.

The group $\mathrm{Spin}(n)$ – generated by products of even number of unit vectors in $\mathbb{R}^n \subset \mathrm{Cl}_n$ – is a subgroup of $Cl_{\mathrm{n}}^\times$ where $Cl_{\mathrm{n}}^\times$ is the group of invertible elements in Cl_n [ML89, p.14]. Let W be an n-dimensional Riemannian spin manifold with metric G. The group $\mathrm{Spin}(n)$ is a double cover of $SO(n)$ and a spin structure of the oriented manifold W defines a principal $\mathrm{Spin}(n)$-bundle $\mathrm{Spin}(W)$ which is a double cover of $SO(W)$, the oriented frame bundle of W. The group $\mathrm{Spin}(n)$ acts on Cl_n by left multiplication so we can define an associated bundle:

Definition 2.2 For any spin manifold W let $\mathrm{Cl}_{\mathrm{Spin}}(W) = \mathrm{Spin}(W) \times_{\mathrm{Spin}(n)} \mathrm{Cl}_n$.

As a rule of thumb, we choose the notation to be consistent with [ML89] since this is the best general reference on the subject.

Suppose that $\pi = \pi_1(W)$. Then $\tilde{W}$ – the universal cover of W – is a principal π-bundle over W. Suppose that π is finite and let $\mathbb{R}\pi$ denote the real group-algebra of π, i. e. $\mathbb{R}\pi = \{f : \pi \to \mathbb{R}\}$. Multiplication is defined by "convolution": $f_1 \cdot f_2(h) := \sum_{g\in\pi} f_1(g) f_2(g^{-1}h)$. We have $\pi \subset \mathbb{R}\pi^{\times}$ if we identify $g \in \pi$ with $\check{g} : \pi \to \mathbb{R}$, $\check{g}(h) = \delta^g_h$. This way we can define an action of π on $\mathbb{R}\pi$ by left multiplication.

Definition 2.3 For a manifold W with $\pi = \pi_1(W)$ finite, let $\mathbb{R}\pi(W) = \tilde{W} \times_\pi \mathbb{R}\pi$

Using that right multiplication commutes with left multiplication, $\mathrm{Cl}_{\mathrm{Spin}}(W)$ is a right Cl_n-module bundle and $\mathbb{R}\pi(W)$ is a right $\mathbb{R}\pi$-module bundle.

We will frequently use $\mathbb{Z}_2$-graded objects:

Definition 2.4 A $\mathbb{Z}_2$-*graded* vector space is an ordered pair (V^0, V^1) of vector spaces. Let $V = V^0 \oplus V^1$. The operator $T : V \to V$ defined by the property that the 1-eigenspace of T is V^0 and the -1-eigenspace of T is V^1. T is called the *grading operator*.

Remark 2.5 Equivalently, we could define a $\mathbb{Z}_2$-graded vector space as a pair (V, T), where V is a vector space and $T : V \to V$ is a linear operator with $T^2 = 1$.

$\mathrm{Cl}_{\mathrm{Spin}}(W)$ admits a connection $\nabla^W : \Gamma(\mathrm{Cl}_{\mathrm{Spin}}(W) \otimes TW) \to \Gamma(\mathrm{Cl}_{\mathrm{Spin}}(W))$ (or ∇ for short) coming from the Levi-Civita connection ∇^{LC} (think ∇^{LC} as a connection on $\mathrm{SO}(W)$ – the orthogonal frame bundle of W. It has a unique lift to the double cover $\mathrm{Spin}(W)$). The pair $(\mathrm{Cl}_{\mathrm{Spin}}(W), \nabla)$ is a so called Dirac bundle (see [ML89, p.114]), so we can define a Dirac operator on it ([ML89, §5]):

Let $\mathrm{Cl}(TW)$ denote the "Cliffordization" of the Euclidean vector bundle TW:

$$\mathrm{Cl}_x(TW) := \mathrm{Cl}(T_xW, q)$$

where $q(v) = \langle v, v\rangle$ and $\langle\cdot,\cdot\rangle$ denotes the scalar product defined by the Riemannian metric of W. $\mathrm{Cl}(TW)$ is an algebra bundle, the fibers are equipped with an algebra structure.

Definition 2.6 [ML89, II.5.2] A *Dirac bundle* over a Riemannian manifold W is a bundle S of left modules over $\mathrm{Cl}(TW)$ together with a metric $\langle\cdot,\cdot\rangle$ and a metric connection ∇^S having the following properties:

1. $\langle e\sigma_1, e\sigma_2\rangle = \langle\sigma_1, \sigma_2\rangle$ for all $\sigma_1, \sigma_2 \in S_x$ and for all $e \in T_xW$ with $||e|| = 1$,

2. $\nabla^S_X(\varphi\sigma) = \nabla_X(\varphi)\sigma + \varphi(\nabla^S_X\sigma)$ for all smooth sections $X \in \Gamma(TW)$, $\varphi \in \Gamma(\mathrm{Cl}(TW))$ and $\sigma \in \Gamma(S)$.

These properties allow us to define a Dirac operator $D(S) : \Gamma(S) \to \Gamma(S)$ by the local formula:

Definition 2.7 $\quad (D(S)\sigma)\,(x) := \left(\sum_i^n e_i \cdot \nabla_{e_i}\right)(x)\sigma$

where $\{e_i\}$ is an orthonormal basis of $T_xW \subset \mathrm{Cl}_x(TW)$ and $\cdot$ denotes the Clifford multiplication.

For example every vector bundle associated to $\mathrm{Spin}(W)$ admits a natural Dirac bundle structure ([ML89, II.5.9]). Also, if S is a Dirac bundle and E is a vector bundle with a metric and a compatible connection then $S \otimes E$ admits a natural Dirac bundle structure.

$\mathbb{R}\pi(W)$ – having a discrete structure group – admits a flat connection ∇^π. Therefore

$$(\mathrm{Cl}_{\mathrm{Spin}}(W) \otimes \mathbb{R}\pi(W), \nabla \otimes \nabla^\pi)$$

is a Dirac bundle, too. Here we follow the notation of [ML89]:

$$(\nabla \otimes \nabla')_X(s \otimes s') := \nabla_X s \otimes 1 + 1 \otimes \nabla'_X s'$$

for any $X \in \Gamma(TW)$.

Definition 2.8 $\quad D(W) := D(\mathrm{Cl}_{\mathrm{Spin}}(W) \otimes \mathbb{R}\pi(W), \nabla \otimes \nabla^\pi)$

This operator is our main interest. $D(W)$ is a first order elliptic differential operator, it acts on the smooth sections of $\mathrm{Cl}_{\mathrm{Spin}}(W) \otimes \mathbb{R}\pi(W)$. By standard methods we can extend it to the Hilbert space of L^2-sections. This is an unbounded self-adjoint operator. $D(W)$ has degree 1: It interchanges the 0- and the 1-part of the space of L^2-sections [ML89]. Degree 1 operators are called *odd operators* as well.

Proposition 2.9 *[ML89, p.115] For a compact Riemannian spin manifold W the vector space* $\mathrm{Ker}(D(W))$ *is finite dimensional.*

Since $D(W)$ commutes with the $\mathrm{Cl}_n \otimes \mathbb{R}\pi$-action, $\mathrm{Ker}(D(W))$ is a $\mathbb{Z}_2$-graded $\mathrm{Cl}_n \otimes \mathbb{R}\pi$-module.

Let's compare $D(W)$ with the classical Dirac operator: if $n \equiv 0 \pmod 2$ then $\mathbb{C}\mathrm{l}_n = \mathrm{Cl}_n \otimes \mathbb{C}$, the complex Clifford algebra has two irreducible $\mathbb{Z}_2$-graded modules:

Definition 2.10 Let $\mathbb{S} = \mathbb{S}_n$ be the unique irreducible $\mathbb{Z}_2$-graded left $\mathbb{C}\mathrm{l}_n$-module for which the action of $\omega_n^{\mathbb{C}} \in \mathbb{C}\mathrm{l}_n$ coincides with the grading operator T, where $\omega_n^{\mathbb{C}}$ is the complex volume form (see [ML89, p. 34]): $\omega_n^{\mathbb{C}} = i^{[\frac{n+1}{2}]} e_1 \cdots e_n$.

Then $\mathrm{Spin}(W) \times_{\mathrm{Spin}(n)} \mathbb{S}$ is the so-called spinor bundle. The corresponding Dirac operator D_{AS} is the Atiyah-Singer Dirac operator. Traditionally the zero-part of the spinor bundle was called positive half spinors and they considered D^0_{AS} – the restriction of D_{AS} to the positive half spinors. (Similarly D^1_{AS} denotes the restriction to the one-part.) So we have:

$$\mathrm{Ker}(D_{AS}) = \Big(\mathrm{Ker}(D(W)) \otimes_{\mathbb{C}\mathrm{l}_n} \mathbb{S}\Big)^{\pi}$$

where the superscript π means the fixpoint set of π, and

$$\mathrm{Index}_{\mathbb{C}} D^0_{AS} = \dim_{\mathbb{C}} \mathrm{Ker}(D^0_{AS}) - \dim_{\mathbb{C}} \mathrm{Coker}(D^0_{AS}) = \dim_{\mathbb{C}} \Big((\mathrm{Ker} D_{AS})^0\Big) - \dim_{\mathbb{C}} \Big((\mathrm{Ker} D_{AS})^1\Big).$$

2.2 Definition of the assembly map and existence of positive scalar curvature metrics

Though we suppressed it in the notation, $\mathrm{Ker}(D(W))$ depends on the metric G. But just slightly:

Definition 2.11 Let $\mathcal{C}$ be a $\mathbb{Z}_2$-graded $\mathbb{R}$-algebra:

$$\mathfrak{M}(\mathcal{C}) := Gr\{\text{finitely generated projective } \mathbb{Z}_2\text{-graded right } \mathcal{C}\text{-modules}\},$$

where Gr stands for Grothendieck group. Let $i^* : \mathfrak{M}(\mathcal{C} \otimes \mathrm{Cl}_{n+1}) \to \mathfrak{M}(\mathcal{C} \otimes \mathrm{Cl}_n)$ be the map which forgets the e_{n+1}-action and

$$KO_n(\mathcal{C}) := \mathfrak{M}(\mathcal{C} \otimes \mathrm{Cl}_n)/i^*\mathfrak{M}(\mathcal{C} \otimes \mathrm{Cl}_{n+1}).$$

Remark 2.12 For finite dimensional semisimple algebras the finitely generated projective $\mathbb{Z}_2$-graded right $\mathcal{C}$-modules are exactly the finite dimensional ones.

Definition 2.13 For a compact spin manifold W with $\pi = \pi_1(W)$ finite let

$$A^{\pi}(W) = [\mathrm{Ker}(D(W))] \in \mathfrak{M}(\mathbb{R}\pi \otimes \mathrm{Cl}_n)/i^*\mathfrak{M}(\mathbb{R}\pi \otimes \mathrm{Cl}_{n+1}) = KO_n(\mathbb{R}\pi).$$

If π is clear from the context we suppress it from the notation. For simply connected manifolds we get an invariant in $KO_n(\mathbb{R}) \cong KO^{-n}(pt)$. This map is the “improved” version of the $\hat{A}$-genus defined by Hitchin ([Hit74] see also in [ML89, p.149]).

We recall now an appropriate form of the Lichnerowicz theorem:

Theorem 2.14 ([Ros86]) *If G is a positive scalar curvature metric on the spin manifold W then* $\mathrm{Ker}(D(W)) = 0$.

As we will see (Theorem 2.30), $A^{\pi}(W)$ is independent of the metric. It implies the following:

Corollary 2.15 *If* $[\mathrm{Ker}(D(W))] \in KO_n(\mathbb{R}\pi)$ *is non-trivial then* W *doesn't admit a positive scalar curvature metric.*

A conjecture of Gromov and Lawson – later modified by Rosenberg – states that this is the only obstruction:

Conjecture 2.16 (Existence Conjecture [RS95]) *Suppose* W^n *is a spin manifold,* $\pi = \pi_1(W)$ *and* $n \geq 5$*. Then* W *admits a positive scalar curvature metric if and only if* $A^\pi(W) = 0$*.*

Remark 2.17 We defined $A^\pi(W)$ only for finite π, but it can be defined for the general case. If $|\pi| = \infty$ then $A^\pi(W) \in KO_n(C_r^*\pi)$ where $C_r^*\pi$ is the real reduced group C^*-algebra of π (see [Ros91, §2]). Recently Thomas Schick found a counter example to the conjecture: a 5-dimensional spin manifold with fundamental group $\mathbb{Z}^4 \times \mathbb{Z}_3$. The proof is based on the hypersurface method of Schoen and Yau [SY79].

The conjecture is known to be true only in a few cases. Even for the case π trivial the proof is quite involved [Sto92]. But a weaker version of the conjecture was proved for a wide class of groups, including all finite groups. It is easy to see that A is multiplicative in the following sense:

Proposition 2.18 $A(M \times N) = A(M) \circ A(N)$ *where*

$$\circ : KO_n(\mathbb{R}\pi_1) \otimes KO_m(\mathbb{R}\pi_2) \to KO_{n+m}(\mathbb{R}(\pi_1 \times \pi_2))$$

is induced by the $\mathbb{Z}_2$*-graded tensor product of modules.*

Proof [ML89, p.150]. We implicitly assumed that the spin structure on $M \times N$ is the product of the spin structures on M and N. It implies that

$$\mathrm{Cl}_{\mathrm{Spin}}(M \times N) \otimes \mathbb{R}(\pi_1 \times \pi_2)(M \times N) \cong pr_M^*(\mathrm{Cl}_{\mathrm{Spin}}(M) \otimes \mathbb{R}\pi_1(M)) \otimes pr_N^*(\mathrm{Cl}_{\mathrm{Spin}}(N) \otimes \mathbb{R}\pi_2(N))$$

where pr_M and pr_N denotes the projections. This implies that $D(M \times N, g_M \times g_N) = D(M, g_M) \otimes 1 + T \otimes D(N, g_N)$ where T is the grading operator and $D^2(M \times N, g_M \times g_N) = D^2(M, g_M) \otimes 1 + 1 \otimes D^2(N, g_N)$. The operator D^2 is a non-negative with $\mathrm{Ker}(D^2) = \mathrm{Ker}(D)$, therefore

$$\mathrm{Ker}D(M, g_M) \otimes \mathrm{Ker}D(N, g_N) \cong \mathrm{Ker}D(M \times N, g_M \times g_N).$$

If the modules are $\mathbb{Z}_2$-graded, then $\otimes$ always denotes the $\mathbb{Z}_2$-graded tensor product. We are interested in the KO-theory of the group algebra $\mathbb{R}\pi$. This is an example of a C^*-algebra:

Definition 2.19 [Goo82, §8] A real $*$-algebra is a real algebra $\mathcal{C}$ equipped with a linear involution $*$ such that $(xy)^* = y^*x^*$. A real Banach $*$-algebra is a real $*$-algebra equipped with a norm making it a real Banach space, such that $||xy|| \leq ||x|| \cdot ||y||$ for all x, y in $\mathcal{C}$. (If $\mathcal{C}$ is unital we also require that $||1|| = 1$.)

Definition 2.20 [Sch93, p.1] A real C^*-algebra $\mathcal{C}$ is a Banach $*$-algebra over the reals which is *-isometrically isomorphic to a norm closed subalgebra of the algebra of continuous linear operators on a real Hilbert space.

Equivalently ([Goo82, §8]) a real C^*-algebra $\mathcal{C}$ is a real Banach $*$-algebra such that $||x^*x|| = ||x||^2$ and $1 + x^*x$ is invertible for every $x \in \mathcal{C}$.

For finite dimensional algebras this condition is equivalent to semisimplicity [Sch93, p. 1], what is clear for $\mathbb{R}\pi$. For completeness we mention that $f^*(g) := f(g^{-1})$ defines a $*$-operation on $\mathbb{R}\pi$. An other example for a real C^*-algebra is $C(X)$ – the space of continuous functions on a compact space X – with the trivial $*$-operation.

Proposition 2.21 *For any real C^*-algebra $\mathcal{C}$ we have $KO_n(\mathcal{C}) \cong KO_{n+8}(\mathcal{C})$ where the isomorphism is induced by $\otimes\mathbf{S}$ for the irreducible $\mathbb{Z}_2$-graded Cl_8-module $\mathbf{S}$, which represents a generator of $KO_8(\mathbb{R}) \cong \mathbb{Z}$ [Sch93].*

We denote by $\mathbf{S}$ the unique irreducible $\mathbb{Z}_2$-graded Cl_8-module which has the property that $\mathbf{S}\otimes\mathbb{C}$ is isomorphic to $\mathbb{S}_8$ as a $\mathbb{C}\mathrm{l}_8$-module.

Proposition 2.22 *[ML89, p.151] There is an 8-dimensional simply connected spin manifold B such that $A(B) = [\mathbf{S}]$.*

B is usually called a Bott manifold since it represents the Bott periodicity. Such a manifold is not unique, but the choice doesn't effect the following. Proposition 2.18 implies that $A(W) = 0$ if and only if $A(W \times B) = 0$.

Definition 2.23 W admits stably a positive scalar curvature metric if there is a k such that $W \times \underbrace{B \times \cdots \times B}_{\text{k times}}$ has positive scalar curvature metric.

Corollary 2.24 *If W admits stably a positive scalar curvature metric then $A(W) = 0$.*

Theorem 2.25 (Stable Existence Theorem – Rosenberg, Stolz [RS95]) *If W is a spin manifold and $\pi_1(W)$ is finite then $A(W) = 0$ if and only if W admits stably a positive scalar curvature metric.*

2.3 Homotopy theoretic description of the assembly map

The assembly map A has strong invariance properties. To see this we first extend the domain of A:

Definition 2.26 Let W be a spin manifold, let $f : W \to B\pi$ be a continuous map and let $\mathbb{R}\pi(B\pi) := E\pi \times_\pi \mathbb{R}\pi$ be the universal $\mathbb{R}\pi$-bundle over $B\pi$. Then

$$A(W,f) := [\mathrm{Ker} D(W,f)] \in KO_n(\mathbb{R}\pi)$$

where $D(W,f) := D(\mathrm{Cl}_{\mathrm{Spin}}(W) \otimes \mathbb{R}\pi(W,f))$ and $\mathbb{R}\pi(W,f) := f^*\mathbb{R}\pi(B\pi)$.

Definition 2.27 $\Omega_n^{\mathrm{Spin}}(X) := \{(W,f) : W^n \text{ spin}, f : W \to X\}/\sim$

where $(W_1,f_1) \sim (W_2,f_2)$ if there are spin manifolds with boundary U_1, U_2 and maps $f_i : U_i \to X$ such that $\partial U_1 \coprod W_1 \cong \partial U_2 \coprod W_2$. Here $\cong$ means a diffeomorphism compatible with the spin structure and the maps to X. We assume that the spin structure on ∂U_i is induced by the spin structure of U_i.

$\Omega_n^{\mathrm{Spin}}(X)$ is a group with addition induced by the operation $\coprod$ ([Swi75, p.226]).

Theorem 2.28 *$A(W,f)$ depends only on $[W,f] \in \Omega_n^{\mathrm{Spin}}(B\pi)$.*

We will prove a much stronger statement. The KO-orientation $u : M\mathrm{Spin} \to KO$ induces a map $U : \Omega_n^{\mathrm{Spin}}(B\pi) \to KO_n(B\pi)$. There is a geometric way to describe this map [KS93, Thm C.]:

Theorem 2.29 *There is a canonical isomorphism U:*

$$KO_n(X) \cong \left(\bigoplus_k \Omega_{n+8k}^{\mathrm{Spin}}(X)\right) / \sim,$$

where the equivalence is generated by

1. *$[E, f \circ p] \sim 0$ if $p : E \to M$ is an $\mathbb{HP}^2$-bundle over a spin manifold M and f is a map from M to X.*
2. *$[M,f] \sim [M \times B, f \circ pr_1]$, where B is the Bott manifold and pr_1 denotes the projection to the first factor.*

Theorem 2.30 *$A(W,f)$ depends only on $U[W,f] \in KO_n(B\pi)$.*

Theorem 2.30 is a consequence of the homotopy theoretic description of A in Theorem 2.37.

This is an important strengthening of Theorem 2.28 since it is much easier to describe $KO_n(B\pi)$ (see Proposition 2.50) than the group $\Omega_n^{\mathrm{Spin}}(B\pi)$. The proof of this theorem is based on a version of the Atiyah-Singer index theorem which gives a topological formula for $A(W,f)$ in terms of $[W,f] \in KO_n(B\pi)$. This formula plays a fundamental role in this work so we present it in details.

Definition 2.31 We will call $A : KO_n(B\pi) \to KO_n(\mathbb{R}\pi)$ the *Assembly map.*

At this point the reader might be interested in what these groups look like:

Proposition 2.32 *[AS69] Let* $\mathrm{Irr}(\mathbb{R}\pi)$ *denote the set of irreducible* $\mathbb{R}\pi$*-modules in other words the* real *representations of* π*. Then*

$$KO_n(\mathbb{R}\pi) \cong \sum_{\alpha \in \mathrm{Irr}(\mathbb{R}\pi)} KO_n\Big(F(\alpha)\Big)$$

where $F(\alpha) = \mathrm{Hom}_{\mathbb{R}\pi}(\alpha, \alpha)$ *is the commuting field of* α *– can be* $\mathbb{R}$*,* $\mathbb{C}$ *or* $\mathbb{H}$ *([AS69]) and*

	0	*1*	*2*	*3*	*4*	*5*	*6*	*7*
$KO_n(\mathbb{R})$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	*0*	$\mathbb{Z}$	*0*	*0*	*0*
$KO_n(\mathbb{C})$	$\mathbb{Z}$	*0*	$\mathbb{Z}$	*0*	$\mathbb{Z}$	*0*	$\mathbb{Z}$	*0*
$KO_n(\mathbb{H})$	$\mathbb{Z}$	*0*	*0*	*0*	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	*0*

Remark 2.33 If π is a group of odd order then $F(\alpha) = \mathbb{C}$ for every non-trivial irreducible $\mathbb{R}\pi$-module α [FH91, p.34].

We want to show that A is induced by a map $\mathbf{A}$ between spectra. $KO_n(B\pi)$ is the n-th homotopy group of the spectrum $KO \wedge B\pi_+$. But it is less clear that $KO_n(\mathbb{R}\pi)$ is the n-th homotopy group of a spectrum as well.

Theorem 2.34 *[Ros95, theorem 2.1] For every real (ungraded)* C^**-algebra* $\mathcal{C}$ *there is a spectrum* $KO(\mathcal{C})$ *such that:*

1. $\pi_n(KO(\mathcal{C})) = KO_n(\mathcal{C})$
2. $(KO(\mathcal{C}))_0 = KO_0(\mathcal{C}) \times BGL(\mathcal{C})$*.* $GL(\mathcal{C})$ *is the limit of* $GL_n(\mathcal{C}) = \mathrm{Aut}_{\mathcal{C}}(\mathcal{C}^n)$ *where* $\mathcal{C}^n$ *considered as a* $\mathcal{C}$*-module.*
3. $KO(\mathcal{C})$ *is 8-periodic i.e. we can assume that* $KO(\mathcal{C})$ *is an* Ω*-spectrum and* $(KO(\mathcal{C}))_8$ *homotopy equivalent to* $(KO(\mathcal{C}))_0$*.*

To describe $\mathbf{A}$, we need one more property of $KO(\mathcal{C})$:

Theorem 2.35 *[Ros95, p. 353]* $KO(\mathcal{C})$ *is naturally a module spectrum over* $KO(\mathbb{R})$ *via tensor product.*

Definition 2.36 $\mathbf{A} : KO \wedge B\pi_+ \to KO(\mathbb{R}\pi), \qquad \mathbf{A} := \mu \circ (\mathrm{Id}_{KO} \wedge Bi_+)$

where $i : \pi \to GL_1(\mathbb{R}\pi) \subset GL(\mathbb{R}\pi)$, so $Bi : B\pi \to BGL(\mathbb{R}\pi) \subset (KO(\mathbb{R}\pi))_0$ and Bi_+ is the extension of Bi sending $+$ to the basepoint of $BGL(\mathbb{R}\pi)$, and $\mu : KO \wedge KO(\mathbb{R}\pi) \to KO(\mathbb{R}\pi)$ is the module spectrum map.

Theorem 2.37 *[Ros91, p. 175]* **A** *induces* A *on the homotopy groups.*

This theorem shows that A is indeed an assembly map in the sense of [WW95]. We show now how to express Theorem 2.37 in terms of the cohomological slant product:

$Bi : B\pi \to BGL_1(\mathbb{R}\pi)$ is the classifying map of the $\mathbb{R}\pi$-module bundle

$$Bi^*\Big(EGL_1(\mathbb{R}\pi) \times_{GL_1(\mathbb{R}\pi)} \mathbb{R}\pi\Big).$$

But pulling back with Bi – where i is a homomorphism of groups – corresponds to the reduction of the structure group (see [Swi75, p. 202]) so we have

$$Bi^*\Big(EGL_1(\mathbb{R}\pi) \times_{GL_1(\mathbb{R}\pi)} \mathbb{R}\pi\Big) \cong E\pi \times_\pi \mathbb{R}\pi = \mathbb{R}\pi(B\pi)$$

For non-compact spaces vector bundles don't always define an element in the corresponding K-theory but in our case $\mathbb{R}\pi(B\pi)$ admits a classifying map – this is how we constructed it – which defines an element $[Bi_+]$ in $KO(\mathbb{R}\pi)^0(B\pi) = [B\pi_+, KO(\mathbb{R}\pi)]$. So let us use the notation $[\mathbb{R}\pi(B\pi)]$ as well for this element.

Let E be a ring spectrum and F a module spectrum over E with module structure $\mu : E \wedge F \to F$:

Definition 2.38 [Swi75, p. 270-271] The (cohomological) *slant product* is a map

$$/ : F^p(Y) \otimes E_q(Y) \to F^{p-q}(pt).$$

Suppose $f : Y_+ \to \Sigma^p(F)$ represents an element $[f] \in F^p(Y)$ and $g : S^q \to E \wedge Y_+$ represent an element $[g] \in E_q(Y)$. Then:

$$[f]/[g] := [h]$$

where

$$\begin{tikzcd} pt_+ = S^0 \arrow[r, "\Sigma^{-q} g"] \arrow[rrr, bend right, "h"'] & \Sigma^{-q} E \wedge Y_+ \arrow[r, "1 \wedge f"] & \Sigma^{-q} E \wedge \Sigma^p F \arrow[r, "\Sigma^{p-q}\mu"] & \Sigma^{p-q} F \end{tikzcd}$$

From the definition we can see that:

Corollary 2.39 *For a spin $B\pi$-manifold (W, f)*

$$\mathbf{A}_*[W, f] = [\mathbb{R}\pi(B\pi)]/[W, f]$$

where $[\mathbb{R}\pi(B\pi)] \in KO(\mathbb{R}\pi)^0(B\pi)$ *and* $[W, f] \in KO_n(B\pi)$.

2.4 The assembly map with $\mathbb{Z}_{p^\infty}$ coefficients

Since A is induced by a map of spectra, we can define the assembly map with any coefficients. In this section we explain how to do this and we describe the map $A(\mathbb{Z}_{p^\infty})$. The fact that this map is an isomorphism (Theorem 2.45) plays a crucial role in the proof of the surjectivity of θ. Also, as a corollary of Theorem 2.45, we get a description of the image of the assembly map A.

Let us recall now the notion of generalized homology theories with coefficients:

Proposition 2.40 *[Ada70, p. 200] For every Abelian group G there is a spectrum MG – called the* Moore spectrum *of G – with the following properties:*

1. $\pi_n(MG) = 0 \qquad$ *if* $n < 0$
2. $\pi_0(MG) \cong G \cong H_0(MG)$
3. $H_n(MG) = 0 \qquad$ *if* $n > 0$.

It is easy to see that such a spectrum is unique up to homotopy. Let E be a spectrum and G be an Abelian group:

Definition 2.41 $\qquad E_n(X;G) := (E \wedge MG)_n(X) = \pi_n(X_+ \wedge E \wedge MG)$

where MG is the Moore spectrum of G.

For a map of spectra $\mathbf{f} : E \to F$ and a group G the maps $\mathbf{f} \wedge \mathrm{Id}_{MG} : E \wedge MG \to F \wedge MG$ induce maps $f(G) : E_n(pt;G) \to F_n(pt;G)$. In particular we can define $A(\mathbb{Z}_{p^\infty}) : KO_n(B\pi;\mathbb{Z}_{p^\infty}) \to KO_n(\mathbb{R}\pi;\mathbb{Z}_{p^\infty})$. First we describe the range of $A(\mathbb{Z}_{p^\infty})$.

For the spectrum $KO(\mathcal{C})$ we want to use the following conventions:

Definition 2.42

$$\begin{aligned} KO_n^{\mathcal{C}}(X) &:= KO(\mathcal{C})_n(X) = \pi_n(X_+ \wedge KO(\mathcal{C})), & KO_n^{\mathcal{C}}(X;G) &:= (KO(\mathcal{C}) \wedge MG)_n(X), \\ KO^n_{\mathcal{C}}(X) &:= KO(\mathcal{C})^n(X) = [X_+, \Sigma^n KO(\mathcal{C})], & KO^n_{\mathcal{C}}(X;G) &:= (KO(\mathcal{C}) \wedge MG)^n(X). \end{aligned}$$

For $X = pt$ we also want to use: $KO_n(\mathcal{C}) := KO_n^{\mathcal{C}}(pt)$ and $KO_n(\mathcal{C};G) := KO_n^{\mathcal{C}}(pt;G)$.

We are mostly interested in the algebra $\mathbb{R}\pi$. $\mathbb{R}\pi$ can be written as a direct sum of algebras: $\mathbb{R}\pi = \tilde{\mathbb{R}}\pi \oplus \mathbb{R}$ where $\mathbb{R}$ is generated by the constant 1 function and $\tilde{\mathbb{R}}\pi$ is the augmentation ideal i.e. the kernel of the augmentation map $\varepsilon : \mathbb{R}\pi \to \mathbb{R}$, $\varepsilon(f) = \sum_{g\in\pi} f(g)$. The algebra $\tilde{\mathbb{R}}\pi$ plays a fundamental role in our considerations. As it can be easily checked, the K-theory of $\tilde{\mathbb{R}}\pi$ is very similar to the K-theory of $\mathbb{R}\pi$:

$$KO_n(\tilde{\mathbb{R}}\pi) \cong \sum_{\alpha \in \widetilde{\mathrm{Irr}}(\mathbb{R}\pi)} KO_n\big(F(\alpha)\big)$$

where $\widetilde{\mathrm{Irr}}(\mathbb{R}\pi)$ denotes the set of non-trivial irreducible $\mathbb{R}\pi$-modules. In other words $KO_n(\mathbb{R}\pi) = KO_n(\tilde{\mathbb{R}}\pi) \oplus KO_n(\mathbb{R})$, so we can think of $\tilde{\mathbb{R}}\pi$ as a "reduced" version of $\mathbb{R}\pi$.

Proposition 2.43 *For any spectrum E a short exact sequence of Abelian groups*

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

induces a cofibration $MA \longrightarrow MB \longrightarrow MC$ which induces a long exact sequence:

$$\xrightarrow{\delta} E_n(X;A) \xrightarrow{\alpha^*} E_n(X;B) \xrightarrow{\beta^*} E_n(X;C) \xrightarrow{\delta} E_{n-1}(X;A) \longrightarrow$$

e.g. for $\quad 0 \longrightarrow \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z}[1/p] \xrightarrow{\beta} \mathbb{Z}_{p^\infty} \longrightarrow 0$ we have:

$$\mathrm{Coker}(\alpha^*) = E_n(X) \otimes \mathbb{Z}_{p^\infty} \quad \text{and} \quad \mathrm{Ker}(\alpha^*) = \mathrm{Tor}_p E_{n-1}(X).$$

The proof of this proposition is essentially the same as of the exactness of the universal coefficient sequence in [Ada70, p. 200].

Corollary 2.44 *For any prime p there is a short exact sequence:*

$$0 \longrightarrow KO_n(\tilde{\mathbb{R}}\pi) \otimes \mathbb{Z}_{p^\infty} \xrightarrow{i} KO_n(\tilde{\mathbb{R}}\pi;\mathbb{Z}_{p^\infty}) \longrightarrow \mathrm{Tor}_p KO_{n-1}(\tilde{\mathbb{R}}\pi) \longrightarrow 0$$

where Tor_p *denotes the p-torsion.*

The torsion term is of course 0 if $p \neq 2$. With the help of this corollary we can calculate $KO_n(\tilde{\mathbb{R}}\pi;\mathbb{Z}_{p^\infty})$: Since we have

$$KO_n(\tilde{\mathbb{R}}\pi;\mathbb{Z}_{p^\infty}) \cong \sum_{\alpha\in\widetilde{\mathrm{Irr}}(\mathbb{R}\pi)} KO_n\Big(F(\alpha);\mathbb{Z}_{p^\infty}\Big),$$

it is enough to look at the following table:

	0	1	2	3	4	5	6	7
$KO_n(\mathbb{R};\mathbb{Z}_{p^\infty})$	$\mathbb{Z}_{p^\infty}$	0	$(\mathbb{Z}_2)$	$(\mathbb{Z}_2)$	$\mathbb{Z}_{p^\infty}$	0	0	0
$KO_n(\mathbb{C};\mathbb{Z}_{p^\infty})$	$\mathbb{Z}_{p^\infty}$	0	$\mathbb{Z}_{p^\infty}$	0	$\mathbb{Z}_{p^\infty}$	0	$\mathbb{Z}_{p^\infty}$	0
$KO_n(\mathbb{H};\mathbb{Z}_{p^\infty})$	$\mathbb{Z}_{p^\infty}$	0	0	0	$\mathbb{Z}_{p^\infty}$	0	$(\mathbb{Z}_2)$	$(\mathbb{Z}_2)$

$$(\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \text{if } p = 2 \\ 0 & \text{otherwise} \end{cases}$$

Theorem 2.45 (J. Rosenberg, S. Stolz [RS]) *For every finite p-group π the map*

$$A(\mathbb{Z}_{p^\infty}) : \widetilde{KO}_n(B\pi;\mathbb{Z}_{p^\infty}) \to KO_n(\tilde{\mathbb{R}}\pi;\mathbb{Z}_{p^\infty})$$

is an isomorphism.

Remark 2.46 This theorem obviously implies that $A(\mathbb{Z}_{p^\infty})\Big(\widetilde{KO}_n(B\pi;\mathbb{Z}_{p^\infty})\Big) \subset KO_n(\tilde{\mathbb{R}}\pi;\mathbb{Z}_{p^\infty})$, and it is easy to see that $A(\widetilde{KO}_n(B\pi)) \subset KO_n(\tilde{\mathbb{R}}\pi)$ holds, too.

Proposition 2.47 $\partial : \widetilde{KO}_n(B\pi;\mathbb{Z}_{p^\infty}) \to \widetilde{KO}_{n-1}(B\pi)$

is an isomorphism where ∂ is the boundary homomorphism in the long exact sequence induced by the short exact coefficient sequence $0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}[1/p] \longrightarrow \mathbb{Z}_{p^\infty} \longrightarrow 0.$

This is a simple consequence of the following fact :

Proposition 2.48 *$\widetilde{KO}_{n-1}(B\pi)$ is a p-torsion group.*

Proof. $B\pi = K(\pi, 1)$ so all homotopy groups of $B\pi$ are p-torsion. By the mod-C Hurewitz theorem [Spa66, p. 512] this implies that all the reduced homology groups of $B\pi$ are p-torsion. The Atiyah-Hirzebruch spectral sequence $E^r_{p,q} \Rightarrow \widetilde{KO}_{n-1}(B\pi)$ in this case has E^2-term:

$$E^2_{p,q} \cong \widetilde{H}_p(B\pi; KO_q(pt)).$$

By the universal coefficient theorem $\widetilde{H}_p(B\pi; KO_q(pt))$ is also p-torsion, so $\widetilde{KO}_{n-1}(B\pi)$ is a p-torsion group.

Remark 2.49 The same argument shows that $\widetilde{\Omega}^{\mathrm{Spin}}_n(B\pi)$ is p-torsion. But Ω^{Spin}_n is zero for $n < 0$, which implies a stronger result: $\widetilde{\Omega}^{\mathrm{Spin}}_n(B\pi)$ is a finite p-group.

Corollary 2.50 *$\widetilde{KO}_{n-1}(B\pi) \cong KO_n(\tilde{\mathbb{R}}\pi;\mathbb{Z}_{p^\infty})$, so it is a sum of $\mathbb{Z}_{p^\infty}$-s and perhaps some $\mathbb{Z}_2$-s.*

Corollary 2.51 ([RS95]) *For $A : \widetilde{KO}_{n-1}(B\pi) \to KO_{n-1}(\tilde{\mathbb{R}}\pi)$ we have:*

(i) $\mathrm{Ker}(A) = \partial \circ \Big(A(\mathbb{Z}_{p^\infty})\Big)^{-1} i(KO_n(\tilde{\mathbb{R}}\pi) \otimes \mathbb{Z}_{p^\infty})$

(ii) $\mathrm{Im}(A) = \mathrm{Tor}_p KO_{n-1}(\tilde{\mathbb{R}}\pi)$.

Proof. Standard diagram chasing in the diagram:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \widetilde{KO}_n(B\pi;\mathbb{Z}_{p^\infty}) \arrow[r, "\partial", "\cong"'] \arrow[d, "A(\mathbb{Z}_{p^\infty})", "\cong"'] & \widetilde{KO}_{n-1}(B\pi) \arrow[r] \arrow[d, "A"] & 0 \arrow[d] \\
KO_n(\tilde{\mathbb{R}}\pi) \otimes \mathbb{Z}_{p^\infty} \arrow[r, "i"] & KO_n(\tilde{\mathbb{R}}\pi;\mathbb{Z}_{p^\infty}) \arrow[r] & \mathrm{Tor}_p KO_{n-1}(\tilde{\mathbb{R}}\pi) \arrow[r] & 0
\end{tikzcd}$$

3 Concordance classes of positive scalar curvature metrics

3.1 Dirac operators on manifolds with boundary

The amazing fact is that the Dirac operator method will give results for the classification problem if we apply it to manifolds with boundary. Suppose that W is an n-dimensional Riemannian spin manifold, $\partial W = M$ and $f : W \to B\pi$. Assume moreover that the metric G is product near the boundary, i.e. it has the form $g + dt^2$ in a collar neighbourhood of the boundary, where g is a metric on M and dt^2 is the standard metric on the line. The same way as for closed spin manifolds in the previous chapter, we can define $\mathrm{Cl}_{\mathrm{Spin}}(W)$ and $\mathbb{R}\pi(W)$ and $D(W)$, but the kernel will be infinite dimensional. There are several ways to solve this problem. We want to use the so called L^2-index, which in our case – positive scalar curvature metric on the boundary – coincides with the the Atiyah-Patodi-Singer index. The θ-map – which is in our main interest – however was defined in terms of a third index [Bun95]. The fact that these indices coincide was proved in [Feh98]. In the meanwhile new developments in the spectral theory of operators on Hilbert-A-modules made it possible to use a new index definition for which both the well definedness of θ and the equality with the classical index are easier to prove (see [Sto]).

Suppose that W is a spin manifold with boundary $\partial W = M$, $f : W \to B\pi$ and G is a metric on W which has product metric near the boundary and $g = G\big|_M$ is a positive scalar curvature metric. Attach an infinite cylinder $M \times [0, \infty)$ to it:

$$\hat{W} := W \cup_M M \times [0, \infty).$$

We extend G to a metric $\hat{G}$ on $\hat{W}$ via the the product metric on $M \times [0, \infty)$ so $(\hat{W}, \hat{G})$ is a complete Riemannian manifold. The map f extends to a map $\hat{f} : \hat{W} \to B\pi$. The metric $\hat{G}$ defines a measure on $\hat{W}$ so it makes sense to speak about L^2-sections. By [APS75a] we can define $D(\hat{W})$ acting on the Hilbert space $L^2(\mathrm{Cl}_{\mathrm{Spin}}(\hat{W}) \otimes \mathbb{R}\pi(\hat{W}))$ and $\mathrm{Ker}D(\hat{W})$ will be finite dimensional.

Definition 3.1 For a spin manifold W with boundary $\mathrm{Ker}D(W) := \mathrm{Ker}D(\hat{W})$.

As we will see this index depends – and only depends – on the concordance class of g. This way it can distinguish different positive scalar curvature metrics. Let us explain this in a more systematic way:

3.2 Positive scalar curvature and the R_n-groups

The best introduction is [Sto]. Here we give a brief summary in the cases we are interested in.

Definition 3.2 Let W be an n-dimensional spin manifold with boundary and let g be a positive scalar curvature metric on ∂W:

$$\mathrm{Riem}^+(W \text{ rel } g) := \left\{ G \;\middle|\; \begin{array}{l} G \text{ is a product metric near the boundary,} \\ G\big|_M = g \text{ and has positive scalar curvature} \end{array} \right\}.$$

If W is closed then $\mathrm{Riem}^+(W)$ denotes the space of positive scalar curvature metrics on W.

Definition 3.3 Let X be an n-dimensional spin manifold, let f be a continuous map to B and let g be a positive scalar curvature metric on X:

$$\mathrm{Pos}_n(B) := \{(X, f, g) | f : X \to B, g \in \mathrm{Riem}^+(X)\}/\sim$$

where (X, f, g) is bordant to (X_1, f_1, g_1) if there is a spin manifold W, a continuous map $F : W \to B$ and a metric $G \in \mathrm{Riem}^+(W \mathrm{rel}\ g \amalg g_1)$ such that W is a spin bordism between X and X_1, and F extends f and f_1.

There is a forgetful map $f : \mathrm{Pos}_n(B) \to \Omega_n^{\mathrm{Spin}}(B)$. In such situations we always have a relative bordism group which fits into a long exact sequence:

$$\ldots \longrightarrow R_{n+1}(B) \longrightarrow \mathrm{Pos}_n(B) \longrightarrow \Omega_n^{\mathrm{Spin}}(B) \longrightarrow R_n(B) \longrightarrow \ldots \tag{1}$$

Definition 3.4

$$R_n(B) := \left\{ (W, f, g) \,\middle|\, \begin{array}{c} W \text{ is an } n\text{-dimensional spin manifold with boundary} \\ f : W \to B \quad \text{and } g \in \mathrm{Riem}^+(\partial W) \end{array} \right\} \Big/ \sim,$$

$(W, f, g) \sim (W_1, f_1, g_1)$ if there is an $n+1$-dimensional spin manifold U, a map $F : U \to B$, an n-dimensional spin manifold V with boundary $\partial W \amalg -\partial W_1$ and a metric $G \in \mathrm{Riem}^+(V \mathrm{rel}\ g \amalg g_1)$ such that

$$\partial U = (W \coprod -W_1) \cup_{\partial W \amalg -\partial W_1} V$$

and F extends f and f_1. It is straightforward to check that $R_n(B)$ is a group and fits into the long exact sequence (1). These groups turned out to be crucial in the existence and classification problem of positive scalar curvature metrics:

Theorem 3.5 (Existence [Haj91]) *Given a connected n-dimensional spin manifold W, $n \geq 5$, and a metric $g \in \mathrm{Riem}^+(\partial W)$, g extends to a positive scalar curvature metric on W (product near the boundary) if and only if*

$$[(W, c, g)] = 0 \in R_n(B\pi)$$

where $\pi_1(W) = \pi$ and $c : W \to B\pi$ is the classifying map of the universal cover of W.

Definition 3.6 Two metrics g, g_1 on a closed manifold M are *concordant* if there is a metric G in $\mathrm{Riem}^+(M \times [0, 1]\ \mathrm{rel}\ g \amalg g_1)$.

Remark 3.7 If two metrics are in the same (path)component of $\mathrm{Riem}^+(M)$ then they are concordant (as an application of the shrinking lemma shows, see Definition 3.11 below). Nothing is known in the other direction.

Theorem 3.8 (Classification [Sto]) *If W is a connected n-dimensional spin manifold, $n \geq 5$, such that $\mathrm{Riem}^+(W\ \mathrm{rel}\ g)$ is not empty, then $R_{n+1}(B\pi)$ acts freely and transitively on the concordance classes of metrics in $\mathrm{Riem}^+(W\ \mathrm{rel}\ g)$, where $\pi = \pi_1(W)$.*

Now we define the map $\theta : R_n(B\pi) \to KO_n(\mathbb{R}\pi)$:

Definition 3.9 Let (W, f, g) be a triple representing an element $[W, f, g]$ in $R_n(B\pi)$:

$$\theta([W, f, g]) := [\mathrm{Ker} D(\mathrm{Cl}_{\mathrm{Spin}}(W) \otimes \mathbb{R}\pi(W, f))].$$

Theorem 3.10 *[Bun95, 1.17] θ is a well defined homomorphism.*

Similarly to the existence problem, it is easier to handle the stable case, when we don't distinguish between M and $M \times B$, where B is the Bott manifold:

Definition 3.11

$$R_n(B\pi)[B^{-1}] := \lim_{\rightarrow} R_{n+8k}(B\pi)$$

where the connecting maps of the inverse system are the following:

$$j_k : R_{n+8k}(B\pi) \to R_{n+8(k+1)}(B\pi), \qquad j_k[W, f, g] = [W \times B, f \times *, \tilde{g}]$$

where $\tilde{g}$ is a positive scalar curvature metric on $\partial W \times B$ constructed the following way:

Take any metric g_B on B and start to rescale it. By [GL80, p. 430] there is a t such that $tg_B \times g$ has positive scalar curvature. This trick is usually called the shrinking lemma.

Proposition 3.12 *The map θ factors through $R_n(B\pi)[B^{-1}]$.*

Proof. We have to show that $\theta(W) = \theta(W \times B)$ under Bott periodicity. This is a slight generalization of Proposition 2.18. Choose a metric g_B on B such that $g \times g_B$ has positive scalar curvature. Then the argument of Proposition 2.18 shows that $\mathrm{Ker}(D(W \times B)) \cong \mathrm{Ker}(D(W)) \otimes \mathrm{Ker}(D(B))$. □

Conjecture 3.13 (Stephan Stolz) *$\theta : R_n(B\pi)[B^{-1}] \to KO_n(C_r^*\pi)$ is an isomorphism for any finitely presented group π.*

Here $C_r^*\pi$ denotes the reduced C^*-algebra of π. For a finite group π we have $C_r^*\pi \cong \mathbb{R}\pi$. Now we are in the position to state our main result:

Theorem 3.14 *For every finite p-group $\theta : R_n(B\pi)[B^{-1}] \to KO_n(\mathbb{R}\pi)$ is* **(p-locally)** *surjective.*

Since $KO_n(\mathbb{R}\pi)$ is the sum of finitely many copies of $\mathbb{Z}$ and $\mathbb{Z}_2$, Theorem 3.14 implies that:

Proposition 3.15 *$\theta : R_n(B\pi) \to KO_n(\mathbb{R}\pi)$ is* **(p-locally)** *surjective for sufficiently large n.*

Remark 3.16 For $\theta : R_n(pt) \to KO_n(pt)$ surjectivity is implied by Carr [Car88] if $n \equiv 0$ and by Hitchin [Hit74] if $n \not\equiv 0$ modulo 4. They show that $A : \Omega_n^{\mathrm{Spin}} \to KO_n(pt)$ is surjective in those dimensions. Since $\theta(W) = A(W)$ for closed manifolds, it is enough to prove the following:

Theorem 3.17 *For every finite p-group $\theta : \tilde{R}_n(B\pi)[B^{-1}] \to KO_n(\tilde{\mathbb{R}}\pi)$ is a surjection where $\tilde{R}_n(B\pi)$ is the kernel of the forgetful map $f : R_n(B\pi) \to R_n(pt)$.*

Remark 3.18 For n odd $KO_n(\mathbb{R}\pi) = \mathrm{Tor}_p KO_n(\mathbb{R}\pi)$ so Corollary 2.51 already proves the surjectivity in odd dimensions.

It would be interesting to find a method to construct elements in $R_n(B\pi)$ with a given value of θ. For cyclic groups such a construction – using lens spaces and bundles of lens spaces – is known [BG95, p.512]. This construction – and the fact that the rational representation ring $\mathcal{R}(\pi) \otimes \mathbb{Q}$ is generated by representations induced from cyclic subgroups for any finite group π – can be used to construct examples such that their image is a subgroup of finite index in $KO_n(\mathbb{R}\pi)$.

3.3 The surjectivity of θ

In this section π denotes a finite p-group. Our main tool for the proof is the following diagram:

$$\begin{tikzcd}
\tilde{R}_n(B\pi)[B^{-1}] \arrow[r, "\partial"] \arrow[d, "\theta"] \arrow[dr, phantom, "1."] & \widetilde{\mathrm{Pos}}_{n-1}(B\pi)[B^{-1}] \arrow[r, "f"] \arrow[d, "\boldsymbol{\eta}"] \arrow[dr, phantom, "2."] & \tilde{\Omega}_{n-1}^{\mathrm{Spin}}(B\pi)[B^{-1}] \arrow[r] \arrow[d, "\overline{\boldsymbol{\eta}}"] \arrow[dr, phantom, "3."] & \tilde{R}_{n-1}(B\pi)[B^{-1}] \arrow[d, "\theta"] \\
KO_n(\tilde{\mathbb{R}}\pi) \arrow[r, "q"] & KO_n(\tilde{\mathbb{R}}\pi; \mathbb{Z}[1/p]) \arrow[r, "r"] & KO_n(\tilde{\mathbb{R}}\pi; \mathbb{Z}_{p^\infty}) \arrow[r, "\delta"] & KO_{n-1}(\tilde{\mathbb{R}}\pi)
\end{tikzcd}$$

Diagram 1

In section 3.3.1 we define the various groups and maps involved in Diagram 1 and we show the exactness of the top row. Then we prove the commutativity of Diagram 1, except square 2, which will be postponed to section 5. In section 3.3.2 we use this diagram to prove the surjectivity of θ. The argument is a slight generalization of the 5-lemma, which can be proved by standard diagram chasing:

Lemma 3.19 *Suppose that in the following commutative diagram all the rows and columns are exact:*

$$\begin{tikzcd}
 & & \bullet \arrow[r, two heads] \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d, "\theta"] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet
\end{tikzcd}$$

Then the map θ is surjective.

Comparison of Diagram 1 with the lemma shows that we have to prove the surjectivity of three maps:

1. $\overline{\boldsymbol{\eta}} : \tilde{\Omega}_n^{\mathrm{Spin}}(B\pi)[B^{-1}] \to KO_{n+1}(\tilde{\mathbb{R}}\pi; \mathbb{Z}_{p^\infty})$ – Proposition 3.25,
2. $\boldsymbol{\eta} : \widetilde{\mathrm{Pos}}_{n-1}(B\pi)[B^{-1}] \to KO_n(\tilde{\mathbb{R}}\pi; \mathbb{Z}[1/p])$ – Proposition 3.26,
3. f restricted to $\mathrm{Ker}(\boldsymbol{\eta})$ gives a surjection: $\mathrm{Ker}(\boldsymbol{\eta}) \to \mathrm{Ker}(\overline{\boldsymbol{\eta}})$ – Proposition 3.27.

3.3.1 The definition of Diagram 1

$\tilde{\Omega}_n^{\mathrm{Spin}}(B\pi)[B^{-1}]$ and $\widetilde{\mathrm{Pos}}_n(B\pi)[B^{-1}]$ have definitions similar to the definition of $\tilde{R}_n(B\pi)[B^{-1}]$:

Definition 3.20

$$\tilde{\Omega}_n^{\mathrm{Spin}}(X) := \mathrm{Ker}(p^* : \Omega_n^{\mathrm{Spin}}(X) \to \Omega_n^{\mathrm{Spin}}(pt)) \quad \text{and} \quad \widetilde{\mathrm{Pos}}_n(X) := \mathrm{Ker}(p^* : \mathrm{Pos}_n(X) \to \mathrm{Pos}_n(pt)).$$

$$\tilde{\Omega}_n^{\mathrm{Spin}}(B\pi)[B^{-1}] := \varinjlim \tilde{\Omega}_{n+8k}^{\mathrm{Spin}}(B\pi),$$

where the connecting maps are the following:

$$j_k : \tilde{\Omega}_{n+8k}^{\mathrm{Spin}}(B\pi) \to \tilde{\Omega}_{n+8(k+1)}^{\mathrm{Spin}}(B\pi), \qquad j_k[M,f] = [M \times B, f \times *]$$

and

$$\widetilde{\mathrm{Pos}}_n(B\pi)[B^{-1}] := \varinjlim \widetilde{\mathrm{Pos}}_{n+8k}(B\pi)$$

where the connecting maps are the following:

$$j_k : \widetilde{\mathrm{Pos}}_{n+8k}(B\pi) \to \widetilde{\mathrm{Pos}}_{n+8(k+1)}(B\pi), \qquad j_k[M,g] = [M \times B, \tilde{g}]$$

where $\tilde{g}$ is defined by shrinking as in Definition 3.11.

Remark 3.21 The top row of Diagram 1 is exact since it is the direct limit of exact sequences.

The bottom row is the long exact sequence induced by the coefficient sequence:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}[1/p] \longrightarrow \mathbb{Z}_{p^\infty} \longrightarrow 0.$$

Definition 3.22 Let $U : \tilde{\Omega}_{n-1}^{\mathrm{Spin}}(B\pi)[B^{-1}] \to \widetilde{KO}_{n-1}(B\pi)$ denote the map induced by the Atiyah orientation and let $\partial : \widetilde{KO}_n(B\pi; \mathbb{Z}_{p^\infty}) \to \widetilde{KO}_{n-1}(B\pi)$ denote the boundary map. Then

$$\overline{\boldsymbol{\eta}} := A(\mathbb{Z}_{p^\infty})\partial^{-1}U.$$

In order to define $\boldsymbol{\eta}$ first we show the commutativity of the already defined square 3.

Proposition 3.23 *Square 3. is commutative.*

Proof. The commutativity of the diagram below implies the commutativity of square 3.

$$\begin{tikzcd}
\tilde{\Omega}^{\mathrm{Spin}}_{n-1}(B\pi)[B^{-1}] \arrow[r] \arrow[d, "U"] & \tilde{R}_{n-1}(B\pi)[B^{-1}] \arrow[ddd, "\theta"] \\
\widetilde{KO}_{n-1}(B\pi) \arrow[d, "\partial^{-1}"] \arrow[ddr, "A"] & \\
\widetilde{KO}_n(B\pi;\mathbb{Z}_{p^\infty}) \arrow[d, "A(\mathbb{Z}_{p^\infty})"] & \\
KO_n(\tilde{\mathbb{R}}\pi;\mathbb{Z}_{p^\infty}) \arrow[r, "\delta"] & KO_{n-1}(\tilde{\mathbb{R}}\pi)
\end{tikzcd}$$

The maps A and θ are defined as the index of the Dirac operator of a manifold representing an element in $\widetilde{KO}_{n-1}(B\pi)$ and $\tilde{R}_{n-1}(B\pi)[B^{-1}]$ respectively. For a closed manifold the two indices coincide which implies the commutativity of the upper square. The commutativity of the lower – triangle shaped – square is the consequence of the fact that A and $A(\mathbb{Z}_{p^\infty})$ are induced by the same spectrum map and the two other sides are the connecting homomorphisms in the long exact sequences induced by the coefficient sequence:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}[1/p] \longrightarrow \mathbb{Z}_{p^\infty} \longrightarrow 0.$$

Strictly speaking it shows the commutativity of the diagram in which ∂^{-1} is replaced with ∂ and the corresponding arrow is reversed, but these two statements are equivalent. $\square$

Corollary 3.24 *There exists a unique map $\boldsymbol{\eta}$ which makes square 1. commutative.*

Proof. $\tilde{\Omega}^{\mathrm{Spin}}_{n-1}(B\pi)$ is p-torsion by Remark 2.49, so for any $x \in \widetilde{\mathrm{Pos}}_{n-1}(B\pi)$ there is a p-power k such that $f(kx) = 0$. By the exactness of the top row of Diagram 1 there exists a $y \in \tilde{R}_n(B\pi)$ such that $\partial y = kx$. The desired commutativity of square 1. then forces us to define

$$\boldsymbol{\eta}(x) := \frac{1}{k} q\theta(y) \in KO_n(\tilde{\mathbb{R}}\pi) \otimes \mathbb{Z}[1/p].$$

(Note that k – being a power of p – is invertible in $\mathbb{Z}[1/p]$.) The commutativity of square 3. implies that $\boldsymbol{\eta}$ is well defined. $\square$

The commutativity of square 2. will be proved in the next chapter.

3.3.2 Three surjective maps

Proposition 3.25 *The homomorphism $\overline{\boldsymbol{\eta}} : \tilde{\Omega}^{\mathrm{Spin}}_n(B\pi)[B^{-1}] \to KO_{n+1}(\tilde{\mathbb{R}}\pi;\mathbb{Z}_{p^\infty})$ is surjective.*

Proof. $A(\mathbb{Z}_{p^\infty})$ and ∂^{-1} are isomorphisms and U is surjective by Theorem 2.29. $\square$

Proposition 3.26 *The homomorphism* $\boldsymbol{\eta}$ *is surjective.*

Proof. **This proof wasn't correct. I am still working on a correct one.**

Proposition 3.27 f *maps* $\mathrm{Ker}(\boldsymbol{\eta})$ *to* $\mathrm{Ker}(\overline{\boldsymbol{\eta}})$ *surjectively.*

We postpone the proof to the next section (after Theorem 4.11), since we need the comparison between $\boldsymbol{\eta}$ and the classical η-invariants. It is possible to give a purely homotopy theoretical proof. As we mentioned in Theorem 2.29 the results of [KS93] imply that if $a \in \mathrm{Ker}(\overline{\boldsymbol{\eta}}) = \mathrm{Ker}(U)$ then a can be represented by the total space of a geometric $\mathbb{HP}^2$-bundle P such that the $B\pi$-structure comes from the base. They also imply that there is an other geometric $\mathbb{HP}^2$-bundle such that for its total space W we have $\partial[W] = kP$, which immediately implies Proposition 3.27, since W admits a positive scalar curvature metric extending the standard positive scalar curvature metric on P therefore $\theta(W) = 0$. The argument is somewhat involved so we preferred the analytic proof. The existence of a homotopy theoretical proof is very interesting since it shows that it is possible to give a proof of the surjectivity of θ without any reference to the classical η-invariant.

This completes the proof of Theorem 3.14.

4 Connection between $\boldsymbol{\eta}$ and the classical eta invariants

4.1 Complexification

In this section we study the complexification map from the KO-groups of $\mathbb{R}\pi$ to the KO-groups of $\mathbb{C}\pi$. We calculate the kernel of the complexification map for different coefficients. In section 4.2 we use these results to compare the map $\boldsymbol{\eta}$ and a map $\boldsymbol{\eta}_{\mathbb{C}}$ which connects $\boldsymbol{\eta}$ with the classical eta invariants. On the other hand in section 5 we show that the index theorem for flat bundles implies a complexified version of the commutativity of square 2. of Diagram 1. These calculations about the complexification map imply that this complexified version can be "lifted" to the real case.

For the spectrum $KO(\mathbb{C}\pi)$ we will also use the notation $K(\mathbb{C}\pi)$ (this is compatible with the usual definition of the K-theory of complex C^*-algebras).

Tensoring with $\mathbb{C}$ induces a map $c : KO_n(\mathbb{R}\pi) \to K_n(\mathbb{C}\pi)$. For a coefficient group G let $c(G)$ denote the induced map:

$$c(G) : KO_n(\mathbb{R}\pi) \otimes G \to K_n(\mathbb{C}\pi) \otimes G$$

Definition 4.1 Let M be a spin manifold and recall that $\hat{\boldsymbol{\eta}} = r \circ \boldsymbol{\eta}$ is the mod $\mathbb{Z}$ reduction of $\boldsymbol{\eta}$.

$$\begin{aligned} A_{\mathbb{C}} &:= c \circ A, & \theta_{\mathbb{C}} &:= c \circ \theta, \\ \boldsymbol{\eta}_{\mathbb{C}} &:= c(\mathbb{Z}[1/p]) \circ \boldsymbol{\eta}, & \hat{\boldsymbol{\eta}}_{\mathbb{C}} &:= c(\mathbb{Z}_{p^\infty}) \circ \hat{\boldsymbol{\eta}}. \end{aligned}$$

Theorem 4.2 *$A_{\mathbb{C}} : K(B\pi) \to K(\mathbb{C}\pi)$ is induced by a map of spectra $\mathbf{A}_{\mathbb{C}}$, and can be represented as slant product with $[\mathbb{C}\pi(B\pi)] \in K^0_{\mathbb{C}\pi}(B\pi)$ where $\mathbb{C}\pi(B\pi) = \mathbb{R}\pi(B\pi) \otimes \mathbb{C}$.*

Proof. Completely analogous to the proof of Theorem 2.37.

At this point we can raise the question how much information is lost by complexification. We use a long exact sequence related to the cofibration of spectra $S^0 \longrightarrow C\eta \longrightarrow S^2$ where η represents the generator of the stable homotopy group $\pi_1^S(S^0)$:

Proposition 4.3 *[Sch93, p.22] For any real C^*-algebra $\mathcal{C}$ we have the long exact sequence:*

$$\longrightarrow KO_{n-1}(\mathcal{C}) \xrightarrow{\gamma_{n-1}} KO_n(\mathcal{C}) \xrightarrow{c} KO_n(\mathcal{C} \otimes_{\mathbb{R}} \mathbb{C}) \longrightarrow KO_{n-2}(\mathcal{C}) \xrightarrow{\gamma_{n-2}} KO_{n-1}(\mathcal{C}) \longrightarrow$$

where γ is multiplication by the generator of $KO_1(\mathbb{R})$.

Corollary 4.4

$$\begin{aligned} &\mathrm{Ker}(c) = \mathrm{Tor}_p KO_n(\mathbb{R}\pi), && \mathrm{Coker}(c) \cong \mathrm{Ker}(\gamma_{n-2}),\\ &\mathrm{Ker}\Big(c(\mathbb{Z}[\tfrac{1}{p}])\Big) = 0, && \mathrm{Ker}\Big(c(\mathbb{Z}_{p^\infty})\Big) = \mathrm{Tor}_p\Big(\mathrm{Ker}(\gamma_{n-2})\Big). \end{aligned}$$

The last statement can be derived by a snake lemma argument (see [Feh98, App.B] for details). What will be important later is that $\mathrm{Ker}\Big(c(\mathbb{Z}_{p^\infty})\Big)$ is a finite group.

4.2 The eta invariants and the APS index theorem for manifolds with boundary

Let us recall now the definition of η_α from [APS76]. We slightly changed the notation to be consistent with ours and of the recent literature.

Definition 4.5 For n even let $\mathbb{S}^+$ be the unique irreducible ungraded left Cl_{n-1}-module for which $\omega^{\mathbb{C}}_{n-1} \in \mathrm{Cl}_{n-1}$ acts as the identity, where $\omega^{\mathbb{C}}_{n-1}$ is the complex volume form $i^{[\frac{n}{2}]}e_1 \cdots e_{n-1}$ and let M be an n-1-dimensional spin manifold with finite fundamental group π:

$$D^+_\alpha(M) := D\Big(\mathrm{Cl}_{\mathrm{Spin}}(M) \otimes_{\mathbb{C}} \alpha(M) \otimes_{\mathrm{Cl}_{n-1}} \mathbb{S}^+\Big) = D\Big(\mathbb{S}^+(M) \otimes_{\mathbb{C}} \alpha(M)\Big)$$

where α is a $\mathbb{C}\pi$-module, $\alpha(M) = \widetilde{M} \times_\pi \alpha$ is equipped with the flat connection and $\mathbb{S}^+(M) = \mathrm{Spin}(M) \times_{\mathrm{Spin}(n-1)} \mathbb{S}^+$.

In the construction of $\mathbb{S}^+(M)$ we used the $\mathrm{Spin}(n-1)$-action coming from the inclusion $\mathrm{Spin}(n-1) \subset \mathrm{Cl}^\times_{n-1} \subset \mathbb{C}\mathrm{l}^\times_{n-1}$.

Definition 4.6 For any self-adjoint operator A :

$$\eta_A(s) := \sum_{\lambda \neq 0} \operatorname{sign}(\lambda)|\lambda|^{-s}$$

where λ runs through the eigenvalues of A counted by multiplicity. Moreover we define:

$$\eta_\alpha(s) := \eta_{D_\alpha^+(M)}(s).$$

This sum doesn't necessarily converge. However:

Theorem 4.7 ([APS75a, 3.10]) *If A is a first order self-adjoint differential operator over the compact odd dimensional manifold M then $\eta_A(s)$ converges absolutely for $Re(s)$ large and extends to a meromorphic function on the whole s-plane with a finite value at $s = 0$.*

This result validates the next definition:

Definition 4.8 Let M be an n-1-dimensional spin manifold with finite fundamental group π and α be an irreducible $\mathbb{C}\pi$-module:

$$\eta_\alpha(M) := -\frac{1}{2}(\eta_\alpha(0) - \eta_N(0))$$

where $\dim_{\mathbb{C}} \alpha = N$ and N also denotes the same dimensional trivial representation.

Remark 4.9 In general $\eta_\alpha(M) = -\frac{1}{2}\Big(\eta_\alpha(0) + \dim \operatorname{Ker}(D_\alpha^+(M)) - \eta_N(0) - N \dim \operatorname{Ker}(D_1^+(M))\Big)$ is the definition but we always assume that the metric on M has positive scalar curvature, so $\dim \operatorname{Ker}(D_\alpha^+(M)) = \dim \operatorname{Ker}(D_1^+(M)) = 0$. Here 1 denotes the trivial 1-dimensional $\mathbb{C}\pi$-module.

Let us recall from 2.10 that for n even we denoted by $\mathbb{S}_n$ the unique irreducible $\mathbb{Z}_2$-graded left $\mathrm{Cl_n}$-module for which the action of $\omega_n^{\mathbb{C}} \in \mathrm{Cl_n}$ coincides with the grading operator T, where $\omega_n^{\mathbb{C}}$ is the complex volume form $i^{[\frac{n+1}{2}]} e_1 \cdots e_n$.

Definition 4.10 Let $\alpha \in \operatorname{Irr}(\mathbb{C}\pi)$ and let $t_\alpha : K_n(\mathbb{C}\pi) \to K_0(\mathbb{C})$ be defined by the rule:

$$t_\alpha[N] := [N \otimes_{\mathrm{Cl_n} \otimes \mathbb{C}\pi} (\mathbb{S} \otimes \alpha)]$$

Theorem 4.11 *$\eta_\alpha(M) = t_\alpha \boldsymbol{\eta}_{\mathbb{C}}(M)$ under the canonical isomorphism $K(pt; \mathbb{Z}[1/p]) \cong \mathbb{Z}[1/p]$.*

Proof. Straightforward calculation comparing the index of $D(M)$ and $D_\alpha^+(M)$ and using the Atiyah-Patodi-Singer index theorem for manifolds with boundary ([APS75a, 4.11, 4.13]).

This theorem shows that the η_α's are the "coordinates" of $\boldsymbol{\eta}_{\mathbb{C}}$ – or of $\boldsymbol{\eta}$, since $\operatorname{Ker}\Big(c(\mathbb{Z}[\frac{1}{p}])\Big) = 0$. So to prove the surjectivity of $f : \operatorname{Ker}(\boldsymbol{\eta}) \to \operatorname{Ker}(\overline{\boldsymbol{\eta}})$ (Proposition 3.27) it is enough to show that:

Proposition 4.12 *If $a \in \mathrm{Ker}(\overline{\boldsymbol{\eta}}) \subset \tilde{\Omega}^{\mathrm{Spin}}_{n-1}(B\pi)$ then there is a manifold T such that $[T] = a$, and a positive scalar curvature metric g on T such that $\eta_\alpha(T, g) = 0$ for all $\mathbb{C}\pi$-module α.*

Proof. By [BGS, lemma 4.12] the total space T of a geometric $\mathbb{HP}^2$-bundle carries a positive scalar curvature metric g, such that $\eta_\alpha(T, g) = 0$ for every α. On the other hand if $a \in \mathrm{Ker}(\overline{\boldsymbol{\eta}}) = \mathrm{Ker}(U)$ then a can be represented by the total space of a geometric $\mathbb{HP}^2$-bundle by Theorem 2.29. □

5 Commutativity of square 2

In this section we finish the the proof of the commutativity of Diagram 1. The missing piece was the commutativity of the second square:

Theorem 5.1 *The following diagram is commutative:*

$$\begin{array}{ccc}
\widetilde{\mathrm{Pos}}_{n-1}(B\pi)[B^{-1}] & \xrightarrow{f} & \tilde{\Omega}^{\mathrm{Spin}}_{n-1}(B\pi)[B^{-1}] \\
\Big\downarrow \boldsymbol{\eta} & & \Big\downarrow \overline{\boldsymbol{\eta}} \\
KO_n(\tilde{\mathbb{R}}\pi; \mathbb{Z}[1/p]) & \xrightarrow{r} & KO_n(\tilde{\mathbb{R}}\pi; \mathbb{Z}_{p^\infty})
\end{array}$$

or equivalently we have to show that:

$$\hat{\boldsymbol{\eta}}(M) = A(\mathbb{Z}_{p^\infty})\partial^{-1}Uf(M),$$

where $\hat{\boldsymbol{\eta}} = r \circ \boldsymbol{\eta}$.

[Feh98] contains a very detailed proof. A different proof can be obtained by developing a certain additivity theorem for families of Dirac operators on manifolds with boundary ([Feh]). Here we give a sketch of the proof in [Feh98]. First we prove a complex analogue of Theorem 5.1:

Theorem 5.2 *For every n-1-dimensional (n even!) spin manifold M with $\pi_1(M) = \pi$ is a finite p-group and for every $\alpha \in \mathrm{Irr}(\mathbb{C}\pi)$:*

$$\hat{\boldsymbol{\eta}}_{\mathbb{C}}(M) = A_{\mathbb{C}}(\mathbb{Z}_{p^\infty})\partial^{-1}Uf(M).$$

Then we show how you can "lift" this result to prove Theorem 5.1. The starting point is the index theorem of Atiyah, Patodi and Singer for flat bundles.

Theorem 5.3 ([APS76, Thm (5.3)]) *Let M be an n-1-dimensional (n even) spin manifold with $\pi_1(M) = \pi$ finite and $\alpha \in \mathrm{Irr}(\mathbb{C}\pi)$. Then*

$$\hat{\eta}_\alpha(M) = \mathrm{Ind}([\sigma(D)] \cdot [\alpha]).$$

Here $[\sigma(D)] \in K^1(TM)$ is the self-adjoint symbol of $D = D(\mathbb{S}^+(M))$ used by Atiyah, Patodi and Singer in [APS76, p.81]. By [APS75b, p.430] there is an element $[\alpha] \in K^{-1}(M;\mathbb{R}/\mathbb{Z})$ such that $\partial[\alpha] = [\alpha(M)] \in K^0(M)$ where ∂ denotes the boundary map induced by the coefficient sequence $0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z} \longrightarrow 0$, and $\alpha(M)$ is the flat bundle over M corresponding to the $\mathbb{C}\pi$-module α and $\pi = \pi_1(M)$. $\cdot$ denotes the pairing

$$K^{-1}(M;\mathbb{R}/\mathbb{Z}) \otimes K^1(TM) \to K^0(TM;\mathbb{R}/\mathbb{Z}).$$

$$\mathrm{Ind} : K^0(TM;\mathbb{R}/\mathbb{Z}) \to K^0(pt;\mathbb{R}/\mathbb{Z}) \cong \mathbb{R}/\mathbb{Z}$$

is the extension of the topological index, in other words the transfer map $p_!$ where $p : TM \to pt$. Since we assume that π is a finite p-group we can replace $\mathbb{R}/\mathbb{Z}$ with $\mathbb{Z}_{p^\infty}$.

One can show that $\sigma(D)$ represents a K-theory Thom class for TM (similarly to [ML89, p.275]) so analogously to [ML89, 16.10]:

$$\mathrm{Ind}([\sigma(D)] \cdot [\alpha]) = q_!([\alpha]),$$

where $q : M \to pt$ and $q_! : K^{-1}(M;\mathbb{Z}_{p^\infty}) \to K^{n-2}(pt;\mathbb{Z}_{p^\infty}) \cong K^0(pt;\mathbb{Z}_{p^\infty})$ is the transfer map corresponding to the map q. (In [ML89] you find the real case and without coefficients but the proof is essentially the same.) Almost by definition:

$$q_!([\alpha]) = [\alpha]/[[M]]$$

where $[[M]] \in K_{n-1}(M)$ is the K-theory fundamental class of M.

We can work with all the $\mathbb{C}\pi$-modules at the same time and, using the fact that

$$\mathbb{C}\pi(M) = \bigoplus \alpha(M) \otimes_{\mathbb{C}} \alpha^*$$

by [AS69, p.15] where α^* is the right $\mathbb{C}\pi$-module dual to α and $\alpha(M) = \tilde{M} \times_\pi \mathbb{C}\pi$, we get:

$$\hat{\boldsymbol{\eta}}_{\mathbb{C}}(M) = [V]/[[M]]$$

where $[V] \in K^{-1}_{\mathbb{C}\pi}(M;\mathbb{Z}_{p^\infty})$ such that $\partial[V] = [\mathbb{C}\pi(M)] \in K^0_{\mathbb{C}\pi}(M)$ and ∂ denotes the boundary map induced by the coefficient sequence $0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}[1/p] \longrightarrow \mathbb{Z}_{p^\infty} \longrightarrow 0$. Also, we used the identification $K(\mathbb{C}\pi;\mathbb{Z}_{p^\infty}) \cong K(\mathbb{C}\pi) \otimes \mathbb{Z}_{p^\infty}$.

Choose an $[N] \in K_n(B\pi;\mathbb{Z}_{p^\infty})$ such that $\partial[N] = c^M_*[M] \in K_{n-1}(B\pi)$. Here $c^M : M \to B\pi$ denotes the classifying map of the universal cover of M so $c^M_*[[M]] = [M]$ is the class M represents in $K_{n-1}(B\pi)$. Such an $[N]$ exists since we assume that $[M] \in \widetilde{K}_{n-1}(B\pi)$ and $\widetilde{K}_{n-1}(B\pi) \subset \mathrm{Im}\partial$.

A quite straightforward calculation (using that $\partial a/b = a/\partial b$) shows that

$$[V]/[[M]] = [\mathbb{C}\pi(M)]/[N].$$

But slant product with $[\mathbb{C}\pi(M)]$ represents the complex assembly map (Theorem 4.2), so

$$[\mathbb{C}\pi(M)]/[N] = A_{\mathbb{C}}(\mathbb{Z}_{p^\infty})\partial^{-1}[M] = A_{\mathbb{C}}(\mathbb{Z}_{p^\infty})\partial^{-1}Uf(M),$$

which proves Theorem 5.2.

Proof of Theorem 5.1. We show that Theorem 5.2 implies Theorem 5.1. First we need the following:

Proposition 5.4 *$\hat{\boldsymbol{\eta}}$ has the following factorization:*

$$\begin{tikzcd}
\widetilde{\mathrm{Pos}}_{n-1}(B\pi)[B^{-1}] \arrow[d, "\hat{\boldsymbol{\eta}}"] \arrow[dr, "U\circ f"] & \\
& \mathrm{Im}(U\circ f) \arrow[dl, dashed, "\varphi"] \\
KO_n(\tilde{\mathbb{R}}\pi)\otimes\mathbb{Z}_{p^\infty} &
\end{tikzcd}$$

Proof. The proposition is equivalent with the statement $\mathrm{Ker}(U\circ f)\subset\mathrm{Ker}(\hat{\boldsymbol{\eta}})$,which is a consequence of the following two observations:

1. $\mathrm{Ker}(f)\subset\mathrm{Ker}(\hat{\boldsymbol{\eta}})$. This is a simple consequence of the definition of $\boldsymbol{\eta}$: changing the metric on M changes $\boldsymbol{\eta}(M)$ by an "integer" – an element in the image of $q: KO_n(\tilde{\mathbb{R}}\pi)\to KO_n(\tilde{\mathbb{R}}\pi)\otimes\mathbb{Z}_{p^\infty}$.

2. For every $M\in\mathrm{Ker}(U)$ there is an element (M,g) in $f^{-1}(M)$ such that $\boldsymbol{\eta}(M,g)=0$. This is just a reformulation of Proposition 3.27. □

To simplify the notation let us denote the map φ by $\hat{\boldsymbol{\eta}}$ and recall (Proposition 3.26) that $\mathrm{Im}(U\circ f)=\mathrm{Ker}(A)$. So we have the following diagram:

$$\begin{tikzcd}
& & & \mathrm{Ker}(A) \arrow[dll, bend right, "\delta"'] \arrow[dl, shift right, "A(\mathbb{Z}_{p^\infty})\partial^{-1}"'] \arrow[dl, shift left, "\hat{\boldsymbol{\eta}}"] \arrow[d, "\hat{\boldsymbol{\eta}}_{\mathbb{C}}"] \\
0 \arrow[r] & \mathrm{Tor}_p\Big(\mathrm{Ker}(\gamma)\Big) \arrow[r] & KO_n(\tilde{\mathbb{R}}\pi)\otimes\mathbb{Z}_{p^\infty} \arrow[r, "c"] & K_n(\tilde{\mathbb{C}}\pi)\otimes\mathbb{Z}_{p^\infty}
\end{tikzcd}$$

where $\delta:=\hat{\boldsymbol{\eta}}-A(\mathbb{Z}_{p^\infty})\partial^{-1}$. Since $\hat{\boldsymbol{\eta}}$ and $A(\mathbb{Z}_{p^\infty})\partial^{-1}$ are both lifts of the same map, δ defines a map to $\mathrm{Tor}_p\Big(\mathrm{Ker}(\gamma)\Big)$. But this is a finite group. On the other hand $\mathrm{Ker}(A)$ is isomorphic to $\bigoplus_m\mathbb{Z}_{p^\infty}$ (Corollary 2.51) and therefore it has no nontrivial homomorphism to a finite group. Which implies that $\delta=0$ and $\hat{\boldsymbol{\eta}}=A(\mathbb{Z}_{p^\infty})\partial^{-1}$. And we finished the proof of Theorem 5.1.

Remark 5.5 This last diagram shows that $\hat{\boldsymbol{\eta}}$ contains more information than $\hat{\boldsymbol{\eta}}_{\mathbb{C}}$ if p=2. The possibility of refining the classical mod $\mathbb{Z}$ reduced eta invariant was first observed in [BGS].Corollary 2.51 implies that $\hat{\boldsymbol{\eta}}$ is an isomorphism, so no further refinement is possible.

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