THOM POLYNOMIALS, SYMMETRIES AND INCIDENCES OF SINGULARITIES

RICHÁRD RIMÁNYI

Abstract. As an application of the generalized Pontryagin-Thom construction [RSz] here we introduce a new method to compute cohomological obstructions of removing singularities — i.e. Thom polynomials [T]. With the aid of this method we compute some sample results, such as the Thom polynomials associated to all stable singularities of codimension ≤ 8 between equal dimensional manifolds, and some other Thom polynomials associated to singularities of maps $N^n \longrightarrow P^{n+k}$ for $k > 0$. We also give an application by reproving a weak form of the multiple point formulas of Herbert and Ronga ([H], [Ro2]). As a byproduct of the theory we define the incidence class of singularities, which — the author believes — may turn to be an interesting, useful and simple tool to study incidences of singularities.

1. Introduction

On the basis of the "generalized Pontryagin-Thom construction" of [RSz] here we present a new method to compute Thom polynomials ([T]) of singularities. Recall that the Thom polynomial $Tp(\eta)$ of a singularity η is a polynomial in the Chern classes of the map f between complex analytic manifolds N, P, and this polynomial equals the Poincaré dual $[\eta(f)]$ of the cycle carried by the closure of

$$\eta(f) = \{x \in N \mid \text{the singularity of } f \text{ at } x \text{ is } \eta\ \}$$

for most maps. About the existence of such a polynomial see [AVGL] or section 6 here. By Chern classes of a map $f : N \longrightarrow P$ we mean the Chern classes of the virtual bundle $f^*TP - TN$ over N. The cohomology class $[\eta(f)]$ is most easily understood when $\eta(f)$ is a submanifold, which is often the case if f has no more complicated singularities than η. In this case $\eta(f)$ carries a fundamental homology class. We take the image of this class in the homology of N and apply Poincaré duality. The resulting class is $[\eta(f)] \in H^*(N;\mathbb{Z})$. Although the definition of $[\eta(f)]$ is not much more difficult when $\eta(f)$ is not a manifold (it has singularities along lower dimensional strata), the interesting thing is that we will not need this. We will only use the definition of $[\eta(f)]$ in the mentioned case. Observe that this is a difference from the desingularization method (see e.g. [G]), where the behavior of $\eta(f)$ near the singular part is studied.

Now we clarify what we will mean by singularities, whose Thom polynomials we are studying. Let $k \geq 0$ be a fixed integer. By *singularity* we will mean an equivalence class of stable germs $(\mathbb{C}^*, 0) \longrightarrow (\mathbb{C}^{*+k}, 0)$ under the equivalence generated by right-left equivalence and suspension (by suspension of a germ κ we mean its trivial unfolding: $(x, v) \mapsto (\kappa(x), v)$). According to the classical Mather theory singularities are in a one-to-one correspondence with (finite dimensional local) $\mathbb{C}$-algebras. So we might as well denote a singularity by its local algebra. Instead, we

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will use — more or less — standard notations as follows: A_i ($= \Sigma^{1_i}$) will stand for the stable germs with local algebra $\mathbb{C}[[x]]/(x^{i+1})$ and we will adapt Mather's notation for the ones of Thom-Boardman type $\Sigma^{2,0}$:

$I_{a,b}$ for those with algebra $\mathbb{C}[[x,y]]/(xy, x^a + y^b)$ $b \geq a \geq 2$,
$III_{a,b}$ for those with algebra $\mathbb{C}[[x,y]]/(xy, x^a, y^b)$ $b \geq a \geq 2$ $(k \geq 1)$.

Recall also that among the germs in a singularity η there is a germ κ (defined up to right-left equivalence) such that all the other germs in η are right-left equivalent to a suspension of κ. We will call such a κ a *prototype* of the singularity η. E.g. a prototype of the singularity A_i (for $k=0$) is

$$(x, u_1 \dots, u_{i-1}) \mapsto (x^{i+1} + \sum_{j=1}^{i-1} u_j x^j, u_1, \dots, u_{i-1}).$$

In [RSz] only those settings were handled where moduli do not occur. For simplicity therefore in this paper we will also avoid the settings where moduli come into the picture (recall e.g. that there is a Thom polynomial of Σ^3 between equal dimensional manifolds, but Σ^3 is not a "singularity" in our sense, it is rather the union of a family of singularities).

The milestones of the history of Thom polynomials are (probably among others) the works of Thom, Levine, Porteous, Ronga, Gaffney, Turnbull, Ohmoto (see the References, or the review on the history of Thom polynomials in [AVGL]). Wherever we present a Thom polynomial in this paper the origin of that result will be indicated — according to the author's best knowledge. The author thanks A. Szűcs, I. Porteous and T. Ohmoto for some helpful conversations and/or encouragement; and the many who pointed out an error in a formal version of this paper.

2. Main theorem

Let $k \geq 0$ and let $\eta : (\mathbb{C}^*, 0) \longrightarrow (\mathbb{C}^{*+k}, 0)$ be a stable singularity with prototype $\kappa : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^{n+k}, 0)$.

Definition 2.1. (The total Chern class and the Euler class associated to a singularity.) *The maximal compact subgroup of the right-left symmetry group*

$$Aut\ \kappa = \{\ (\phi, \psi) \in Diff(\mathbb{C}^n, 0) \times Diff(\mathbb{C}^{n+k}, 0) \mid \psi \circ \kappa \circ \phi^{-1} = \kappa\ \}$$

of κ will be denoted by G_η. Its representations (projections) on the source $\mathbb{C}^n$ and the target $\mathbb{C}^{n+k}$ spaces will be $\lambda_1(\eta)$ and $\lambda_2(\eta)$. The vector bundles associated to the universal principal G_η-bundle $EG_\eta \longrightarrow BG_\eta$ using the representations $\lambda_1(\eta)$ and $\lambda_2(\eta)$ will be called $\bar{\xi}_\eta$ and ξ_η. The total Chern class of the singularity η is defined as

$$c(\eta) := \frac{c(\xi_\eta)}{c(\bar{\xi}_\eta)} \in H^{\Pi}(BG_\eta, \mathbb{Z})$$

where $c(\xi) = 1 + c_1(\xi) + c_2(\xi) + \dots$ is the total Chern class of the vector bundle ξ. Let the Euler class $e(\eta) \in H^{2codim\ \eta}(BG_\eta, \mathbb{Z})$ be the Euler class of the bundle $\bar{\xi}_\eta$.

Let us make a few remarks about this definition. It is clear that $Aut\ \kappa$ is much too big to be a (finite dimensional) Lie group. Still, the concept of its maximal compact subgroup (up to conjugacy) can be defined in a sensible way, see [W], [R1] — in fact, not only the definition but the computation of G_η is feasible. It is clear that G_η can be chosen so that the images of its projection to its factors $Diff(\mathbb{C}^n)$, $Diff(\mathbb{C}^{n+k})$ are linear, so we can indeed talk about the *representations* $\lambda_i(\eta)$ $(i=1,2)$. The ring $H^{\Pi}(X)$ is almost like the usual cohomology ring $H^*(X)$ of the space X with the only difference that in its elements we allow infinitely many non-zero coefficients, see [MS] p.39. With the exception of section 7 from now on all cohomology is meant with integer coefficients without any further notice.

Let $\eta \neq \zeta : (\mathbb{C}^*, 0) \longrightarrow (\mathbb{C}^{*+k}, 0)$ be stable singularities.

Definition 2.2. (The hierarchy of singularities.) *The singularity η will be called* more difficult *than ζ if in any neighbourhood of 0 in the source space of (a representative of) η there is a point where the singularity of (the representative of) η is ζ. In this case we will write $\zeta < \eta$. Let us adapt the convention $\eta \not< \eta$, too.*

For instance in the case $k = 0$ we have the following relations: $A_k > A_l$ iff $k > l$; $I_{2,2} > A_1, A_2, A_3$; $I_{2,3} \geq A_1, A_2, A_3, A_4, I_{2,2}$ etc. Some more results on the hierarchy is e.g. in [La]. A trivial but key observation is that a more difficult singularity must have larger codimension, so:

Proposition 2.3. *codim* $\zeta \geq$ *codim* $\eta \qquad \Rightarrow \qquad \zeta \not< \eta.$ □

Suppose we want to compute the Thom polynomial $Tp(\eta)$. Our main theorem gives its value under some substitutions.

Theorem 2.4. (Main theorem.)

$$Tp(\eta)\Big(c(\zeta)\Big) = \begin{cases} e(\zeta) & if\ \zeta = \eta \\ 0 & if\ \zeta \not> \eta\ and\ \zeta \neq \eta. \end{cases}$$

The proof will be given in the next chapter, and applications in chapter 5. Observe that this theorem does not state anything about the value of $Tp(\eta)\Big(c(\zeta)\Big)$ for $\zeta > \eta$, c.f. chapter 8.

Sometimes it will be convenient not to work with the whole maximal compact subgroup G_η but with a subgroup $G'_\eta \leq G_\eta$. For such a G'_η one can define $c'(\eta)$ and $e'(\eta)$ in the same way as in definition 2.1. The following corollary of the main theorem will be useful in section 5.

Corollary 2.5.

$$Tp(\eta)\Big(c'(\zeta)\Big) = \begin{cases} e'(\zeta) & if\ \zeta = \eta \\ 0 & if\ \zeta \not> \eta\ and\ \zeta \neq \eta \end{cases}$$

for any subgroup $G'_\zeta \leq G_\zeta$.

Proof. These equations are homomorphic images of the equations of theorem 2.4 at the natural homomorphism: $H^{\Pi}(BG_\eta; \mathbb{Z}) \longrightarrow H^{\Pi}(BG'_\eta, \mathbb{Z})$. □

Of course, if we choose G'_ζ too small (e.g. the trivial subgroup) then the statement of the corollary is trivial — so to get reasonable statement we have to choose G'_ζ as 'close' to G_ζ as possible.

3. Proof of the main theorem

The proof uses the generalized Pontryagin-Thom construction [RSz], whose idea goes back to earlier works of A. Szűcs (e.g. [Sz1]), see references therein. Let us now recall the main results. Meanwhile (and it the whole paper) we will use the following notation: if ξ is a vector bundle then $D(\xi)$, $S(\xi)$ will denote its ball and sphere bundle, respectively. If A is a submanifold in B then $\nu(A \subset B)$ or simply ν_A will denote the normal bundle of the embedding. The closure of a subspace X will be called *closure* X. Note that overline does not mean closure.

Let $k \geq 0$ be fixed and let τ be a set of multi-singularities $(\mathbb{C}, S) \longrightarrow (\mathbb{C}^{*+k}, 0)$ (S a finite set) satisfying that $\zeta < \eta$, $\eta \in \tau \Rightarrow \zeta \in \tau$. For this latter notion, of course, we have to extend the definition of $<$ to multi-singularities as in [RSz]. A map $f : N \longrightarrow P$ is called a τ-map if for any $y \in P$ the singularity of f at $f^{-1}(y)$ is from τ. The result of [RSz] is a concrete description of a universal τ-map $f\tau : Y\tau \longrightarrow X\tau$. The topological spaces $Y\tau$ and $X\tau$ are glued together from so called blocks, corresponding to elements of τ. The blocks in $Y\tau$ and $X\tau$ corresponding to η are constructed as follows. Let G_η be the maximal compact subgroup of the right-left automorphism group of κ (the prototype of η), with the representations $\lambda_1(\eta)$ and $\lambda_2(\eta)$ on the source and target spaces. We can associate vector bundles to the universal principal G_η-bundle $EG_\eta \longrightarrow BG_\eta$ using these representations, we obtain $\bar{\xi}_\eta$, ξ_η. The block in $Y\tau$ corresponding to η is the disc bundle of $\bar{\xi}_\eta$ and the block in $X\tau$ corresponding to η is the disc bundle of ξ_η. One can also define a natural map $D(\bar{\xi}_\eta) \longrightarrow D(\xi_\eta)$ which is η in each fibre, obtaining the block

$$\begin{array}{ccccc} D(\bar{\xi}_\eta) & \longrightarrow & D(\xi_\eta) & & \\ \downarrow & & \downarrow & & \\ \bar{K}_\eta & \longrightarrow & K_\eta & = & BG_\eta \end{array}$$

Remark 3.1. Let us remark here that in case of a multisingularity, i.e. if S has more than 1 element, some notions in the above paragraph should be meant more general. In this case $\lambda_1(\eta)$ is a generalized representation, that is a representation on some disjoint ($|S|$ copies of) vector spaces (permutations allowed). Thus the result of the association to $EG_\eta \longrightarrow BG_\eta$ with $\lambda_1(\eta)$ will not be a vector bundle, but a composition of a vector bundle (the left-hand arrow in the diagram above) with an $|S|$-sheeted covering (the bottom arrow). If $|S| = 1$, then $\bar{K}_\eta = BG_\eta$, too, and the bottom arrow is a diffeomorphism.

The gluing of these blocks are defined recursively. Suppose we have already constructed $f\tau' : Y\tau' \longrightarrow X\tau'$ and want to attach the block $D(\bar{\xi}_\eta) \longrightarrow D(\xi_\eta)$, where both τ' and $\tau' \cup \{\eta\}$ satisfy the condition on τ above. The way of gluing is detailed in [RSz], here we will only need that $S(\bar{\xi}_\eta)$ is glued to $Y\tau'$ and $S(\xi_\eta)$ is glued to $X\tau$.

Now we describe what we mean by $f\tau$ being a "universal τ-map". In [RSz] — where the real case is considered — theorem 1 has two parts: (A) and (B). Although the proof of part (B) contains smooth techniques, which do not work in the complex case, part (A) holds with no change. According to this for any τ-map $N \longrightarrow P$ there are maps g and h such that the diagram

$$\begin{array}{ccc} P & \xrightarrow{g} & X\tau \\ \uparrow{\scriptstyle f} & & \uparrow{\scriptstyle f\tau} \\ N & \xrightarrow{h} & Y\tau \end{array}$$

is commutative. Also for all $\eta \in \tau$ we have that

- $\eta(f) = h^{-1}(\bar{K}_\eta)$ and the normal bundle of $\eta(f)$ in
$$N \setminus \{\ \zeta(f) \mid \zeta > \eta\ \}$$
is isomorphic to $h^*(\bar{\xi}_\eta)$ (for simplicity here h is written instead of its restriction $h|_{\bar{K}_\eta}$);
- $f(\eta(f)) = g^{-1}(K_\eta)$ and the normal bundle of $f(\eta(f))$ in
$$P \setminus \{\ f(\zeta(f)) \mid \zeta > \eta\ \}$$
is isomorphic to $g^*(\bar{\xi}_\eta)$ (for simplicity here g is written instead of its restriction $g|_{K_\eta}$).

Now we start the proof of the main (2.4) theorem. Let us fix η and let τ be the following set of (multi)singularities $\tau = \{\zeta | \zeta \not> \eta\}$. For a τ-map $f : N \longrightarrow P$ the subset $\eta(f)$ is a submanifold, so its cohomology class $[\eta(f)]$ (the Poincaré dual of the fundamental homology class it carries) is defined. The minimum that we expect from the Thom polynomial associated to η is
$$Tp(\eta)\big(c(f)\big) = [\eta(f)]\ \in H^*(N)$$
for all τ-maps. Now let $\zeta \in \tau$ different from η. If we restrict the above cohomological identity to $\eta(f)$ and $\zeta(f)$ we get

$$\begin{array}{rcll} Tp(\eta)\big(c(\nu_{f(\eta(f))} - \nu_{\eta(f)})\big) & = & e(\nu_{\eta(f)}) & \in H^*(\eta(f)) \quad \text{and} \\ Tp(\eta)\big(c(\nu_{f(\zeta(f))} - \nu_{\zeta(f)})\big) & = & 0 & \in H^*(\zeta(f)). \end{array} \tag{1}$$

Here we implicitly identified $\eta(f)$ with its f-image and used some standard facts from differential topology, such as

- $c(f) = c(f^*TP - TN)$ restricted to a submanifold $M \subset N$ for which $f|_M$ is a diffeomorphism onto $f(M)$ is
$$\frac{c(TM \oplus \nu_{f(M)})}{c(TM \oplus \nu_M)} = \frac{c(\nu_{f(M)})}{c(\nu_M)} = c(\nu_{f(M)} - \nu_M).$$
- The cohomology class of a submanifold restricted to the submanifold itself is the Euler class of its normal bundle.
- The cohomology class of a submanifold M restricted to a subset which is disjoint from M is 0.

Now we use the result of the generalized Pontryagin-Thom construction: the map $f : N \longrightarrow P$ induces the maps $g : P \longrightarrow X\tau$, $h : N \longrightarrow Y\tau$. By abuse of notation let us denote the restriction of these maps to some subsets (e.g. $f(\eta(f))$, $\eta(f)$, $f(\zeta(f))$ and $\zeta(f)$) by the same letter. We have that
$$\nu_{f(\eta(f))} = g^*\xi_\eta, \qquad \nu_{\eta(f)} = h^*\bar{\xi}_\eta, \qquad \nu_{f(\zeta(f))} = g^*\xi_\zeta, \qquad \nu_{\eta(f)} = h^*\bar{\xi}_\zeta,$$
and therefore formula (1) (after appropriate identifications) yields
$$Tp(\eta)\big(h^*c(\bar{\xi}_\eta - \xi_\eta)\big) = e(h^*\bar{\xi}_\eta) \qquad Tp(\eta)\big(h^*c(\bar{\xi}_\zeta - \xi_\zeta)\big) = 0.$$
Keeping our eyes on the definition of $c(\eta)$, $c(\zeta)$, $e(\eta)$, we have
$$Tp(\eta)\big(h^*c(\eta)\big) = h^*e(\eta) \qquad Tp(\eta)\big(h^*c(\zeta)\big) = 0,$$
that is
$$h^*\Big(\ Tp(\eta)\big(c(\eta)\big) - e(\eta)\ \Big) = 0 \qquad h^*\Big(\ Tp(\eta)\big(c(\zeta)\big)\ \Big) = 0.$$

Since this holds for all τ-maps f (and the map h is induced from f) we have the same identities without h^*. This proofs the theorem. $\square$

4. Symmetries of singularities

To effectively use theorem 2.4 we need to study the maximal compact subgroup of some singularities. In this section we will use the following notations: ρ_i will mean the standard i-dimensional representation of the unitary group $U(i)$. The powers of representations are meant *tensor powers.* The following theorem is essentially from [W] (see also [R1]).

Theorem 4.1. *Let η be a singularity whose prototype is $\kappa : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^{n+k}, 0)$. The germ κ is the miniversal unfolding of another germ $\beta : (\mathbb{C}^m, 0) \longrightarrow (\mathbb{C}^{m+k}, 0)$ with $d\beta = 0$. The group G_η is a subgroup of the maximal compact subgroup of the algebraic automorphism group of the local algebra Q_η times $U(k-D)$, where D is the defect of Q_η.* [1] *With β well chosen G_η acts as a right-left symmetry group on β with representations μ_1 and μ_2. The representations λ_1, λ_2 are equal to $\mu_1 \oplus \mu_V$, $\mu_2 \oplus \mu_V$ respectively, where μ_V is the representation of G_η on the unfolding space $V \cong \mathbb{C}^{n-m}$ with the formula*

$$\begin{aligned} (\phi, \psi) \cdot \alpha := \psi \circ \alpha \circ \phi^{-1} \quad & \textit{for} \quad \alpha \in V \leq Germs\big((\mathbb{C}^m, 0) \longrightarrow (\mathbb{C}^{m+k}, 0)\big) \\ & \textit{and} \quad (\phi, \psi) \in G_\eta \leq Aut\ \beta \leq Diff(\mathbb{C}^m, 0) \times Diff(\mathbb{C}^{m+k}, 0). \end{aligned}$$

$\square$

Remark 4.2. Observe that the formulas given for λ_1 and λ_2 imply that one does not have to determine μ_V (so does not have to work with κ, but only with β) when wants to compute $c(\eta)$. Observe also that β's source and target dimensions are usually much smaller than those of κ.

Let us see these notions on examples. Consider first the case $k = 0$, i.e. singularities between equal dimensional manifolds. Let η be the set of stable singularities of type A_i ($i > 0$). Its prototype κ maps from $(\mathbb{C}^i, 0)$ to $(\mathbb{C}^i, 0)$ and is the miniversal unfolding of the map $\beta : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}, 0)$, $x \mapsto x^{i+1}$ ($d\beta = 0$). The maximal compact subgroup of the automorphism group of the algebra $\mathbb{C}[[x]]/(x^{i+1})$ is $U(1)$ (defect$=0$), so $G_{A_i} = U(1)$ and it acts on β (as right-left symmetry group) with the representations ρ_1, ρ_1^{i+1}. An appropriately chosen unfolding space V is generated by the germs $g_j : x \mapsto x^j$ for $j = 1, 2, \ldots, i-1$. The formula in the theorem implies that $G_\eta = U(1)$ acts on the line spanned by g_j with the representation ρ_1^{-j+i+1}. Therefore we have proved the following theorem.

Theorem 4.3. *For $k = 0$*

$$G_{A_i} = U(1), \qquad \lambda_1(A_i) = \bigoplus_{j=1}^{i} \rho_1^j, \qquad \lambda_2(A_i) = \bigoplus_{j=2}^{i+1} \rho_1^j.$$

$\square$

Now we consider the case $\eta = I_{a,b}$ where $a, b \geq 2$, $a \neq b$. Its prototype maps from $(\mathbb{C}^{a+b}, 0)$ to $(\mathbb{C}^{a+b}, 0)$ and is the miniversal unfolding of $\beta : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^2, 0)$, $(x, y) \mapsto (xy, x^a + y^b)$ ($d\beta = 0$). The maximal compact subgroup of $Aut(\mathbb{C}[[x, y]]/(xy, x^a + y^b))$ is $U(1)$ and it acts on β (as right-left symmetry group) with the representations

$$\rho_1^{\frac{b}{d}} \oplus \rho_1^{\frac{a}{d}}, \qquad \rho_1^{\frac{a+b}{d}} \oplus \rho_1^{\frac{ab}{d}},$$

[1] The defect of an algebra is the integer D such that if we present the algebra as a quotient of a formal power series algebra with minimal number of relations then $D = \#\{relations\} - \#\{generators\}$.

where d is the greatest common divisor of a and b. An appropriately chosen unfolding space V is generated by the germs $f_j : (x, y) \mapsto (0, x^j)$ for $j = 1, 2, \dots, a-1$ and $g_j : (x, y) \mapsto (0, y^j)$ for $j = 1, 2, \dots, b-1$. The formula in the theorem implies that $G_\eta = U(1)$ acts on the line spanned by f_j with the representation $\rho_1^{b(a-j)/d}$ and on the line spanned by g_j with the representation $\rho_1^{a(b-j)/d}$. Therefore we have proved the following theorem.

Theorem 4.4. *For $\eta = I_{a,b}$ $(a \neq b)$ between equal dimensional manifolds $G_\eta = U(1)$, $\lambda_1(\eta) = \mu_1 \oplus \mu_V$, $\lambda_2(\eta) = \mu_2 \oplus \mu_V$, where*

$$\mu_1 = \rho_1^{\frac{b}{d}} \oplus \rho_1^{\frac{a}{d}}, \qquad \mu_2 = \rho_1^{\frac{a+b}{d}} \oplus \rho_1^{\frac{ab}{d}}, \qquad \mu_V = \bigoplus_{j=1}^{a-1} \rho_1^{b(a-j)/d} \oplus \bigoplus_{j=1}^{b-1} \rho_1^{a(b-j)/d}.$$

□

The following theorems use the same argument, so we give only the results.

Theorem 4.5. *For $\eta = I_{a,a}$, $a > 2$ between equal dimensional manifolds the group G_η has an index 2 subgroup $G'_\eta = U(1)$. The formulas for the representations restricted to this subgroup are the same as in theorem 4.4 (with $b, d = a$).* □

Theorem 4.6. *For $\eta = I_{2,2}$ between equal dimensional manifolds the group G_η has an index 2 subgroup $G'_\eta = U(1) \times U(1)$. The representations restricted to this subgroup are $\lambda_1 = \mu_1 \oplus \mu_V$, $\lambda_2 = \mu_2 \oplus \mu_V$, where*

$$\mu_1 = \rho_{1(1)} \oplus \rho_{1(2)}, \qquad \mu_2 = \rho_{1(1)}^2 \oplus \rho_{1(2)}^2, \qquad \mu_V = \rho_{1(1)}^2 \otimes \rho_{1(2)}^{-1} \oplus \rho_{1(2)}^2 \otimes \rho_{1(1)}^{-1}.$$

(The number i in the bracket refers to a representation of the i'th factor.)

□

Now we are ready to compute the total Chern classes and Euler classes associated to the singularities mentioned.

Corollary 4.7. *For $k = 0$*

$$\begin{array}{lcll}
c(A_i) & = & \frac{1+(i+1)x}{1+x} = 1 + ix - ix^2 + ix^3 - \dots & \in \mathbb{Z}[[x]] \\
e(A_i) & = & i! \cdot x^i & \in \mathbb{Z}[x] \\
c'(I_{2,2}) & = & \frac{(1+2x)(1+2y)}{(1+x)(1+y)} = & \\
 & & 1 + (x+y) + (-x^2 + xy - y^2) + (x^3 - x^2y - xy^2 + y^3) + \dots & \in \mathbb{Z}[[x,y]] \\
e'(I_{2,2}) & = & xy(2x-y)(2y-x) & \in \mathbb{Z}[x,y] \\
c(I_{a,b})_{a\neq b} & = & \frac{(1+\frac{a+b}{d}\cdot x)(1+\frac{ab}{d}\cdot x)}{(1+\frac{b}{d}\cdot x)(1+\frac{a}{d}\cdot x)} & \in \mathbb{Z}[[x]] \\
e(I_{a,b})_{a\neq b} & = & \frac{b}{d}x \cdot \frac{a}{d}x \cdot \Pi_{j=1}^{a-1}\left(\frac{b(a-j)}{d}x\right) \cdot \Pi_{j=1}^{b-1}\left(\frac{a(b-j)}{d}x\right) = \frac{a!b!a^{b-1}b^{a-1}}{d^{a+b}} \cdot x^{a+b} & \in \mathbb{Z}[x] \\
c'(I_{a,a})_{a>2} & = & \frac{(1+2x)(1+ax)}{(1+x)(1+x)} & \in \mathbb{Z}[[x]] \\
e'(I_{a,a})_{a>2} & = & (a-1)!^2 \cdot x^{2a} & \in \mathbb{Z}[x] \square
\end{array}$$

The letters c' and e' instead of c, e mean that we used a subgroup G'_η (given above) instead of the whole G_η.

This method to compute maximal compact symmetry groups and thus to compute total Chern classes and Euler classes of singularities works for $k > 0$, too. As an example let us present without proof the result on Morin singularities A_n $(n > 0)$ for any $k \geq 0$.

Theorem 4.8. *Let $k \geq 0$. The prototype of $A_n : (\mathbb{C}^*, 0) \longrightarrow (\mathbb{C}^{*+k}, 0)$ is the miniversal unfolding of $\kappa : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}^{k+1}, 0)$, $(x) \mapsto (x^{n+1}, 0, \dots, 0)$. So $G_{A_n} = U(1) \times U(k)$ with*

$$\mu_1 = \rho_1, \qquad \mu_2 = \rho_1^{n+1} \oplus \rho_k, \qquad \mu_V = \bigoplus_{j=2}^{n} \rho_1^j \oplus \bigoplus_{j=1}^{n} (\rho_k \otimes \rho_1^{-j}).$$

Hence

$$\begin{array}{rcll} c(A_n) & = & \frac{1+(n+1)x}{1+x}(1 + y_1 + y_2 + \dots + y_k) & \in \mathbb{Z}[[x, y_1, y_2, \dots, y_k]] \\ e(A_n) & = & n! x^n \Pi_{j=1}^n (y_k - jxy_{k-1} + j^2x^2y_{k-2} - \dots + (-1)^k j^k x^k) & \in \mathbb{Z}[x, y_1, y_2, \dots, y_k]. \end{array}$$

□

5. Computation of Thom polynomials

In this section we will use theorem 2.4 and the computations in section 4 to compute some Thom polynomials. Let us start with the setting where the most activity took place in the past, i.e. singularities of maps between equal dimensional manifolds.

Theorem 5.1. *The polynomials in this table are Thom polynomials associated to the given singularities between equal dimensional manifolds. By $c = 1 + c_1 + c_2 + \dots$ we mean $c(TP - f^*TN)$.*

codim 1

A_1	c_1	[T]

codim 2

A_2	$c_1^2 + c_2$	[folklore]

codim 3

A_3	$c_1^3 + 3c_1c_2 + 2c_3$	[folklore]

codim 4

A_4	$c_1^4 + 6c_1^2c_2 + 2c_2^2 + 9c_1c_3 + 6c_4$	[G]
$I_{2,2}$	$c_2^2 - c_1c_3$	[P1]

codim 5

A_5	$c_1^5 + 10c_1^3c_2 + 25c_1^2c_3 + 10c_1c_2^2 + 38c_1c_4 + 12c_2c_3 + 24c_5$	[Tu]
$I_{2,3}$	$2c_1c_2^2 - 2c_1^2c_3 + 2c_2c_3 - 2c_1c_4$	[P2]

codim 6

A_6	$c_1^6 + 15c_1^4c_2 + 55c_1^3c_3 + 30c_1^2c_2^2 + 141c_1^2c_4 + 79c_1c_2c_3 + 5c_2^3 +$ $202c_1c_5 + 55c_2c_4 + 17c_3^2 + 120c_6$	
$I_{2,4}$	$2c_1^2c_2^2 + 3c_2^3 - 2c_1^3c_3 + 2c_1c_2c_3 - 3c_3^2 - 5c_1^2c_4 + 9c_2c_4 - 6c_1c_5$	
$I_{3,3}$	$c_1^2c_2^2 - c_2^3 - c_1^3c_3 + 3c_1c_2c_3 + 3c_3^2 - 2c_1^2c_4 - 3c_2c_4$	[P2]

codim 7

A_7	$c_1^7 + 21c_1^5c_2 + 105c_1^4c_3 + 70c_1^3c_2^2 + 399c_1^3c_4 + 301c_1^2c_2c_3 + 35c_1c_2^3 + 960c_1^2c_5 +$ $467c_1c_2c_4 + 139c_1c_3^2 + 58c_2^2c_3 + 1284c_1c_6 + 326c_2c_5 + 154c_3c_4 + 720c_7$
$I_{2,5}$	$2c_1^3c_2^2 + 9c_1c_2^3 - 2c_1^4c_3 + 14c_2^2c_3 - 17c_1c_3^2 - 9c_1^3c_4 + 29c_1c_2c_4 - 10c_3c_4 -$ $26c_1^2c_5 + 34c_2c_5 - 24c_1c_6$
$I_{3,4}$	$2c_1^3c_2^2 - c_1c_2^3 - 2c_1^4c_3 + 8c_1^2c_2c_3 - 2c_2^2c_3 + 13c_1c_3^2 - 7c_1^3c_4 - 5c_1c_2c_4 + 10c_3c_4 -$ $6c_1^2c_5 - 10c_2c_5$
(x^2, y^3)	$2c_1c_2^3 - 2c_1^2c_2c_3 + 2c_2^2c_3 + 2c_1c_3^2 - 4c_1c_2c_4 + 2c_3c_4 - 2c_2c_5$ [P2]

codim 8

A_8	$c_1^8+28c_1^6c_2+140c_1^4c_2^2+140c_1^2c_2^3+14c_2^4+182c_1^5c_3+868c_1^3c_2c_3+501c_1c_2^2c_3+$ $642c_1^2c_3^2+202c_2c_3^2+952c_1^4c_4+2229c_1^2c_2c_4+364c_2^2c_4+1559c_1c_3c_4+332c_4^2+$ $3383c_1^3c_5+3455c_1c_2c_5+954c_3c_5+7552c_1^2c_6+2314c_2c_6+9468c_1c_7+5040c_8$
$I_{2,6}$	$2c_1^4c_2^2+18c_1^2c_2^3+7c_2^4-2c_1^5c_3-4c_1^3c_2c_3+57c_1c_2^2c_3-53c_1^2c_3^2+4c_2c_3^2-$ $14c_1^4c_4+60c_1^2c_2c_4+74c_2^2c_4-90c_1c_3c_4-3c_4^2-71c_1^3c_5+166c_1c_2c_5-$ $39c_3c_5-154c_1^2c_6+162c_2c_6-120c_1c_7$
$I_{3,5}$	$2c_1^4c_2^2+3c_1^2c_2^3-4c_2^4-2c_1^5c_3+8c_1^3c_2c_3+10c_1c_2^2c_3+13c_1^2c_3^2+24c_2c_3^2-$ $11c_1^4c_4+7c_1^2c_2c_4-32c_2^2c_4+34c_1c_3c_4-20c_4^2-26c_1^3c_5-2c_1c_2c_5+$ $60c_3c_5-24c_1^2c_6-40c_2c_6$
$I_{4,4}$	$c_1^4c_2^2-c_1^2c_2^3+2c_2^4-c_1^5c_3+6c_1^3c_2c_3-5c_1c_2^2c_3+18c_1^2c_3^2-11c_2c_3^2-5c_1^4c_4-$ $9c_1^2c_2c_4+13c_2^2c_4+27c_1c_3c_4+23c_4^2-6c_1^3c_5-29c_1c_2c_5-21c_3c_5-2c_2c_6$
(x^2+y^3,xy^2)	$2c_1^2c_2^3+c_2^4-2c_1^3c_2c_3+4c_1c_2^2c_3+2c_1^2c_3^2+2c_2c_3^2-7c_1^2c_2c_4+2c_2^2c_4+$ $5c_1c_3c_4+c_4^2-9c_1c_2c_5+3c_3c_5-4c_2c_6$

Proof. Consider the following table of singularities up to codimension 8 between equal dimensional manifolds, together with their Thom-Boardman class (although it it not important here, we count codimension in the source), see e.g. [WP]:

$codim_{\mathbb{C}}$	Σ^0	Σ^1	$\Sigma^{2,0}$			$\Sigma^{2,1}$
0	A_0					
1		A_1				
2		A_2				
3		A_3				
4		A_4	$I_{2,2}$			
5		A_5	$I_{2,3}$			
6		A_6	$I_{2,4}$	$I_{3,3}$		
7		A_7	$I_{2,5}$	$I_{3,4}$		(x^2,y^3)
8		A_8	$I_{2,6}$	$I_{3,5}$	$I_{4,4}$	(x^2+y^3,xy^2)

Let η be one of these singularities. If $codim\ \zeta \leq codim\ \eta$ and $\zeta \neq \eta$ then $\zeta \not> \eta$ (proposition 2.3) and therefore $Tp(\eta)\Big(c^{(')}(\zeta)\Big)=0$ which gives a restriction on the possible values of the coefficients in $Tp(\eta)$. (The sign $^{(')}$ means that c is meant with or without the prime according to which is computed in section 4.) In fact if $G_\zeta^{(')}=U(1)$ and hence its cohomology ring is a polynomial ring in one variable then this restriction is a linear equation. If $G_\zeta^{(')}$ is $U(1)\times U(1)$ then it gives a number of linear equations. Anyway, considering all these restrictions we get a system of linear equations on the coefficients of $Tp(\eta)$. We can add the additional linear equation(s) coming from $Tp(\eta)(c^{(')}(\eta))=e^{(')}(\eta)$. These all turn to be uniquely solvable in the cases given in the theorem. The actual computations were carried out by the computer algebra package called *Maple*. A detailed description of the calculation for $Tp(A_4)$ can be found in [R2]. ∎

Carrying out the computations mentioned, one can see that although in some cases there are some redundancies in the system of linear equations but typically the equations are "quite independent". That is e.g. to compute $Tp(A_8)$ it is not enough to consider only e.g. A_1–A_8 for

ζ, or those and $I_{a,b}$ ($a+b \leq 8$), one has to consider all singularities with codimension at most 8. Thus our method heavily depends on the classification of singularities. Since this classification is known much further than *codim* 8 (see [WP]), the method is probably applicable to compute further Thom polynomials. Now we are presenting a result which gives a partial information on the Thom polynomials of *all* Morin singularities.

Theorem 5.2. *$k=0$. Let $\pi(r)$ denote the set of partitions of r and $\pi(r;i)$ the set of partition in exactly i terms; let σ_j be the j'th elementary symmetric polynomial ($\sigma_0 = 1$).*

- *If $Tp(A_r) = \sum_{I\in\pi(r)} \alpha_I c_I$, then*

$$\sum_{I\in\pi(r;i)} \alpha_I = \sigma_{r-i}(1,2,\dots,r-1) \qquad \textit{for all} \;\; i=1,2,\dots r.$$

- *Let η be a codimension r singularity different from A_r. If $Tp(\eta) = \sum_{I\in\pi(r)} \alpha_I c_I$, then*

$$\sum_{I\in\pi(r;i)} \alpha_I = 0 \qquad \textit{for all} \;\; i=1,2,\dots r.$$

Proof . For both A_r and η we have $A_1, A_2, \dots, A_{r-1} \not> A_r, \eta$. So theorem 2.4 gives $r-1$ linear equations for the coefficients both of $Tp(A_r)$ and $Tp(\eta)$. We can add one more equation: $Tp(A_r)\big(c(A_r)\big) = e(A_r)$ in case of A_r and $Tp(\eta)\big(c(A_r)\big) = 0$ in case of η. So in both case we have r linear equations on the coefficients of the relevant Thom polynomial, whose general solution is given in the theorem. □

Remark 5.3. Theorem 5.2 is sufficient to determine $Tp(A_1)$, $Tp(A_2)$, $Tp(A_3)$ completely. From A_4 on, however, it is not enough, e.g. for A_4 it only gives:

$$c_1^4 + 6c_1^2c_2 + Ac_2^2 + Bc_1c_3 + 6c_4, \qquad \text{where} \qquad A+B=11,$$

cf. remark 6.4.

Corollary 5.4.
- *In the polynomial $Tp(A_r)$ the coefficient of c_1^r is 1, that of c_r is $(r-1)!$, that of $c_1^{r-2}c_2$ is $r(r-1)/2$ ($r\geq 2$).*
- *In the polynomial $Tp(\eta)$ ($\eta \neq A_r$ is of codimension r) the coefficients of the terms c_1^r, c_r, $c_1^{r-2}c_2$ are all 0.*

□

Theorem 5.2 assures that there must be negative coefficients in $Tp(\eta)$ if $\eta \neq A_r$ (unless $Tp(\eta) = 0$, is it possible?). Since the converse is true for A_r till it is computed (and not only for $k=0$) let put it as a conjecture.

Conjecture 5.5. *All the non-zero coefficients of $Tp(A_r)$ (for any $k \geq 0$) are positive.*

Since our method works for $k>0$, too, let us give a few results in this setting.

Theorem 5.6. *The following are Thom polynomials for $k=1$*

A_1	c_2	[T], [P1]
A_2	$c_1c_3+c_2^2+2c_4$	[Ro1]
A_3	$2c_1^2c_4+3c_1c_2c_3+10c_1c_5+c_2^3+7c_2c_4+c_3^2+12c_6$	
A_4	$c_2^4+6c_1c_2^2c_3+2c_1^2c_3^2+4c_2c_3^2+9c_1^2c_2c_4+16c_2^2c_4+17c_1c_3c_4+$ $+11c_4^2+6c_1^3c_5+53c_1c_2c_5+21c_3c_5+54c_1^2c_6+76c_2c_6+156c_1c_7+144c_8$	
$III_{2,2}$	$c_3^2-c_2c_4$	[P1]
$I_{2,2}$	$c_1c_3^2-c_1c_2c_4+2c_3c_4-2c_2c_5$	
$III_{2,3}$	$2c_2c_3^2-2c_2^2c_4+2c_1c_3c_4-2c_1c_2c_5-4c_2c_6+6c_3c_5-2c_4^2$	

Proof. The proof is essentially the same as that of theorem 5.1 — with the only difference that here we have to consider the hierarchy basically implied by the following table

$codim_{\mathbb{C}}$	Σ^0	Σ^1	Σ^2
0	A_0		
1			
2		A_1	
3			
4		A_2	
5			
6		A_3	$III_{2,2}$
7			$I_{2,2}$
8		A_4	$III_{2,3}$

and of course, we have to consider the symmetries of these singularities (some of them is given in theorem 4.8). □

Carrying out these kind of calculations for singularities of *codim* >8 ($k=1$) or for singularities with a concrete $k>1$ seems to be a question of computer capacity. However, another challenge is to find Thom polynomials consisting k as a parameter. The easiest is the computation of $Tp(A_1)$ for any $k\geq 0$, which was the first Thom polynomial computed, and is very easy with the former methods ([T], [P1]). Still let us show how it is proved with our techniques.

Theorem 5.7. $Tp(A_1)=c_{k+1}$.

Proof. According to the main theorem $Tp(A_1)=e(A_1)$, hence according to theorem 4.8, we know that substituting $c_1=-a+b_1$, $c_2=-a^2-ab_1+b_2$, $\ldots$, $c_k=-a^k-a^{k-1}b_1\ldots+b_k$, $c_{k+1}=a^{k+1}-a^kb_1-\ldots-ab_k$ into the polynomial $Tp(A_1)$ we obtain $-a(a^k+a^{k-1}b_1+\ldots+b_k)$. Observe that this latter is c_{k+1} and that c_1, c_2, $\ldots$, c_k, c_{k+1} are polynomially independent in $\mathbb{Z}[a,b_1,\ldots,b_k]$, so $Tp(A_1)$ must be c_{k+1}. □

A bit more delicate but still managable calculation gives the result for A_2 for any $k\geq 0$: the linear equations on the coefficients of $Tp(A_2)$ implied by $Tp(A_2)(c(A_1))=0$ and $Tp(A_2)(c(A_2))=e(A_2)$ have the unique solution

$$Tp(A_2)=c_{k+1}^2+\sum_{j=0}^{k}2^{k-j}c_{2k+2-j}c_j.$$

Since this result has already been known by Ronga [Ro1] and because our proof is pure linear algebra, we omit the proof.

The author was not able to similarly organize the solution of the linear equations implied by

$$Tp(A_3)(c(A_1)) = 0, \quad Tp(A_3)(c(A_2)) = 0, \quad Tp(A_3)(c(A_3)) = e(A_3), \quad Tp(A_3)(c(III_{2,2})) = 0$$

to obtain a general formula for $Tp(A_3)$ — although it can easily be done for concrete k's.

6. Existence

In this section we are discussing the existence of Thom polynomials in terms of our techniques. This will be different from the one in the literature, see e.g. [AVGL] p. 187.

Let $\eta : (\mathbb{C}^*, 0) \longrightarrow (\mathbb{C}^{*+k}, 0)$ be a singularity of codimension d and let τ be the set of multisingularities $\tau = \{\zeta \mid \zeta \not> \eta\}$. Consider them fixed until the final remark of this section.

Let $p(c_1, c_2, \dots, c_d)$ be a weighted homogeneous degree $2d$ polynomial (the weight of c_i is $2i$). Our goal is to analyze the logical connections between the following four statements.

(A) For almost all maps $f : N^n \longrightarrow P^{n+k}$ the cohomology class $[\eta(f)] \in H^{2d}(N)$ is equal to $p(c_1(f), c_2(f), \dots, c_d(f))$, i.e. p is the Thom polynomial associated to η.

(B) For all τ-maps $f : N^n \longrightarrow P^{n+k}$ the cohomology class $[\eta(f)] \in H^{2d}(N)$ is equal to $p(c_1(f), c_2(f), \dots, c_d(f))$.

(C) $[\bar{K}_\eta] = p(c_1(f\tau), c_2(f\tau), \dots, c_d(f\tau)) \;\in\; H^{2d}(Y\tau)$.

(D) $0 = p(c_1(\zeta), c_2(\zeta), \dots, c_d(\zeta)) \;\in\; H^{2d}(\bar{K}_\zeta)$ for all $\zeta \in \tau$, $\zeta \neq \eta$ and $e(\eta) = p(c_1(\eta), c_2(\eta), \dots, c_d(\eta)) \;\in\; H^{2d}(\bar{K}_\eta)$.

The first thing we have to observe is that in (C) $[\bar{K}_\eta]$ and the $c_i(f\tau)$'s are not defined (yet), since the spaces $Y\tau$, $X\tau$ are not manifolds (even less finite dimensional ones). So we first define these notions. Let $[\bar{K}_\eta]$ be the image of the Thom class of $\bar{\xi}_\eta$ under the composition

$$H^{2d}(\bar{\xi}_\eta, \bar{\xi}_\eta - \bar{K}_\eta) \overset{excision}{\cong} H^{2d}(Y\tau, Y\tau - \bar{K}_\eta) \overset{restriction}{\longrightarrow} H^{2d}(Y\tau).$$

(Here we identified vector bundles with their total spaces.) To justify this definition recall that in (B) $[\eta(f)]$ could be defined similarly as the image of the Thom class of $\eta(f)$'s normal bundle under the composition

$$H^{2d}(\nu_{\eta(f)}, \nu_{\eta(f)} - \eta(f)) \overset{excision}{\cong} H^{2d}(N, N - \eta(f)) \overset{restristion}{\longrightarrow} H^{2d}(N).$$

The definition of $c_i(f\tau)$ is a bit more delicate and uses a conjecture which holds for all singularities mentioned in the previous sections.

Conjecture 6.1. *Let ζ be a singularity. The multiplication by $e(\zeta)$ in $H^*(BG_\zeta)$ is a monomorphism.*

In fact, in all the singularities occurred above the ring $H^*(BG_\zeta)$ is a polynomial ring and the element $e(\zeta)$ is different from 0. (Let us remark that unfortunately $e(\zeta) = 0$ quite often over the reals.) Now we define $c_i(f\tau)$ by recursion on τ. Let $c_i(f\{A_0\})$ be the i'th universal Chern class in $H^{2i}(Y\{A_0\}) = H^{2i}(BG_{A_0}) = H^{2i}(BU(k))$. Now suppose we have already defined $c_i(f\tau')$ and want to define $c_i(f\tau)$ where $\tau = \tau' \cup \{\zeta\}$. According to conjecture 6.1 map 4 (and so map 3) in the Gysin sequence

$$\begin{array}{ccccccc}
H^{2i-1}(D\bar{\xi}_\zeta) & \xrightarrow{1} & H^{2i-1}(S\bar{\xi}_\zeta) & \xrightarrow{2} & H^{2i}(D\bar{\xi}_\zeta, S\bar{\xi}_\zeta) & \xrightarrow{3} & H^{2i}(D\bar{\xi}_\zeta) \\
 & & & & || & & || \\
 & & & & H^{2i-2codim\ \zeta}(BG_\zeta) & \xrightarrow{4} & H^{2i}(BG_\zeta)
\end{array}$$

is monomorphism, so map 2 is 0-map, and therefore map 1 is surjective. Then map 5 in the exact sequence

$$H^{2i-1}(Y\tau')\oplus H^{2i-1}(D\bar{\xi}_\zeta)\ \xrightarrow{5}\ H^{2i-1}(S\bar{\xi}_\zeta)\ \xrightarrow{6}\ H^{2i}(Y\tau)\ \xrightarrow{7}\ H^{2i}(Y\tau')\oplus H^{2i}(D\bar{\xi}_\zeta)$$

is also surjective and therefore map 7 is injective. So we can define $c_i(f\tau)$ as the unique inverse image of $(c_i(f\tau'),c_i(\zeta))$ (which maps to 0 in $H^{2i}(S\bar{\xi}_\zeta)$) at map 7.

Remark 6.2. There is another way to define $c_i(f\tau)$ — suggested to the author by A. Szűcs, as follows. One can identify the map $f\tau : Y\tau \longrightarrow X\tau$ with an ideal end-object of the category of all τ-maps, where the morphisms are the *pull-back diagrams* defined in [RSz]. Under such a pull-back morphism Chern classes of τ-maps are mapped to each other. So the Chern classes of the "ideal end-object" can be defined as their direct limit. Of course, some more work is needed to make this definition precise.

Now we study the properties (A) – (D). Clearly (A) $\Rightarrow$ (B), since τ-maps must be among 'most maps'. The definition of $c_i(f\tau)$ and $[\bar{K}_\eta]$ assures that $(B)\Leftrightarrow(C)$ holds. What is proved in the proof of the main theorem is that $(C)\Rightarrow(D)$ since (D) is in fact the restriction of (C) to the $\bar{K}_\zeta$'s $(\zeta\in\tau)$. Now we prove that provided conjecture 6.1 holds then $(D)\Rightarrow(C)$. This will be done by a Mayer-Vietoris type argument. We will use induction: first observe that (C) restricted to $Y\{A_0\}$ holds, this is (D) for $\zeta=A_0$. Now we will prove that if (C) holds restricted to $Y\tau'$ where $\tau'\subset\tau$ then it holds restricted to $Y\tau''$ where $\tau''=\tau\cup\{\zeta\}$ for a $\zeta\in\tau\setminus\tau'$. That is we want to show that the element $[\bar{K}_\eta]-p(c_1(f\tau),c_2(f\tau),\dots,c_d(f\tau))$ in $H^d(Y\tau)$ is 0. Our hypotheses and condition (C) for ζ says that this element is mapped to $(0,0)$ at map 10 in the exact sequence

$$H^{2d-1}(Y\tau')\oplus H^{2d-1}(D\bar{\xi}_\zeta)\ \xrightarrow{8}\ H^{2d-1}(S\bar{\xi}_\zeta)\ \xrightarrow{9}\ H^{2d}(Y\tau)\ \xrightarrow{10}\ H^{2d}(Y\tau')\oplus H^{2d}(D\bar{\xi}_\zeta).$$

Just like in the argument above it is clear that map 8 is surjective (provided conjecture 6.1 holds) and therefore map 10 is injective. This proves our claim.

Conclusions. On one hand we have the partial converse of the main theorem, that is the implication $(D)\Rightarrow(B)$. On the other hand, if there are polynomials satisfying (D), then the implication $(D)\Rightarrow(B)$ can be interpreted as a proof of the existence of Thom polynomials for τ-maps. If the polynomial satisfying (D) is unique and we know the existence of a polynomial satisfying (A) then clearly the one satisfying (D) is the Thom polynomial. Since this is the case in all cases studied by the author let us put it as a conjecture.

Conjecture 6.3. *Let η be singularity. The polynomial satisfying the conditions of theorem 2.4 is* unique.

Remark 6.4. In fact, one could go further and define the set of Thom polynomials $Tp(\eta;\tau)$ of a singularity η modulo a set τ (satisfying that $\eta\in\tau$ and a closeness assumption: $\zeta<\eta\Rightarrow\zeta\in\tau$). One can e.g. ask for the Thom polynomials of A_n considering only Morin-maps, i.e. τ-maps for $\tau_{Morin}=\cup_{m=0}^{\infty}\cup_{i=0}^{m}(a_0A_0+a_1A_1+\ldots+a_mA_m)$. The arguments in this section prove that the set $Tp(\eta;\tau)$ coincides with the set of polynomials satisfying condition (D) for $\zeta\in\tau$. E.g. computations show that

$$Tp(A_4;\tau_{Morin})=\{c_1^4+6c_1^2c_2+Ac_2^2+Bc_1c_3+6c_4 \mid A+B=11\},$$

or even more generally, the polynomials in $Tp(A_n;\tau_{Morin})$ are exactly the polynomials satisfying the equations of the first part of theorem 5.2.

7. Multiple points

The "universal map method" which we used to compute Thom polynomials is capable to determine multiple point formulas (see e.g. [K], [Sz3], [Sz4]), too — this will be shown in a subsequent paper. In this section we just show how to use our method to (re)prove the most basic multiple point formula (of Herbert [H] and Ronga [Ro1]) with rational coefficients. Proving it with integer coefficients needs finer cohomological analysis. In this section all cohomologies are meant with rational coefficients.

Definition 7.1. *For a self-transverse immersion $f : N^n \looparrowright P^{n+k}$ let $\bar{\Delta}_r \subset N$ denote the (possibly open) submanifold of r-tuple points of f, and let Δ_r be its f-image.*

Theorem 7.2. (Herbert, Ronga) *For any self-transverse immersion $f : N \looparrowright P$ and any $r \geq 0$*

$$[closure\ \bar{\Delta}_r] = f^*[closure\ \Delta_{r-1}] - e(f)\cdot[closure\ \bar{\Delta}_{r-1}] \qquad \in H^{2(r-1)k}(N)$$

where $e(f)$ is the top Chern class (top Stiefel-Whitney class in the real case) of the map – or what is the same, of the normal bundle of the immersion.

Proof. Let k be fixed and let τ be the set of the multisingularities of self-transverse immersions, i.e. $\tau = \{A_0, 2A_0, 3A_0, \dots\}$. Associated to this τ we can construct the universal τ-map $f\tau : Y\tau \longrightarrow X\tau$, i.e. the universal self-transverse immersion, which we now describe.

Both $Y\tau$ and $X\tau$ are glued together from blocks corresponding to the elements of τ. The block corresponding to sA_0 in $Y\tau$ is the disc bundle of a vector bundle $\bar{\xi}_{sA_0}$ over the bases space $\bar{K}_{sA_0}$, where

$$\bar{K}_{sA_0} = B\Big[U(k)\times\big(U(k)^{s-1}\rtimes S_{s-1}\big)\Big] \qquad\qquad \bar{\xi}_{sA_0} = \big(\gamma_2^k\times\ldots\times\gamma_s^k\big)\rtimes S_{s-1}.$$

Here S_i means the symmetric group on i elements and its action on the other factor is permutation. The bundle γ_i^k is the universal k-bundle over the i'th $U(k)$ factor.

The block corresponding to sA_0 in $X\tau$ is the disc bundle of a vector bundle ξ_{sA_0} over the base space K_{sA_0}, where

$$K_{sA_0} = B\Big[U(k)^s\rtimes S_s\Big] \qquad\qquad \xi_{sA_0} = \big(\gamma_1^k\times\ldots\gamma_s^k\big)\rtimes S_s.$$

The pull back of ξ_{sA_0} by the s-sheeted covering $(f\tau)|_{\bar{K}_{sA_0}}$ is

$$(f\tau)|^*_{\bar{K}_{sA_0}}(\xi_{sA_0}) = \gamma_1^k\times\Big[\big(\gamma_2^k\times\ldots\times\gamma_s^k\big)\rtimes S_{s-1}\Big].$$

To prove the formula of the theorem we only have to prove it for $f\tau$, that is

$$(2)\qquad [closure\ \bar{K}_{rA_0}] = (f\tau)^*[closure\ K_{(r-1)A_0}] - e(f\tau)\cdot[closure\ \bar{K}_{(r-1)A_0}]\ \in H^{2(r-1)k}(Y\tau).$$

Our strategy is to prove this first restricted to $\bar{K}_{sA_0}$ for all $s = 1,2,\dots$ (I) and then use a Mayer-Vietoris argument to prove it in $Y\tau$ (II).

(I) We need to determine the restrictions of $[closure\ \bar{K}_{rA_0}]$, $(f\tau)^*[closure\ K_{rA_0}]$ and $e(f\tau)$ to $H^*(\bar{K}_{sA_0})$. Since this latter cohomology ring is not easy to determine, we go further and pull back the mentioned classes by the $(s-1)!$-sheeted covering $\iota : B\big[U(k)^s\big] \longrightarrow \bar{K}_{sA_0}$ and compute them in the polynomial ring

$$H^*\Big(B\big[U(k)^s\big]\Big) = \mathbb{Z}[v_1^{(1)}, \dots, v_k^{(1)}, v_1^{(2)}, \dots, v_k^{(2)}, \dots, v_1^{(s)}, \dots, v_k^{(s)}].$$

Here $v_i^{(j)}$ is a generator in dimension i, the upper index is just a distinguishing sign.

Lemma 7.3.

(i)	ι^* *is injective*		
(ii)	ι^* $[closure\ \bar{K}_{rA_0}]\|_{\bar{K}_{sA_0}}$	$=$	$\sigma_{r-1}(v_k^{(2)}, \dots, v_k^{(s)})$
(iii)	ι^* $(f\tau)^*[closure\ K_{rA_0}]\|_{\bar{K}_{sA_0}}$	$=$	$\sigma_r(v_k^{(1)}, \dots, v_k^{(s)})$
(iv)	ι^* $e(f\tau)\|_{\bar{K}_{sA_0}}$	$=$	$v_k^{(1)}$

Here, again, σ_i is the i'th elementary symmetric polynomial, with the convention that $\sigma_0($ anything $) = 1$ and $\sigma_i($ less than i elements $) = 0$.

Proof of the lemma. Statement (i) is clear since ι is a covering, and in the coefficient group (rationals) one can divide by any integer. Statement (iv) comes from the above description of $f\tau$ as follows:

$$\iota^* c(f\tau)|_{\bar{K}_{sA_0}} = \frac{\big(1+v_1^{(1)}+\dots+v_k^{(1)}\big)\cdot\ldots\cdot\big(1+v_1^{(s)}+\dots+v_k^{(s)}\big)}{\big(1+v_1^{(2)}+\dots+v_k^{(2)}\big)\cdot\ldots\cdot\big(1+v_1^{(s)}+\dots+v_k^{(s)}\big)} = 1+v_1^{(1)}+\dots+v_k^{(1)},$$

so $\iota^* e(f\tau)|_{\bar{K}_{sA_0}} = v_k^{(1)}$.

Now let us turn to the computation of $\iota^*[closure\ \bar{K}_{rA_0}]|_{\bar{K}_{sA_0}}$ for $r \le s$, since the $r > s$ case is trivial (indeed, for $r > s$ the subspaces *closure* $\bar{K}_{rA_0}$ and $\bar{K}_{sA_0}$ are disjoint). The reader here is advised to picturize first a special case, e.g. $k=1$, $r=3$, $s=4$. Recall that the neighbourhood of $\bar{K}_{sA_0}$ in $Y\tau$ is the total space of the bundle $(\gamma_2^k \times \dots \times \gamma_s^k) \rtimes S_{s-1}$. Pulling back this bundle by the map ι we get the bundle $\gamma_2^k \times \dots \times \gamma_s^k$. In this bundle we can identify the inverse images $L_i := \iota^{-1}(\bar{K}_{iA_0})$ as follows.

$$\begin{array}{lcll}
L_s & = & \text{0-section} & \\
L_{s-1} & = & \gamma_2^k \cup \gamma_3^k \cup \dots \cup \gamma_s^k & \setminus\ L_s \\
L_{s-2} & = & span(\gamma_2^k, \gamma_3^k) \cup span(\gamma_2^k, \gamma_4^k) \cup \dots \cup span(\gamma_{s-1}^k, \gamma_s^k) & \setminus\ closure\ L_{s-1} \\
\dots & & & \\
L_1 & = & span(\gamma_2^k, \gamma_3^k, \dots, \gamma_s^k) & \setminus\ closure\ L_2
\end{array}$$

So the closure of L_r has $\binom{s-1}{s-r}$ branches through L_s. These all considered we have that

$$\iota^*[closure \bar{K}_{rA_0}]|_{\bar{K}_{sA_0}} = [closure\ L_r]|_{L_s} = e\Big(\nu(closure\ L_r \subset closure\ L_1)\Big)|_{L_s} =$$

$$= e\Big(\nu(closure\ L_r \subset closure\ L_1)|_{L_s}\Big) = e\Big(\bigoplus(\gamma_{i_1}^k \times \dots \times \gamma_{i_{r-1}}^k)\Big) =$$

$$= \sum v_k^{(i_1)} \cdot \ldots \cdot v_k^{(i_{r-1})} = \sigma_{r-1}(v_k^{(2)}, \dots, v_k^{(s)}).$$

(The $\bigoplus$ and the $\sum$ is taken for all the $r-1$-element subsets $\{i_1, \dots, i_{r-1}\}$ of $\{2, 3, \dots, s\}$. The proof of (iii) is similar. $\square$

To complete the proof of part (I) we only have to substitute the cohomology classes computed in the lemma (keeping in mind that ι^* is injective), i.e. we have to check the equation

$$\sigma_{r-1}(v_k^{(2)},\dots,v_k^{(s)})=\sigma_{r-1}(v_k^{(1)},\dots,v_k^{(s)})-v_k^{(1)}\cdot\sigma_{r-2}(v_k^{(2)},\dots,v_k^{(s)})$$

in $\mathbb{Z}[v_1^{(1)},\dots,v_k^{(1)},\dots,v_1^{(s)},\dots,v_k^{(s)}]$. This is an identity, so part (I) is proved.

(II) Now we are going to use a Mayer-Vietoris type argument to prove formula (2) in $H^{2(r-1)k}(Y\tau)$. This will be done by induction. The formula clearly holds restricted to the first block, i.e. restricted to $Y\tau_1$, where $\tau_1=\{A_0\}$, since this is the result of part (I) for $s=1$. Now we are going to prove that if the formula holds for $Y\tau_s$, $\tau_s=\{A_0,2A_0,\dots,sA_0\}$ then it must hold for $Y\tau_{s+1}$, too. Consider the following portion of the Mayer-Vietoris sequence:

$$H^{2(r-1)k-1}(Y\tau_s)\oplus H^{2(r-1)k-1}(D\bar{\xi}_{(s+1)A_0})\xrightarrow{1}H^{2(r-1)k-1}(Y\tau_s\cap D\bar{\xi}_{(s+1)A_0})\xrightarrow{2}$$

$$H^{2(r-1)k}(Y\tau_s\cup D\bar{\xi}_{(s+1)A_0})\xrightarrow{3}H^{2(r-1)k}(Y\tau_s)\oplus H^{2(r-1)k}(D\bar{\xi}_{(s+1)A_0}).$$

According to the induction hypothesis the cohomology class

$$\Big(\ [closure\ \bar{K}_{rA_0}]-(f\tau)^*[closure\ K_{(r-1)A_0}]+e(f\tau)\cdot[closure\ \bar{K}_{(r-1)A_0}]\ \Big)|_{Y\tau_{s+1}}$$

in $H^{2(r-1)k}(Y\tau_s\cup D\bar{\xi}_{(s+1)A_0})=H^{2(r-1)k}(Y\tau_{s+1})$ maps to $(0,0)$ at map 3. Now we prove that map 2 is the 0-map, which then shows our claim. To show this we will prove that map 1 is surjective, in fact already the map $H^{2(r-1)k-1}(D\bar{\xi}_{(s+1)A_0})\longrightarrow H^{2(r-1)k-1}(Y\tau_s\cap D\bar{\xi}_{(s+1)A_0})$ is surjective. Observe that this latter cohomology group is equal to $H^{2(r-1)k-1}(S\bar{\xi}_{(s+1)A_0})$ (where S means "sphere bundle"), and the map is contained in the Gysin sequence (as map 4)

$$H^{2(r-1)k-1}(D\bar{\xi}_{(s+1)A_0})\xrightarrow{4}H^{2(r-1)k-1}(S\bar{\xi}_{(s+1)A_0})\xrightarrow{5}$$

$$H^{2(r-1)k}(D\bar{\xi}_{(s+1)A_0},S\bar{\xi}_{(s+1)A_0})\xrightarrow{6}H^{2(r-1)k}(D\bar{\xi}_{(s+1)A_0}).$$

Map 6 is — after appropriate identifications — a multiplication by the Euler class e of $\bar{\xi}_{(s+1)A_0}$ in the cohomology ring of $\bar{K}_{(s+1)A_0}$. Considering the commutative diagram

$$\begin{array}{ccc} H^*(\bar{K}_{(s+1)A_0}) & \xrightarrow{\cdot e} & H^{*+2sk}(\bar{K}_{(s+1)A_0}) \\ \downarrow{\scriptstyle \iota^*} & & \downarrow{\scriptstyle \iota^*} \\ H^*(BU(k)^r) & \xrightarrow{\cdot\iota^*(e)} & H^{*+2sk}(BU(k)^r) \end{array}$$

as well as the facts that in the bottom line we have a polynomial ring, and $\iota^*(e)=v_k^{(1)}\neq 0$, we see that map 6 is injective. Therefore map 5 is a 0-map, so map 4 is surjective. The proof is complete. $\square$

Remark 7.4. Let us remark that the history of the universal immersion used in this section (or at least its target space) goes back to works of Szűcs much before [RSz], see e.g. [Sz2].

8. Incidences

Let η and ζ be singularities (non-multi ones for now), i.e. sets of germs $(\mathbb{C}^*,0)\longrightarrow(\mathbb{C}^{*+k},0)$, with the same k. Let their codimensions be d and b, respectively.

Definition 8.1. *The incidence class of η, ζ is defined as*

$$I(\eta,\zeta):=[\bar{K}_\eta]|_{\bar{K}_\zeta}\in H^{2d}(BG_\zeta).$$

Although this definition assumes familiarity with the generalized Pontryagin-Thom construction, according to our theory an alternative definition can be given without any mentioning of [RSz]:

Proposition 8.2. $I(\eta,\zeta)=Tp(\eta)\Big(c(\zeta)\Big)$. □

Our main theorem in these terms can be composed as

Proposition 8.3.
- *If $\zeta\not>\eta,\zeta\neq\eta$ then $I(\eta,\zeta)=0$;*
- *$I(\eta,\eta)=e(\eta)$.* □

Let us remark, that at least one of $\zeta\not>\eta$ and $\eta\not>\zeta$ necessarily holds, so at least one of $I(\eta,\zeta)=0$, $I(\zeta,\eta)=0$ necessarily holds.

While definition 8.1 gives us a geometric idea of $I(\eta,\zeta)$, proposition 8.2 is easily computable. This duality can be used in both ways. The *geometric* $\Rightarrow$ *computable* direction was, in fact, our method to compute Thom polynomials. One can, however, consider the other direction, too. We are going to show it on an example.

Let ζ be the singularity between equal dimensional manifolds corresponding to the local algebra $\mathbb{C}[[x,y]]/(x^2,y^3)$. Its codimension is 7, so proposition 2.3 allows it to be $>$ than A_6 whose codimension is 6. Still, if we compute

$$I(A_6,\zeta)=Tp(A_6)\Big(c(\zeta)\Big)=Tp(A_6)\Big(\frac{(1+2a)(1+3b)}{(1+a)(1+b)}\Big)$$

we find that it is 0, so we might suspect that $\eta\not>A_6$. In fact it it true, and can be proved by long algebraic computations (the author used the computer algebra package Maple/grobner). The same method can be used to calculate $I(A_8,(x^2,y^4))=0$ and so conjecture that $(x^2,y^4)\not>A_8$ although $codim(x^2,y^4)=10$. Just like above, long calculation can prove that indeed $(x^2,y^4)\not>A_8$.

There are two directions which the author believes may turn to be interesting in the study of incidences of singularities this way. The first challenge is a deeper *geometric* understanding of the incidence class. The other is the possible proof of the converse of proposition 8.3:

Conjecture 8.4. *If $I(\eta,\zeta)=0$ then $\zeta\not>\eta$.*

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Department of Analysis, ELTE TTK, Rákóczi út 5., Budapest 1088, Hungary
E-mail address: `rimanyi@cs.elte.hu`