THOM POLYNOMIALS WITH INTEGER COEFFICIENTS

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1. Introduction

Thom polynomials of real singularities have been calculated mainly with $\mathbb{Z}_2$-coefficients. The exceptions are the Σ^i (see [Ron71]) and some $\Sigma^{i,j}$ (see [And82]) Thom-Boardman singularities. In this paper we would like to demonstrate how the methods of the theory of Thom polynomials for group actions (see [FR]) can be be used in this case. We concentrate on the case of (contact class) singularities between manifolds of equal dimension which have been studied the most. We calculate the Thom polynomials up to codimension 8.

It turned out that the difficult part is not to calculate these Thom polynomials, but to find out "who" has a Thom polynomial. Vassiliev defined a cochain complex in [Vas88, $8] where the cochains are linear combinations of cooriented orbits (singularities in our case) and showed that exactly the cocycles admit Thom polynomials (see also [Kaz97] and [FR]). In cases where every orbit is cooriented and even codimensional—e.g. the case of complex singularities—the differential of the Vassiliev complex is trivial. Such Thom polynomials are calculated in e.g. [Rim01b], see also references therin.

Calculation of Thom polynomials of real singularities with $\mathbb{Z}_2$-coefficients is easier due to a result of Borel and Haefliger (see [BH61]). It implies that we can get the Thom polynomial of a real singularity η by replacing Chern classes to the corresponding Stiefel-Whitney classes in the Thom polynomial of the complexification of η.

In [Rim01b] the second author established a method of calculating Thom polynomials by solving a system of linear equations. We will see that these equations are *not* enough in the case of real singularities. However, knowing also their $\mathbb{Z}_2$-reductions and properties of the Kazarian spectral sequence we can calculate them.

In Section 2 we calculate the Vassiliev complex. These calculations are fairly complicated. An extra difficulty compared to the $\mathbb{Z}_2$-case (which was done in [Ohm94]) is to determine the signs in the differential. In Section 3 we calculate the Thom polynomials. In Section 4 we study the connection between the real and complex case which leads us to finding obstructions to avoid certain singularities.

2. The Vassiliev complex

The n-cochains in the Vassiliev complex are linear combinations of the n-codimensional cooriented orbits. (We assume here that every orbit is simple in the codimension-range we

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are interested in.) Following Vassiliev we can calculate the coefficients of the differential d_1 as follows: Let ξ be an n-codimensional cooriented orbit and

$$d_1(\xi) = \sum_{\operatorname{codim}\eta=n+1} c(\xi,\eta)\eta.$$

Then $c(\xi,\eta)$ is "the number of ξ-curves entering η" counted with sign. More precisely if we take a normal slice N_ξ to the orbit ξ, then the intersection $N_\xi \cup \eta$ is one dimensional i.e. disjoint union of curves L_i. For every curve L_i we calculate its sign: Choose a point $x_i \in L_i$ and a normal slice N_i. In other words

$$(*) \qquad N_\xi = N_i \oplus T_{x_i}L_i.$$

Notice that N_i is also a normal slice to η therefore oriented (defined by the coorientation of η). N_ξ is also oriented and we can give an orientation to $T_{x_i}L_i$ by choosing a vector pointing to the "ξ-direction". If the three orientations fit we give plus sign to L_i and a minus sign otherwise.

So first we have to find the coorientable singularities, then the curves L_i, and finally we have to calculate the signs.

2.1. **Coorientable singularities.** The top (up to codimension 9) of the classification of stable singularities between equal dimensional spaces is as follows (e.g. [PW95]):

codim	0	1	2	3	4	5	6	7	8	9
	A_0	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9
					$I_{2,2}$	$I_{2,3}$	$I_{2,4}$	$I_{2,5}$	$I_{2,6}$	$I_{2,7}$
					$II_{2,2}$		$II_{2,4}$		$II_{2,6}$	
							$I_{3,3}$	$I_{3,4}$	$I_{3,5}$	$I_{3,6}$
									$I_{4,4}$	$I_{4,5}$
									$II_{4,4}$	
							IV_3		IV_4	
								(x^2,y^3)	(x^2+y^3,xy^2)	(x^2+y^3,y^4)
										(x^2+y^4,xy^2)
										Σ^3

Here by 'singularity' we mean a stratum (satisfying the Vassiliev conditions [Vas88], [FR]) of the following group action: The group $\mathrm{Diff}(\mathbb{R}^\infty,0)\times\mathrm{Diff}(\mathbb{R}^\infty,0)$ (diffeomorphism germs at 0) acts on $\mathcal{E} := \{$ stable $(\mathbb{R}^\infty,0)\to(\mathbb{R}^\infty,0)$ germs$\}$ by $(\psi,\varphi)\cdot f := \varphi\circ f\circ\psi^{-1}$. In fact, all but Σ^3 is an orbit, the latter is a 1-parameter family of orbits.

Orbits of this group action are characterized by their local algebras [Mat69]. So the above symbols encode local algebras as follows: A_i stands for the singularity with local algebra $\mathbb{R}[[x]]/(x^{i+1})$. The symbols I–IV stand for algebras corresponding to $\Sigma^{2,0}$ singularities as in [Mat71]. In the other cases we indicated the ideal in $\mathbb{R}[[x,y]]$ which is to be factored out to get the algebra. The stratum Σ^3 corresponds to the 1-parameter family of algebras

$$\mathbb{R}[[x,y]]/(x^3+\lambda yz, y^3+\lambda xz, z^3+\lambda xy), \quad \lambda(\lambda^3-1)(8\lambda^3+1)\neq 0.$$

Definition 2.1. *A stable germ $\kappa:(\mathbb{R}^n,0)\to(\mathbb{R}^n,0)$ is called a* prototype *of the singularity η if the infinite trivial unfolding of κ is in η and n is minimal.*

Let η be a singularity and κ is a prototype of η. Then we can consider the right-left symmetry group of κ:

$$\{(\varphi,\psi)\in\mathcal{A}=\mathrm{Diff}(\mathbb{R}^n,0)\times\mathrm{Diff}(\mathbb{R}^n,0):\psi\circ\kappa\circ\varphi^{-1}=\kappa\}$$

or its maximal compact subgroup G_η (see more details in [Rim], [FR]). If κ is well chosen then by the definition of maximal compact subgroup for subgroups of $\mathcal{A}$, the group G_η acts linearly on the source and target spaces. We denote these representations by λ_0,λ_1 respectively. If $\lambda_0(G_\eta)\subset GL^+(n)$ then we call η *coorientable*. This condition is equivalent to the coorientability of the stratum in $\mathcal{E}$ since the source space of a prototype can be identified with a normal slice to the orbit η. Geometrically it means that if $\eta(f)\subset N$ is the set of η-points of a stable map $f:N\to P$ then the normal bundle of $\eta(f)$ in N is orientable.

Theorem 2.2. *Among the above singularities exactly the following are coorientable:*

codim	0	1	2	3	4	5	6	7	8	9
Σ^0	A_0									
Σ^1				A_3	A_4			A_7	A_8	
$\Sigma^{2,0}$					$I_{2,2}$	$I_{2,3}$		$I_{2,5}$	$I_{2,6}$	$I_{2,7}$
					$II_{2,2}$				$II_{2,6}$	
										$I_{3,6}$
									IV_4	
$\Sigma^{2,1}$									(x^2+y^3,xy^2)	(x^2+y^3,y^4)

The problem of calculating the maximal compact symmetry group and its representation λ_0 is solved in [Rim96], [Rim], here we show an example. For more detailed discussion see [Rim01a].

A prototype κ of (x^2,y^3) is the miniversal unfolding of $(x,y)\mapsto(x^2,y^3)$:

$$\kappa:(x,y,\underline{v})\mapsto(x^2+v_1y+v_2y^2,y^3+v_3x+v_4y+v_5xy,\underline{v})$$

where $\underline{v}=v_1,\dots,v_5$. Its maximal compact symmetry group $G_{(x^2,y^3)}\cong\mathbb{Z}_2\times\mathbb{Z}_2$ and $\lambda_0=\alpha\oplus\beta\oplus\beta\oplus 1\oplus\alpha\beta\oplus 1\oplus\alpha$ where α and β are the nontrivial irreducible representations of the first and second $\mathbb{Z}_2$-factor. So (x^2,y^3) is not coorientable.

Remark 2.3. From now on the symbols of coorientable singularities we will mean the given singularity with a chosen coorientation. We don't specify these coorientations. It leaves some sign indeterminacy in our final results. Calculations of the coorientations would be a tedious job presenting no theoretical novelties so we decided to omit them.

2.2. **Computation of the differentials.** The easiest case in the computation of the Vassiliev coefficients $c(\xi,\eta)$ is when *near a ξ-point there are no η-points* at all, i.e. the germ of the set $\eta(\kappa)$ is empty for the prototype κ of ξ.

Theorem 2.4. *The following Vassiliev coefficients are all 0.*

1. $c(A_8,I_{2,5})$, $c(I_{2,7},(x^2{+}y^3,xy^2))$, $c(I_{3,6},(x^2{+}y^3,xy^2))$,
2. $c(II_{2,2},A_3)$, $c(II_{2,6},A_7)$, $c(IV_4,II_{2,5})$, $c(I_{2,7},IV_4)$, $c(I_{3,6},IV_4)$,
3. $c((x^2{+}y^3,xy^2),A_7)$, $c((x^2{+}y^3,y^4),A_8)$.

Proof. Case 1 holds because of Thom-Boardman symbols, i.e. if J,I are Thom-Boardman symbols and $J<I$ in the lexicographic order then near a J-point there are no I points. Case

2 follows from the work of Lander [Lan76] saying that the appropriate set germs are empty. Case 3 follows from the notion of multiplicity (e.g. [AVGL91, p.161.]). The multiplicity of (x^2+y^3, xy^2) is 7 since it is the dimension of its local algebra. Geometrically this means that the preimage (at the complexified map) of a general point near 0 in the target consists of 7 points. The multiplicity of A_7 is 8 so it can not be near 0 of a (x^2+y^3, xy^2)-germ. Similarly the multiplicities for (x^2+y^3, y^4) and A_8 are 8 and 9, respectively. $\square$

Now let us consider $c(A_4, A_3)$. A prototype of A_4 is

$$\kappa : (x, y_3, y_2, y_1) \mapsto (x^5 + y_3x^3 + y_2x^2 + y_1x, y_3, y_2, y_1)$$

with maximal compact symmetry group $\langle g\rangle \cong \mathbb{Z}_2$ acting as $\alpha\oplus 1\oplus\alpha\oplus 1$ on the source. Calculating the partial derivatives shows that the A_3-points of κ are parametrized as $(t, -10t^2, 20t^3, -15t^4)$, which is a non-singular curve having thus two intersection with a sphere centered at the origin. To determine the signs associated to these intersection points we would need clear definition of the coorientation of A_4 and A_3. Although we have not specified the chosen coorientations we can still see that the signs associated to the two intersection points must coincide by the following lemma, because an orientation preserving diffeomorphism germ, namely $\alpha(g)$ interchanges them:

Lemma 2.5. *If the symmetry group G_ξ of the orbit ξ interchanges the curves L_i and L_j then they have the same sign.*

Proof. Suppose that $gL_i = L_j$ for a $g \in G$. Then by choosing $x_j := gx_i$ we have $gN_i = N_j$. Since ξ and η are cooriented and $g\xi = \xi$ the symmetry g preserves the orientations in the decomposition $(*)$. $\square$

So we can state that $c(A_4, A_3) = \pm 2$. Similar computation shows that $c(A_8, A_7) = \pm 2$, too. The key in these computations was our ability to write down the equations of the 'nearby singularity types' and the luck that the obtained points on the sphere are permuted by the symmetry group of the singularity at 0. The equations of the singularities near $\Sigma^{2,0}$-points are described in [Lan76]. The symmetry groups of them are computed in [Rim]. Luckily enough in the following cases Lemma 2.5 applies, so—as above—we can determine the absolute values of the Vassiliev coefficients:

Theorem 2.6.

$$\begin{array}{llllll}
c(I_{2,2}, A_3) & = \pm 4, & c(I_{2,3}, A_4) & = \pm 2, & c(I_{2,3}, I_{2,2}) & = \pm 1, \\
c(I_{2,3}, II_{2,2}) & = \pm 1, & c(I_{2,6}, A_7) & = \pm 4, & c(I_{2,6}, I_{2,5}) & = \pm 2, \\
c(II_{2,6}, I_{2,5}) & = \pm 2, & c(IV_4, A_7) & = \pm 8, & c(I_{2,7}, A_8) & = \pm 2, \\
c(I_{2,7}, I_{2,6}) & = \pm 1, & c(I_{2,7}, II_{2,6}) & = \pm 1, & c(I_{3,6}, A_8) & = \pm 2, \\
c(I_{3,6}, I_{2,6}) & = \pm 1, & c(I_{3,6}, II_{2,6}) & = \pm 1. & &
\end{array}$$

$\square$

There only remains five Vassiliev coefficients to be calculated: $c((x^2+y^3, xy^2), I_{2,5})$ and $c((x^2+y^3, y^4), \eta)$ with $\eta = I_{2,6}, II_{2,6}, IV_4, (x^2+y^3, xy^2)$. Here again the great work is to determine the equations for the η-points in the source of a prototype of ξ. This can be done using the Hilbert functions of the local algebras: $h : i \mapsto \dim M^i/M^{i+1}$, where M is the unique maximal ideal. We will sketch the procedure in one particular case: $c((x^2+y^3,y^4),IV_4)$. A prototype of (x^2+y^3,y^4) is

$$\kappa:(x,y,\underline{u},\underline{v})\mapsto(x^2+y^3+u_1y+u_2y^2,y^4+v_1x+v_2y+v_3xy+v_4y^2+v_5xy^2,\underline{u},\underline{v}),$$

where $\underline{u}=(u_1,u_2)$ and $\underline{v}=(v_1,\dots,v_5)$. By differentiating we get the following equations for the Σ^2-points in the source:

$$x=0,\qquad u_1=-3y^2-2u_2y,\qquad v_1=-v_3y-v_5y^2,\qquad v_2=-4y^3-2v_4y,$$

so it is a graph of a map $\mathbb{R}^5(y,u_2,v_3,v_4,v_5)\longrightarrow\mathbb{R}^4(x,u_1,v_1,v_2)$, so it is smooth. Let us choose a point p on this graph. So p is of the form:

$$p:=(0,\bar{y},(-3\bar{y}^2-2\bar{u}_2\bar{y}),\bar{u}_2,(-\bar{v}_3\bar{y}-\bar{v}_5\bar{y}^2),(-4\bar{y}^3-2\bar{v}_4\bar{y}),\bar{v}_3,\bar{v}_4,\bar{v}_5).$$

The germ of κ at p, i.e. the germ of $\kappa((x,y,\underline{u},\underline{v})+p)-\kappa(x,y,\underline{u},\underline{v})$ at 0 is the unfolding of

$$\mu_p:(x,y)\to(x^2+y^3+(\bar{u}_2+3\bar{y})y^2,\quad y^4+\bar{v}_5xy^2+(2\bar{v}_5\bar{y}+\bar{v}_3)xy+(4\bar{y})y^3+(6\bar{y}^2+\bar{v}_4)y^2).$$

The local algebra Q_p of κ at p is $\mathbb{R}[[x,y]]/I$, where I is the ideal generated by the two coordinate functions of μ_p. Our task is to obtain the values of p for which the local algebra Q_p is isomorphic to that of IV_4. The algebra of IV_4 has Hilbert function $(h(0),h(1),\dots)=(1,2,2,2,1,0,\dots)$. For Q_p we have $h(0)=1,h(1)=2$ for any values of

$$A:=\bar{u}_2+3\bar{y},\qquad B:=\bar{v}_5,\qquad C:=2\bar{v}_5\bar{y}+\bar{v}_3,\qquad D:=4\bar{y},\qquad E:=6\bar{y}^2+\bar{v}_4,$$

but

$$h(2)=3-\operatorname{rank}\begin{pmatrix}1&0&A\\0&C&E\end{pmatrix},$$

so the condition for $h(2)$ to be equal to 2 is $C=E=0$. Similarly,

$$h(3)=7-\operatorname{rank}\begin{pmatrix}1&.&A&.&.&.&1\\.&C&E&.&.&B&D\\.&.&.&1&.&A&.\\.&.&.&.&1&.&A\\.&.&.&.&C&E&.\\.&.&.&.&.&C&E\end{pmatrix}$$

so the condition for $h(3)$ to be equal to 2 (using $C=E=0$) is $B=D=0$. So there is a 1-dimensional curve (as A varies) in the source of κ where the Hilbert function starts as $(1,2,2,2)$. Studying the Hilbert functions of the singularities of codimension 8 we see that the singularities on this curve are either IV_4 or $I_{4,4}$. (Let us remark that the complexification of these two are the same so one needs real arguments to distinguish them.) In our case the algebra $(x^2+y^3+Ay^2,y^4)$ is isomorphic to the algebra of IV_4 if $A>0$, and isomorphic to that of $I_{4,4}$ if $A<0$ (in the latter case use the substitution $x=u+v,y=u-v$). So we have that the IV_4-points of κ are on a ray coming out of 0 in the source of κ, so $c((x^2+y^3,y^4),IV_4)=\pm1$. Using similar arguments we can compute the remaining Vassiliev coefficients (up to sign):

Theorem 2.7.

$$c((x^2+y^3,xy^2),I_{2,5}) = \pm 2, \quad c((x^2+y^3,y^4),I_{2,6}) = \pm 2, \quad c((x^2+y^3,y^4),(x^2+y^3,xy^2)) = \pm 2,$$
$$c((x^2+y^3,y^4),IV_4) = \pm 1, \quad c((x^2+y^3,y^4),II_{2,6}) = 0.$$

□

Since we determined only the signs of the coefficients, apparently we lack some information to write down the Vassiliev complex exactly. However, we know that the signs are to be distributed so that $d_1 \circ d_1 = 0$ is satisfied, and in fact (again) we are lucky, because *essentially* there is only one way to distribute the signs satisfying this condition. Consequently we get the cohomology groups of the Vassiliev complex:

Theorem 2.8.

$$H^1 = H^2 = H^3 = H^5 = H^6 = H^7 = 0, H^0 = \mathbb{Z}\langle A_0\rangle,$$
$$H^4 = \mathbb{Z}\langle I_{2,2} + II_{2,2}\rangle \oplus \mathbb{Z}_2\langle A_4 + 2I_{2,2}\rangle$$
$$H^8 = \mathbb{Z}\langle(x^2+y^3,xy^2) - 2IV_4\rangle \oplus \mathbb{Z}_2\langle I_{2,6} + II_{2,6} + (x^2+y^3,xy^2)\rangle \oplus \mathbb{Z}_2\langle A_8 + 2I_{2,6} + 4IV_4\rangle,$$

where in the brackets we indicated possible generators (which mean singularities with some (determinable but undetermined) coorientations.

□

3. Calculation of the Thom polynomials

Let us briefly recall the definition of the Thom polynomial: The Vassiliev complex $\mathbf{V}C$ is the 0^{th} row of the E_2-table of the Kazarian spectral sequence (see [Kaz97]), so we have an edge homomorphism:

$$\mathrm{Tp} : \mathbf{V}C \to H^*(BG).$$

In the case of real singularities $G = \mathrm{Diff}(\mathbb{R}^\infty,0) \times \mathrm{Diff}(\mathbb{R}^\infty,0)$ so the Thom polynomials live in $H^*(BO \times BO;\mathbb{Z})$. In classical terms, given a smooth map $f : N \to P$ the Poincaré dual of the singular points of type ξ can be expressed as a polynomial of characteristic classes of TN and f^*TP.

Proposition 3.1. $\mathrm{Tp}(\xi)$ *depends only on the characteristic classes of* $f^*TP \ominus TN$.

The proof can be found in [FR] for the complex case but it applies word by word for the real case as well. We adopt the notation $\mathrm{tp}(\xi)$ for the corresponding element in $H^*(BO;\mathbb{Z})$.

We need a convenient notation for the elements of $H^*(BO;\mathbb{Z})$:

Theorem 3.2. [MS74, Pr.15C]

$$H^*(BO;\mathbb{Z}) \cong \mathbb{Z}[p_1,\dots,p_i,\dots] \oplus \mathrm{Im}\,\mathrm{Sq}^1.$$

We will use the notation $\sum v_I$ for the unique 2^{nd} order element in $H^*(BO;\mathbb{Z})$ such that $r(\sum v_I) = \sum w_I$ where r denotes the mod 2 reduction and w_I is a monomial of Stiefel-Whitney classes corresponding to the multiindex I. So we write elements of $H^*(BO;\mathbb{Z})$ in the form $\sum a_I p_I + \sum b_I v_I$ where I runs through multiindices and p_I, v_I are the corresponding monomials. We can assume that $\sum b_I w_I \in \mathrm{Im}\,\mathrm{Sq}^1$.

Our major tool to the calculations is the following theorem of Borel and Haefliger:

Theorem 3.3 ([BH61]). *Let $\eta_{\mathbb{C}}$ be the complexification of a real singularity η. Suppose that* $\mathrm{tp}(\eta_{\mathbb{C}}) = \sum a_I c_I$. *Then* $\mathrm{tp}(\eta;\mathbb{Z}_2) = \sum a_I w_I$.

For a cooriented singularity η we have $r\big(\mathrm{tp}(\eta;\mathbb{Z})\big) = \mathrm{tp}(\eta;\mathbb{Z}_2)$ and $\mathrm{Ker}(r) = 2{\cdot}\mathbb{Z}[p_1,\dots,p_i,\dots]$ so we need some extra information to find out the coefficients of the Pontryagin classes.

$I_{22}+II_{22}$**:** This Thom polynomial coincides with the Thom polynomial of the Σ^2 Thom-Boardman singularity which was calculated by Ronga ([Ron71]):

$$\mathrm{tp}(I_{22}+II_{22}) = p_1 + v_1v_3.$$

A_4+2I_{22}**:** This cocycle has order 2 in the Vassiliev complex, so the Thom polynomial cannot contain Pontryagin classes. So by the Borel-Haefliger theorem:

$$\mathrm{tp}(A_4+2I_{22}) = v_1^4 + v_1v_3.$$

$I_{26}+II_{26}+(x^2{+}y^3,xy^2)$**:** This cocycle also has order 2 in the Vassiliev complex, so:

$$\mathrm{tp}(I_{26}+II_{26}+(x^2{+}y^3,xy^2)) = v_1^2v_2v_4 + v_1v_2v_5 + v_1v_3v_4 + v_1v_2^2v_3 + v_1^2v_3^2 + v_1^3v_5$$

$A_8+2I_{26}+4IV_4$**:** This cocycle also has order 2 in the Vassiliev complex, so:

$$\mathrm{tp}(A_8+2I_{26}+4IV_4) = v_1^8 + v_1^3v_5 + v_1^2v_2v_4 + v_1v_2v_5 + v_1v_3v_4 + v_1v_2^2v_3$$

$\eta = (x^2{+}y^3,xy^2) - 2IV_4$**:** Also by the Borel-Haefliger theorem it is enough to calculate rationally, i.e. to find the coefficients A and B for $\mathrm{tp}(\eta;\mathbb{Q}) = Ap_1^2 + Bp_2$. First we try to get equations by the method of restriction (see [FR]), which was the method for calculating Thom polynomials in the complex case. In the real case however the symmetry groups are "too small". The only useful one is $G_{II_{22}} \cong O(2)$. As it is explained in [FR] we need the representations λ_0 and λ_1 of $O(2)$ on the source and target space of II_{22}. They were calculated in [Rim]: $\lambda_0 = \rho_1\oplus\rho_3$ and $\lambda_1 = \rho_2\oplus\rho_3$, where ρ_n is the unique 2-dimensional representation of $O(2)$ which restricts to α^n on $U(1)\cong SO(2)$ where α is the standard representation of $U(1)$. So, using the notation of [FR]:

$$p(\lambda_1\ominus\lambda_0) = \frac{1+4p_1}{1+p_1} = 1 + 3p_1 - 3p_1^2 + \cdots$$

and

$$j^*_{II_{22}}\,\mathrm{tp}(\eta;\mathbb{Q}) = A(3p_1)^2 + B(-3p_1^2) = 0,$$

which implies that $B = 3A$. We need one more equation. To find it we look at the Kazarian spectral sequence. Choosing $\mathbb{Z}[\frac{1}{2}]$ as coefficient group greatly simplifies the E^∞-table. On the 8^{th} diagonal there remains only two nonzero entries:

$$E^\infty_{4,4} \cong H^4(BO;\mathbb{Z}[\tfrac{1}{2}]) \cong \mathbb{Z}[\tfrac{1}{2}] \qquad\text{and}\qquad E^\infty_{0,8}\cong\mathbb{Z}[\tfrac{1}{2}].$$

The second isomorphism follows from our calculation of H^8 for the Vassiliev complex. Since the Kazarian spectral sequence converges to $H^*(BO;\mathbb{Z}[\frac{1}{2}])$ and $H^8(BO;\mathbb{Z}[\frac{1}{2}]) \cong \mathbb{Z}[\frac{1}{2}][p_1^2,p_2]$ we get:

$$E^\infty_{4,4} \cong \mathbb{Z}[\tfrac{1}{2}][p_1^2,p_2]/\langle \mathrm{tp}(\eta;\mathbb{Z}[\tfrac{1}{2}])\rangle.$$

So $E^\infty_{4,4}$ has an element of order g.c.d(A,B) which implies that $A=\pm1$. So for the proper coorientations we have:

$$\mathrm{tp}((x^2{+}y^3,xy^2) - 2IV_4) = p_1^2 + 3p_2 + v_1^2v_2v_4 + v_1v_2v_5 + v_1v_3v_4 + v_3v_5.$$

4. Complex versus real

In this section we discuss a complexification technique (and its relation to Thom polynomials) different from the Borel-Haefliger one [BH61] as suggested to us by A. Szűcs and R. Szőke. Given a smooth map $f : N^n \to P^p$ between real manifolds consider its complexification $f_{\mathbb{C}} : N_{\mathbb{C}} \to P_{\mathbb{C}}$ in the Bruhat-Whitney sense [BW59] as follows.

First choose real analytic atlases for N and P and perturb f to be real analytic [Hir76, Thm. 5.1]. Then change the real coordinate charts to complex ones and glue them with the original gluing maps now considered as complex analytic maps. In fact these gluing maps can be defined only in a neighbourhood of $\mathbb{R}^n$ in $\mathbb{C}^n$, so consider only these tubes. Also f, now considered as a complex analytic map on each coordinate chart is defined only in a (possibly smaller) tube. Choosing these appropriately small tubes we get a map $f_{\mathbb{C}} : N_{\mathbb{C}} \to P_{\mathbb{C}}$. Since $N_{\mathbb{C}} \cong TN$ we identify the cohomology rings of N and $N_{\mathbb{C}}$.

Now let η be a real singularity of codimension c which is the complete real form of its complexification $\eta_{\mathbb{C}}$ (e.g. $\eta = A_i$, $\eta_{\mathbb{C}} = A_i^{\mathbb{C}}$ or $\eta = I_{2,2} \cup II_{2,2}$, $\eta_{\mathbb{C}} = I_{2,2}^{\mathbb{C}}$). Also suppose that η defines a cocycle in the Vassiliev complex, i.e. the set of η-points $\eta(f) \subset N$ defines a cohomology class $[\eta(f)]$ in N.

In the next lemma η is a coorientable singularity.

Lemma 4.1. *If $\eta(f)$ is closed then $[\eta(f)]^2 = [\eta_{\mathbb{C}}(f_{\mathbb{C}})] \in H^{2c}(N)$.*

Proof. A tubular neighbourhood of $S := \eta(f)$ in $N_{\mathbb{C}}$ is diffeomorphic to its normal bundle. However

$$\nu(S \subset N_{\mathbb{C}}) = \nu(S \subset N) \oplus TN|_S = \nu(S \subset N) \oplus \Big(TS \oplus \nu(S \subset N)\Big),$$

and the diffeomorphism can be chosen so that $\eta_{\mathbb{C}}(f_{\mathbb{C}})$ is the total space of the middle term. The cohomology class $\eta(f)$ is represented by the total space of the 2^{nd} and 3^{rd} term. Since the intersection of $\eta(f)$ and its perturbation $(0, t, n) \mapsto (n, t, 0)$ is exactly $\eta_{\mathbb{C}}(f_{\mathbb{C}})$ the lemma is proved. □

Using this lemma we can construct an obstruction for avoiding singularities more complicated than η:

Definition 4.2. *Let η be a real singularity of codimension d admitting a complexification $\eta_{\mathbb{C}}$.*

$$s(\eta) := i^* \operatorname{tp}(\eta_{\mathbb{C}}) - \operatorname{tp}^2(\eta) \in H^{2d}(BO; \mathbb{Z})$$

where $i : BO \to BU$ is the map induced by the embedding $O \to U$.

We mention that Theorem 3.3 and Lemma 4.1 can be generalized easily to the more general setting of Thom polynomials for group actions.

Lemma 4.1 implies the following:

Corollary 4.3. *If $s(\eta)(f) \neq 0$ then there exists a singularity ξ more complicated than η such that $\xi(f)$ is not empty.*

We can see that Theorem 3.3 is equivalent to the following:

Theorem 4.4. *$s(\eta)$ is even i.e. its mod 2 reduction is 0.*

It would be interesting to find a direct proof of this theorem.

Example 4.5. To see that $s(\eta)$ is not always 0 consider maps of codimension 1, i.e. maps $N^* \to P^{*+1}$ and let $\eta = A_2 (= \Sigma^{1,1})$. The Thom polynomial of its complexified is ([Ron72], [Rim01b]):

$$\mathrm{tp}(A_2^{\mathbb{C}}) = c_2^2 + c_1c_3 + 2c_4.$$

By Lemma 4.1:

$$\mathrm{tp}(A_2; \mathbb{Z}) = Ap_1 + v_1v_3,$$

where A is an odd integer. (In fact $A = 1$ as Toru Ohmoto and András Szűcs explained to us.) So

$$s(A_2) = p_1^2 + v_1^2v_3^2 + 2p_2 - (Ap_1 + v_1v_3)^2 = (1 - A^2)p_1^2 + 2p_2,$$

which is nonzero.

We would like to speculate on the question whether $s(\eta)$ is a new obstruction. We expect that $s(\eta)$ is a linear combination of (possibly higher) Thom polynomials of singularities more complicated than η. First we define higher Thom polynomials after Kazarian [Kaz97]:

Definition 4.6. *Suppose that a representation of the Lie group G is given. Then the corresponding Kazarian spectral sequence converges to $H^*(BG)$. It implies that $H^d(BG)$ has well defined subgroups K_i^d corresponding to $\bigoplus_{k=0}^{i} E_\infty^{d-k,k}$. Elements of K_i^d for $i < d$ are called* higher *Thom polynomials.*

So K_0^d is the set of usual Thom polynomials and $K_d^d = H^d(BG)$.

Conjecture 4.7.

$$s(\eta) \in K_{d-1}^{2d}.$$

Example 4.8. The case of Thom-Boardman singularities $\Sigma^i(k)$—where k refers to maps $\mathbb{R}^n \to \mathbb{R}^{n+k}$—has some interesting properties. They are coorientable if i and k are even. Their Thom polynomials were calculated in [Ron71] and [And82]. One can also calculate them by the method of restriction (which is somewhat surprising in the light of the previous calculations in this paper), the calculation is completely analogus to the complex case in [FR]. The other—probably related—phenomenon is that $s(\Sigma^i(k)) = 0$. Conjecture 4.7 doesn't explain this, since K_{d-1}^{2d} is nontrivial in this case.

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