Contemporary Mathematics

Expressions for resultants coming from the global theory of singularities

G. Bérczi, L. M. Fehér, and R. Rimányi

Abstract. We present formulas for the Thom polynomials of $\Sigma^{1,1}$ and $\Sigma^{1,1,1}$ singularities in all relative dimensions k (Theorem 3.2, 4.2) and relate these formulas to resultant identities (Theorem 5.4, 5.5).

1. Introduction

The global behavior of singularities is governed by their so called Thom polynomials. Once the Thom polynomial of a singularity η is known, one can compute the cohomology class represented by the η-points of a map. This approach of enumerative geometry is turning out to be more and more fruitful. The problem of calculating Thom polynomials has essentially been solved in [**Rim01**]. The way of obtaining the sought Thom polynomial there is through the solution of a system of linear equations, which is fine when we want to find one concrete Thom polynomial.

However, if we want to find the Thom polynomials of a series of singularities, we have to solve a series of systems of linear equations simultaneously.

When trying to organize these solutions for the singularity series

- $\Sigma^{1,1}: N^n \to P^{n+k}$, $k=0,1,2,\dots$ and
- $\Sigma^{1,1,1}: N^n \to P^{n+k}$, $k=0,1,2,\dots$

the authors found surprising expressions for the resultant of two polynomials. The authors hope that a better understanding of the underlying algebra (or analysis) will yield a closer connection between the theory of resultants and Thom polynomials as well as a possibility to find many more (infinite series) of Thom polynomials.

The new result in this paper is the formula for the Thom polynomials of $\Sigma^{1,1,1}$ singularities (Theorem 4.2) and its relation to a resultant identity (Theorem 5.5).

2. Thom polynomials

Let $\eta : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+k}, 0)$ be a singularity, i.e. an equivalence class of analytic germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^{n+k}, 0)$ under the equivalence of analytic reparametrizations

2000 *Mathematics Subject Classification.* 14N10, 57R45.

Key words and phrases. Classes of degeneracy loci, singularities, Thom polynomials, global singularity theory, resultants.

Supported by FKFP0055/2001 as well as OTKA T029759 (third author).

(biholomorphisms) of the source and the target. For a summary of the theory of singularities, see e.g. [**AVGL91**]. If η is such a singularity then for any map $f : N^n \to P^{n+k}$ one can identify the points in N where the map has singularity η. For generic f this set is a submanifold of N, whose codimension is called the codimension of η. *Usually* the closure of this set carries a fundamental cohomology class, whose determination is our goal. It is a classical theorem of Thom that this class can be obtained as the value of a polynomial $\mathrm{Tp}_\eta(c_1, c_2, c_3, \ldots)$ (the Thom polynomial—depending only on η) when we substitute the characteristic classes of the map f, i.e.

$$1 + c_1 + c_2 + \ldots = \frac{f^*c(TP)}{c(TN)} \quad \Big(= \frac{c(f^*TP)}{c(TN)} = c(f^*TP - TN)\Big),$$

where $c(TP)$ and $c(TN)$ stand for the total Chern classes of the manifolds N and P.

As said, this is the case "usually"; in fact this is definitely the case with complex singularities in the region where moduli (continuous families) of singularities do not occur (see [**Kaz97**] or [**FR**] for the general case). In this paper we will study singularities in this easier setting: Namely, we will compute the Thom polynomials of the complex singularities called $\Sigma^{1,1}$ (also called A_2) and $\Sigma^{1,1,1}$ (also called A_3) for $(\mathbb{C}^n, 0) \to (\mathbb{C}^{n+k}, 0)$. In this dimension setting the moduli of singularities start at codimension $> 6k + 8$ whereas the codimension of A_2 and A_3 are $2k + 2$, $3k + 3$ respectively.

Since [**Rim01**] the main technique of computing Thom polynomials is the method of restriction equations. The essence of this method is that when computing the Thom polynomial of η, one has to deal with the singularities ζ of codimension at most $\operatorname{codim}\eta$. For any such singularity ζ, using the symmetries of ζ one gets a certain set of equations on the coefficients of Tp_η. The form of these equations and their proof will not be given in this paper. They and all the necessary calculations are explicitly given in [**Rim01**], see Theorem 2.4, Section 4 and the last two paragraphs of Section 5. We will give and solve these equations in case of A_2 and A_3 singularities in the next two chapters.

3. Thom polynomial of A_2 singularities

In this section we consider the Thom polynomial of A_2 singularities, originally determined by Ronga ([**Ron72**]). Our goal is to show the connection with the theory of resultants.

Consider the theory of singularities of maps $\mathbb{C}^n \to \mathbb{C}^{n+k}$. There are three singularities with codimension at most $2k + 2$: A_0 (codim 0), A_1 (codim $k + 1$), A_2 (codim $2k + 2$). So looking for the Thom polynomial of A_2 we will obtain three sets of equations on the coefficients of

$$\mathrm{Tp}_{A_2} = \alpha \cdot c_1^{2k+2} + \beta \cdot c_1^{2k} c_2 + \ldots + \omega \cdot c_{2k+2},$$

as follows ([**Rim01**])

$$(A0) \qquad \mathrm{Tp}_{A_2}\Big(\prod_{i=1}^k (1 - g_i t)\Big) = 0 \qquad \in \mathbb{Z}[g_i]$$

$$(A1) \qquad \mathrm{Tp}_{A_2}\Big(\tfrac{1-2xt}{1-xt} \prod_{i=1}^k (1 - g_i t)\Big) = 0 \qquad \in \mathbb{Z}[x, g_i]$$

$$(A2) \qquad \mathrm{Tp}_{A_2}\Big(\tfrac{1-3xt}{1-xt} \prod_{i=1}^k (1 - g_i t)\Big) = 2x^2 \prod_{i=1}^k ((2x - g_i)(x - g_i)) \quad \in \mathbb{Z}[x, g_i].$$

These equations mean that what we really substitute in Tp_{A_2} are the Taylor coefficients of the fractions (with variable t) given in brackets. Clearly the equation (A0) is contained in (A1) so we really have to solve (A1) and (A2).

We know from theory ([**FR**, Th. 3.5]) that the solution is unique (once k is fixed), so we only have to show that a certain polynomial satisfies the conditions.

THEOREM 3.1. $\mathrm{Tp}_{A_2} = c_{k+1}^2 + \sum_{i=1}^{k+1} 2^{i-1} c_{k+1-i} c_{k+1+i}$.

Although it is easy to find this polynomial directly using elementary algebra, we will give a proof using the resultant formulas from Section 5; see the discussion after Theorem 5.4.

There is an equivalent way to characterize these polynomials independently from k as follows. Let us shift the indices by $k+1$, i.e. let $d_i = c_{i+k+1}$ (so $d_i = 0$ for $i < -k-1$). Then we have

THEOREM 3.2. *For every k we have* $\mathrm{Tp}_{A_2} = d_0^2 + \sum_{i=1}^{\infty} 2^{i-1} d_{-i} d_i$.

4. Thom polynomial of A_3 singularities

For all k, the case of $A_3 : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+k}, 0)$ singularities is essentially harder than that of A_2. The reason is that in this case there is a singularity of lower codimension than A_3, which is not from the A_i series (not a *Morin singularity*). So Tp_{A_3} has to satisfy a condition which comes from this extra singularity (called $III_{2,2}$), and this extra condition is very different in nature. This phenomenon was observed by Gaffney in [**Gaf83**]— there he had to deal with the "competing singularities" A_4, $I_{2,2}$ for $k = 0$.

So let us recall from [**Rim01**] the conditions Tp_{A_3} has to satisfy.

$$
\begin{array}{lll}
(A0) & \mathrm{Tp}_{A_3}\left(\prod_{i=1}^{k}(1-g_i t)\right) = 0 & \in \mathbb{Z}[g_i] \\
(A1) & \mathrm{Tp}_{A_3}\left(\frac{1-2xt}{1-xt}\prod_{i=1}^{k}(1-g_i t)\right) = 0 & \in \mathbb{Z}[x, g_i] \\
(A2) & \mathrm{Tp}_{A_3}\left(\frac{1-3xt}{1-xt}\prod_{i=1}^{k}(1-g_i t)\right) = 0 & \in \mathbb{Z}[x, g_i] \\
(A3) & \mathrm{Tp}_{A_3}\left(\frac{1-4xt}{1-xt}\prod_{i=1}^{k}(1-g_i t)\right) = & \\
 & \quad -6x^3\prod_{i=1}^{k}((3x-g_i)(2x-g_i)(x-g_i)) & \in \mathbb{Z}[x, g_i] \\
(III_{2,2}) & \mathrm{Tp}_{A_3}\left(\frac{(1-2xt)(1-2yt)(1-(x+y)t)}{(1-xt)(1-yt)}\prod_{i=1}^{k-1}(1-g_i t)\right) = 0 & \in \mathbb{Z}[x, y, g_i].
\end{array}
$$

REMARK 4.1. The interested reader can find these conditions in [**Rim01**] p. 513, though we reversed the sign convention. Also the "total Chern class" of the singularity $III_{2,2}$ is not included explicitly there, one has to follow the instructions of [**Rim01**, Section 4].

Here, again, by theory we know that for each k there is only one polynomial that satisfies all these equations. Clearly, the last one is more difficult to deal with. As a start one might try whether the first three already determine Tp_{A_3}, but this is not the case: there is a family of polynomials satisfying the first three, and the task is really to find the one from this family satisfying the last one. It turns out that this task again translates to a surprising expression on resultants.

To state the solution we will need some terminology. Let us modify the Pascal triangle by putting the 3-powers on the edges (instead of 1's), but let us keep the rule of the Pascal triangle, as shown:

$$\begin{array}{ccccccccc} &&&& a_{0,0} &&&& \\ &&& a_{1,0} && a_{0,1} &&& \\ && a_{2,0} && a_{1,1} && a_{0,2} && \\ & a_{3,0} && a_{2,1} && a_{1,2} && a_{0,3} & \\ a_{4,0} && a_{3,1} && a_{2,2} && a_{1,3} && a_{0,4} \end{array} = \begin{array}{ccccccccc} &&&& 0 &&&& \\ &&& 1 && 1 &&& \\ && 3 && 2 && 3 && \\ & 9 && 5 && 5 && 9 & \\ 27 && 14 && 10 && 14 && 27 \end{array}$$

Let $a_{i,j}$ stand for the number which stands in the place of $\binom{i+j}{j}$ in this modified Pascal triangle, i.e. let

$$\sum_{i,j} a_{i,j} u^i v^j = \frac{u\frac{1-u}{1-3u} + v\frac{1-v}{1-3v}}{1-(u+v)}.$$

THEOREM 4.2. *By shifting the indices $d_i = c_{i+k+1}$ the Thom polynomial of A_3* for any *k is given by*

$$\mathrm{Tp}_{A_3} = \sum_{i=0}^{\infty} 2^i d_{-i} d_0 d_i + \frac{1}{3}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} 2^i 3^j d_{-i} d_{-j} d_{i+j} + \frac{1}{2}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{i,j} d_{-i-j} d_i d_j.$$

The verification that this polynomial satisfies the conditions (A0)–(A3) can be done directly (for this one only needs that $2^r + \sum_{i+j=r} a_{i,j} = 4\cdot 3^{r-1}$, not the actual values of $a_{i,j}$)—although the authors did it using topological arguments which are beyond the scope of this paper. However, the fact that the given polynomial satisfies the last condition turned out to be more difficult than expected, and we still do not have an illuminating proof which would give the *real* reason. See Theorem 5.5 and the discussion after.

5. Expressions for the resultant

In this section we review some well known facts about resultants of two polynomials and give the two new expressions, which seem interesting in their own right and are needed for the proofs of the preceding sections.

DEFINITION 5.1. *Let $\underline{x} = (x_1, \dots, x_k)$ and $\underline{y} = (y_1, \dots, y_l)$ where $x_1, \dots, x_k$ and $y_1, \dots, y_l$ are elements from some ring. Then we define their resultant as* $\mathrm{Res}(\underline{x}|\underline{y}) = \prod_i \prod_j (x_i - y_j)$.

If A and B are the polynomials whose roots are x_i and y_j respectively (or rather $1/x_i$, $1/y_j$ for convenient terminology), then the resultant can be obtained as a determinant whose entries are the coefficients of A and B (i.e. the elementary symmetric polynomials of x_i and y_j).

THEOREM 5.2. (Sylvester-matrix) *Let*

$$A(t) = 1 + \sum_{i=1}^{k} a_i t^i = \prod_{i=1}^{k} (1 - x_i t) \qquad \textit{and} \qquad B(t) = 1 + \sum_{i=1}^{l} b_j t^j = \prod_{j=1}^{l} (1 - y_j t).$$

Then

$$\operatorname{Res}(\underline{x}|\underline{y}) = \begin{vmatrix} 1 & a_1 & a_2 & \dots & a_k & 0 & \dots & & 0 \\ 0 & 1 & a_1 & a_2 & \dots & a_k & 0 & \dots & 0 \\ \vdots & & \ddots & & & & \ddots & & \vdots \\ 0 & \dots & 0 & 1 & a_1 & \dots & & & a_k \\ 1 & b_1 & \dots & b_l & 0 & & \dots & & 0 \\ 0 & 1 & b_1 & \dots & b_l & 0 & & \dots & 0 \\ \vdots & & \ddots & & & & \ddots & & \vdots \\ \vdots & & & & & & & & 0 \\ 0 & \dots & & 0 & 1 & b_1 & \dots & & b_l \end{vmatrix}_{(k+l)\times(k+l)}$$

□

Now let the Taylor coefficients of $1/B(t)$ be $1, \bar{b}_1, \bar{b}_2, \dots$. If we multiply the Sylvester matrix from the right with the upper triangular matrix whose (i,j)-entry is $\bar{b}_{j-i}$, then we get a very useful expression for the resultant.

THEOREM 5.3. *Let* $1 + c_1 t + c_2 t + \ldots = A(t)/B(t)$. *Then*

$$\operatorname{Res}(\underline{x}|\underline{y}) = (-1)^{kl} \begin{vmatrix} c_k & c_{k+1} & c_{k+2} & \dots \\ c_{k-1} & c_k & c_{k+1} & \ddots \\ & \ddots & \ddots & \ddots \\ c_{k-l+1} & \dots & c_{k-1} & c_k \end{vmatrix}_{l\times l}$$

□

The determinant of this latter matrix (i.e. whose (i,j)-entry is c_{k+j-i}) is called the k^l-Schur-determinant, and is denoted by Δ_{k^l}.

We want to abbreviate the formula of the last Theorem as

$$\operatorname{Res}(\underline{x}|\underline{y}) = (-1)^{kl}\Delta_{k^l}\Big(\frac{\underline{x}}{\underline{y}}\Big),$$

by which it is clear what we mean: the variables that we substitute into Δ_{k^l} are the Taylor coefficients of $\prod(1-x_i t)/\prod(1-y_j t)$. In this language the dual statement is

$$\operatorname{Res}(\underline{x}|\underline{y}) = \Delta_{l^k}\Big(\frac{\underline{y}}{\underline{x}}\Big).$$

The interesting fact is that when there are algebraic relations among the roots, then there are other nice expressions for the resultants—the coming Theorems 5.4 and 5.5—and these involve exactly the Thom polynomials of Sections 3, 4.

THEOREM 5.4. *For the polynomial* Tp_{A_2} *given in Theorem 3.2 we have*

$$\operatorname{Res}(x, 2x | g_1, \dots, g_{k+1}) = \mathrm{Tp}_{A_2}\Big(\frac{g_1, \dots, g_{k+1}}{x}\Big).$$

PROOF. I. The proof we give here imitates the proof of Theorem 5.3. Let us multiply the Sylvester matrix of $((1-xt)(1-2xt), \prod(1-g_i t))$ with the upper triangular matrix whose (i,j)'th entry is the $j-i$'th Taylor coefficient of $1/(1-x)$.

We get

$$\operatorname{Res}(x,2x|g_1,\dots,g_{k+1})=\begin{vmatrix} 1 & -2x & & & & & \\ & 1 & -2x & & & & \\ & & \ddots & \ddots & & & \\ & & & \ddots & \ddots & & \\ & & & & 1 & -2x & 0 \\ d_{-k-1} & d_{-k} & & \dots & & d_0 & d_1 \\ & d_{-k-1} & d_{-k} & \dots & & d_{-1} & d_0 \end{vmatrix},$$

where $d_{-k-1}t^{-k-1}+d_{-k}t^{-k}+\ldots=1/t^{k+1}\cdot\prod(1-g_it)/(1-xt)$. Now applying the Lagrange expansion with respect to the last two rows we obtain

$$\sum_0^\infty(2x)^i\begin{vmatrix} d_{-i} & d_1 \\ d_{-i-1} & d_0\end{vmatrix}=\sum_0^\infty 2^i\begin{vmatrix} d_{-i} & x^id_1 \\ d_{-i-1} & x^id_0\end{vmatrix}.$$

Now observe that $d_i=xd_{i-1}$ for $i=1,2,\ldots$, so we get

$$\sum_0^\infty 2^i\begin{vmatrix} d_{-i} & d_{i+1} \\ d_{-i-1} & d_i\end{vmatrix}=d_0^2+\sum_{i=1}^\infty 2^{i-1}d_{-i}d_i=\operatorname{Tp}_{A_2}\Big(\frac{g_1,\dots,g_{k+1}}{x}\Big).$$

□

Let us sketch another proof not using determinants.

PROOF. II. Observe that Tp_{A_2} is the constant term of $1/2\cdot f(2t)f_+(t)+1/2\cdot d_0^2$ where $f(t)=1/t^{k+1}\cdot\prod(1-g_it)/(1-xt)$, and $f_+(t)$ is the positive (power) part of $f(t)$, $f_+(t)=d_0+d_1t+d_2t^2+\ldots$. It is easy to see that $f_+(t)=v\frac{1}{1-xt}$, where $v=\prod_{i=1}^{k+1}(x-g_i)$. Then

$$\operatorname{Tp}_{A_2}\Big(\frac{g_1,\dots,g_{k+1}}{x}\Big)=\frac{1}{2}v\frac{1}{(1-2xt)(1-xt)}\prod_{i=1}^{k+1}(1-g_it)\Big|_{k+1\text{st coeff.}}+\frac{1}{2}v^2.$$

It is clearly divisible by v and when we substitute $g_{k+1}=2x$, we get

$$v\Big(\frac{1}{2}\frac{1}{(1-xt)}\prod_{i=1}^{k}(1-g_it)\Big|_{k+1\text{st coeff.}}+\frac{1}{2}(-x)\prod_{i=1}^{k}(x-g_i)\Big).$$

Observe that the $k+1$'st coefficient of $\frac{1}{(1-xt)}\prod(1-g_it)$ is $x\prod(x-g_i)$ so this last expression is 0. So the original is divisible by $\operatorname{Res}(x,2x|g_1,\dots,g_{k+1})$. To get that it is in fact equal to it one can check the coefficient of e.g. x^{2k+2}. □

Now we show how this theorem implies that Tp_{A_2}—as given in Theorems 3.1, 3.2—satisfies the $(A1)$ and $(A2)$ conditions (see Section 3). For $(A1)$ we need to calculate

$$\operatorname{Tp}_{A_2}\Big(\frac{1-2xt}{1-xt}\prod_{i=1}^{k}(1-g_it)\Big),$$

which is by the theorem $=\operatorname{Res}(x,2x|g_1,\dots,g_{k+1},2x)$. Because of the $2x$ on both sides this is 0. For $(A2)$ we identify

$$\operatorname{Tp}_{A_2}\Big(\frac{1-3xt}{1-xt}\prod_{i=1}^{k}(1-g_it)\Big)$$

with $\mathrm{Res}(x, 2x|g_1, \dots, g_{k+1}, 3x)$, which is exactly the right hand side of the $(A2)$ condition.

We have given two proofs for Theorem 5.4, an algebraic and a more analytic one. However, neither easy algebraic or analytic proofs have been found for the next theorem, so we do not have a satisfactory proof for Theorem 4.2 either. The only proof was obtained by the first author, and that involves extremely long calculations—so we omit it. The authors believe that there must be an illuminating proof which shows the underlying algebra or analysis. It would particularly be interesting because a better understanding of this theorem would probably open the way to find the Thom polynomials for more complicated singularities, as well as a deeper relation between Thom polynomials and resultants.

THEOREM 5.5. *For the polynomial* Tp_{A_3} *given in Theorem 4.2 we have*

$$\mathrm{Res}(a, b, a+b|g_1, \dots, g_k, 0) = -\,\mathrm{Tp}_{A_3}\Big(\frac{2a, 2b, g_1, \dots, g_k}{a, b}\Big).$$

Now we show how this theorem implies that Tp_{A_3} satisfies the $(III_{2,2})$ condition (in fact, it easily proves the $(A1)$ and $(A3)$ conditions, as well). We need to calculate

$$\mathrm{Tp}_{A_3}\Big(\frac{(1-2xt)(1-2yt)(1-(x+y)t)}{(1-xt)(1-yt)}\prod_{i=1}^{k-1}(1-g_it)\Big),$$

which is, according to our theorem, $-\mathrm{Res}(x, y, x+y|g_1, \dots, g_k, x+y)$. Since there is an $x+y$ on both sides, it is 0, which is what we wanted to prove.

References

[AVGL91] V. I. Arnold, V. A. Vassiliev, V. V. Goryunov, and O. V. Lyashko. *Singularities. Local and global theory.* Enc. Math. Sci. Dynamical Systems VI. Springer, 1991.

[FR] L. Fehér and R. Rimányi. Calulation of Thom polynomials and other cohomological obstructions for group actions. preprint, www.math.ohio-state.edu/~rimanyi.

[Gaf83] T. Gaffney. The Thom polynomial of $\overline{\Sigma^{1111}}$. In *Singularities*, Proc. Symp. Pure Math. 40. Part I, pages 399–408. AMS, 1983.

[Kaz97] M. É. Kazarian. Characteristic classes of singularity theory. In V. I. Arnold et al., editors, *The Arnold-Gelfand mathematical seminars: Geometry and singularity theory*, pages 325–340, 1997.

[Rim01] R. Rimányi. Thom polynomials, symmetries and incidences of singularities. *Inv. Math.*, 143:499–521, 2001.

[Ron72] F. Ronga. La calcul des classes duales aux singularités de Boardman d'ordre 2. *Comm. Math. Helv.*, 47:15–35, 1972.

DEPARTMENT OF ANALYSIS, EOTVOS UNIVERSITY, BUDAPEST
E-mail address: berczig@cs.elte.hu

DEPARTMENT OF ANALYSIS, EOTVOS UNIVERSITY, BUDAPEST
E-mail address: lfeher@renyi.hu

DEPARTMENT OF MATHEMATICS, THE OHIO STATE UNIVERSITY
E-mail address: rimanyi@math.ohio-state.edu