Analytic Number Theory WS14/15
Homework 1 (Due 24.10.2014)

October 13, 2014

Problem 1. Show that $f(n)$ is multiplicative iff $S_f(n)$ is multiplicative.(*You can use Lemma 1.(i) from Lecture 1 in the one direction and induction on n in the other.*)

Problem 2. Define the *Dirichlet product* of the arithmetic functions f and g by

$$f * g(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right) .$$

Let

$$e(n) = \begin{cases} 1 & \text{, if } n = 1 \\ 0 & \text{, if } n > 1 \end{cases}$$

and

$$I(n) = 1 \text{ for all } n \in \mathbb{N} .$$

Let $\mathcal{M}$ denote the set of all multiplicative arithmetic functions.

1. Prove that for $f, g \in \mathcal{M}$ we have $f * g \in \mathcal{M}$. Deduce again that if $f \in \mathcal{M}$, then $S_f \in \mathcal{M}$.
2. Show that the Dirichlet product is associative, commutative and $e(n)$ is the identity element.
3. Show that every function $f \in \mathcal{M} \setminus \{f : f(1) = 0\}$ has an inverse element. (*You can do it recursively expressing $f^{-1}(n)$ by $f^{-1}(k)$ for some $k < n$.*)

This shows that the set $\mathcal{M} \setminus \{f \equiv 0\}$ is a group under the Dirichlet product.

Problem 3. Prove the *Möbius inversion formula* $f = \mu * S_f$.