HW2, Problem 3*:
Use Dirichlet hyperbola method to show that

$$\sum_{n\le x} \tau(n^2+1) = \frac{3}{\pi} x\log(x) + \mathcal{O}(x)\,.$$

This note presents the different ideas suggested by the students Daniel Klocker, Jürgen Steininger, Stefania Ebli and Valerie Roitner for solving this problem.

PROOF: Let

$$\rho(d) = \#\left\{m \in \mathbb{Z}/d\mathbb{Z} : m^2+1 \equiv 0 \pmod d\right\} \tag{1}$$

denote the number of solutions modulo d of the congruence $n^2+1 \equiv 0 \pmod d$. The actual application of the Dirichlet hyperbola method is only in the first part of the proof, namely in

Claim 1. *We have the following identity*

$$\sum_{n\le x} \tau(n^2+1) = 2x\sum_{d\le x}\frac{\rho(d)}{d} + \mathcal{O}\left(\sum_{d\le x}\rho(d)\right).$$

Proof. We can rewrite $\sum_{n\le x}\tau(n^2+1)$ as

$$\sum_{n\le x}\tau(n^2+1) = \sum_{n\le x}\sum_{d|n^2+1} 1 = 2\sum_{n\le x}\sum_{\substack{d\le n\\ d|n^2+1}} 1 = 2\sum_{d\le x}\sum_{\substack{n\le x\\ n^2+1\equiv 0 \pmod d}} 1$$

following from Dirichlet hyperbola method. The factor 2 arises since every divisor d of n^2+1 with $d \le \left[\sqrt{n^2+1}\right] = n$ has a complimentary divisor q with $dq = n^2+1$ and $q > n$ and vice versa (there exists a bijection between the divisors d of n^2+1 with $d \le n$ and the divisors $d > n$.)

Since the condition $n^2+1 \equiv 0 \pmod d$ is periodic in n with period d the above sum can be rewritten as

$$2\sum_{d\le x}\left(\frac{x}{d}\rho(d) + \mathcal{O}(\rho(d))\right)$$

where $\rho(d)$ denotes the number of the solutions of $n^2+1=0$ in $\mathbb{Z}/d\mathbb{Z}$. Hence

$$\sum_{n\le x} \tau(n^2+1) = 2x\sum_{d\le x}\frac{\rho(d)}{d} + \mathcal{O}\left(\sum_{d\le x}\rho(d)\right).$$

□

1 Analyses of $\rho(d)$

The hardest part of the proof is to understand the function $\rho(d)$ and the sums $\sum_{d\le x}\frac{\rho(d)}{d}$ and $\sum_{d\le x}\rho(d)$. Here is where the ideas split and we will present three different methods, the text following closely the solutions of the students.

1.1 Idea of Valerie

The first method is presented in most consice way, however it is the least elementary of all that will be given here, requiring some knowledge in algebraic number theory.

The function $\rho(n)$ is multiplicative in n, while $f(n) = \tau(n^2+1)$ is not. From Fermat's two square theorem, we know that $\rho(p) = 2$ if $p \equiv 1 \pmod 4$ and $\rho(p) = 0$ if $p \equiv 3 \pmod 4$. With Hensel's lemma ρ can be computed at prime powers. The multiplicity of ρ follows from the Chinese Remainder Theorem.

We now want to estimate the sums

$$\sum_{d \le x} \frac{\rho(d)}{d} \quad \text{and} \quad \sum_{d \le x} \rho(d).$$

Using a result from [1] we get

$$\sum_{d \le x} P(d) = \frac{6H^*(\Delta)}{\pi\sqrt{-\Delta}} x + \mathcal{O}(\sqrt{x}\log x)$$

where $P(x)$ is the number of solutions of the equation $n^2 + bn + c = 0$ with $\Delta = b^2 - 4c < 0$ and $H^*(\Delta)$ is the Hurwitz class number. The Hurwitz class number is defined as

$$H^*(\Delta) = \sum_{\substack{(A,B,C) \text{ reduced} \\ B^2 - 4AC = \Delta}} \frac{2}{w(A,B,C)},$$

where (A,B,C) is a poistive definite quadratic form $Ax^2 + Bxy + Cy^2$ with the same discriminat Δ as the equation $n^2 + bn + c$ and $w(A,B,C)$ is the size of the automorohism group of the form (A,B,C). We have that $w(A,A,A) = 6$, $w(A,0,A) = 4$ and for other reduced forms $w(A,B,C) = 2$. (The main idea of McKee's proof is to find an explicit expression for $\rho(d)$ using the automorphism group of a quadratic form and then, using this explicit expression, reformulation $\sum_{d \le x} \rho(d)$ as a nested sum of Möbius functions, which after some changes of inner sums then can be computed explicitly).

In our case we have the equation $n^2 + 1 = 0$, $\Delta = -4$ and $H^*(-4) = \frac{1}{2}$. Hence

$$\sum_{d \le x} \rho(d) = \frac{3}{2\pi} x + O(\sqrt{x}\log x) = \mathcal{O}(x).$$

Using summation by parts we get

$$\sum_{d \le x} \frac{\rho(d)}{d} = \frac{3}{2\pi}\log x + \mathcal{O}(1).$$

Combining everything we finally get

$$\sum_{n \le x} \tau(n^2+1) = 2x \sum_{d \le x} \frac{\rho(d)}{d} + \mathcal{O}\left(\sum_{d \le x} \rho(d)\right) = \frac{3}{\pi} x \log x + \mathcal{O}(x).$$

1.2 Idea of Stefania

This method uses the intriguing relation of the function $\rho(d)$ to the Gauss circle problem.

Define the functions

$$r(n) = \#\left\{(x,y) \in \mathbb{Z}^2 : (x,y) = 1 \text{ and } x^2 + y^2 = n\right\}$$

counting the proper representations of n as a sum of two squares, and

$$P(n) = \#\left\{(x,y) \in \mathbb{Z}^2 : (x,y) = 1,\ x > 0, y \ge 0 \text{ and } x^2 + y^2 = n\right\}$$

which counts the proper representations in the first quadrant.

Recall that the Gauss circle problem counts the lattice points in a circle of radius $\sqrt{x}$ by the asymptotic formula

$$\sum_{n\leq x}\sum_{x^2+y^2=n} 1 = \pi x + \mathcal{O}(\sqrt{x})\,. \tag{2}$$

As the density of coprime integers is $6/\pi^2$, one expects that the primitive circle problem gives

$$\sum_{n\leq x} r(n) = \frac{6}{\pi}x + \mathcal{O}(x^{1/2+\varepsilon})\,. \tag{3}$$

The main statement importing $\rho(d)$ in the context of the circle problem is the following

Lemma 1. *(Theorem 3.21, [2]) We have $r(n) = 4\rho(n)$ for every positive integer n. In particular $\rho(n) = P(n)$.*

Proof. Consider any solution of $x^2 + y^2 = n$, where $n > 0$. Of the four solutions (x, y), $(-y, x)$, $(-x, -y)$, $(y, -x)$) exactly one of them has both the first and the second coordinate positive. Thus, let $P(n)$ be defined as above, we will have $r(n) = 4P(n)$ and we will now prove that $\rho(n) = P(n)$. Suppose that n is a given positive integer, we shall exhibit one-to-one correspondence between the representations $x^2 + y^2 = n$ with $x > 0$, $y \geq 0$, $(x, y) = 1$ and the solutions s of the congruence $s^2 \equiv -1 \pmod{n}$.

We will do this in three steps. First we define a function from the appropriate pairs (x, y) to the appropriate residue class $s \pmod{n}$. Second, we will show that the function is one-to-one. Third, we prove that the function is onto.

To define the function, suppose that x and y are integers such that $x^2 + y^2 = n$, $x > 0$ and $y \geq 0$, and $(x, y) = 1$. Then $(x, n) = 1$, so there exists a unique $s \pmod{n}$ such that $xs \equiv y \pmod{n}$. More precisely, if $\tilde{x}$ is chosen so that $x\tilde{x} \equiv 1 \pmod{n}$, then $s \equiv \tilde{x}y \pmod{n}$. Since $x^2 \equiv y^2 m \pmod{n}$, on multiplying both sides by $\tilde{x}^2$ we deduce that $s^2 \equiv -1 \pmod{n}$.

We now show that our function is one-to-one. Suppose that for $i = 1, 2$ we have $n = x_i^2 + y_i^2$ with $x_i > 0$ $y_i \geq 0$, $(x_i, y_i) = 1$ and $x_i s_i \equiv y_i \pmod{n}$. We show that if $s_1 \equiv s_2 \pmod{n}$ then $x_1 = x_2$ and $y_1 = y_2$. Suppose $s_1 \equiv s_2 \pmod{n}$, as $x_1y_2s_1 \equiv y_1y_2 \equiv x_2y_1s_2 \pmod{n}$ it follows that $x_1y_2 \equiv x_2y_1 \pmod{n}$ since $(s_i, n) = 1$. But $0 < x_i^2 \leq n$ so that $0 < x_i \leq \sqrt{n}$ and similary $0 < y_i \leq \sqrt{n}$. From this inequalities we dedeuce that $0 \leq x_1y_2 \leq \sqrt{n}$ and similary $0 \leq x_2y_1 \leq \sqrt{n}$. As these two numbers are congruent modulo n and both lie in the interval $[0, n)$ we can conclude that $x_1y_2 = x_2y_1$. Thus $x_1 | x_2y_1$. But $(x_1, y_1) = 1$ and $(x_2, y_2) = 1$, so it follows that $x_1 | x_2$ and $x_2 | x_1$. As the x_i are positive we deduce that $x_1 = x_2$ and hence $y_1 = y_2$. This completes the proof that the function is one-to-one.

To complete the proof we just need to show that our function is onto. That is, for each s such that $s \equiv -1 \pmod{n}$ there is a representation $x^2 + y^2 = n$ with $x > 0$ and $y \geq 0$, $(x, y) = 1$ and $xs \equiv y \pmod{n}$. Suppose that such s is given, then there is an integer c such that $(2s)^2 - 4nc = -4$. Thus $g(x, y) = nx^2 + 2sxy + cy^2$ is a positive definite binary quadratic form of discriminant -4. In the proof of Theorem 3.20[2] it is shown that all these forms are equivalent. Thus there is a matrix $M \in \Gamma$ that takes the form $f(x, y) = x^2 + y^2$ to the form g. From definition (3.7a)[2] we see that $m_{11}^2 + m_{21}^2 = n$ and $(m_{11}, m_{21}) = 1$ since $det(M) = m_{11}m_{22} - m_{12}m_{21} = 1$. From (3.7b) we see that $s = m_{11}m_{12} + m_{21}m_{22}$. Hence:

$$\begin{aligned} m_{11}^2 &= m_{11}^2 + m_{11}m_{21}m_{22} \\ &= -m_{21}^2 m_{12} + m_{11}m_{21}m_{22} \pmod{n} && (\text{since } m_{11}^2 \equiv -m_{21}^2 \pmod{n}) \\ &= -m_{21}^2 m_{12} + m_{21}(1 + m_{21}m_{12}) && (\text{since } m_{11}m_{22} - m_{21}m_{12} = 1) \\ &= m_{21} \end{aligned}$$

In the case $m_{11} > 0$ and $m_{21} \le 0$ it is suffices to take $x = m_{11}$ and $y = m_{21}$. In case these inequalities do not hold, then we take the point (x, y) to be one of the point $(-m_{21}, m_{11})$, $(-m_{11}, -m_{21})$, (m_{21}, m_{11}). From the congruence $m_{11}s \equiv m_{21} \pmod{n}$, $s^2 \equiv -1 \pmod{n}$ we deduce that $-m_{21}s \equiv m_{11} \pmod{n}$. Thus $xs \equiv y \pmod{n}$ in any of these case. This complete the proof that $r(n) = 4\rho(n)$.

□

Equation (2) shows us the asymptotic behaviour of the number of integer points inside a circle of radius $\sqrt{x}$. In the following lemma we will find the asymptotic behaviour of integer points (x, y) inside a circle of radius $\sqrt{x}$ such that $(x, y) = 1$, i.e. we will prove (3).

Lemma 2. *We have*

$$Q(x) := \sum_{n=1}^{x} r(n) = \frac{6}{\pi}x + \mathcal{O}(\sqrt{x}\log x)\,.$$

In particular

$$\sum_{n=1}^{x} P(n) = \sum_{n=1}^{x} \rho(n) = \frac{3}{2\pi}x + \mathcal{O}(\sqrt{x}\log x)\,.$$

Proof. From the definition of $Q(x)$ we can write

$$Q(x) = \sum_{\substack{u,v \\ 1\le u^2+v^2\le x \\ (u,v)=1}} 1$$

and

$$\begin{aligned} B(x) &:= \sum_{\substack{u,v \\ 1\le u^2+v^2\le x}} 1 \\ &= \sum_{1\le d\le\sqrt{x}} \sum_{\substack{u,v \\ 1\le u^2+v^2\le x \\ (u,v)=d}} 1 \\ &= \sum_{1\le d\le\sqrt{x}} \sum_{\substack{u',v' \\ 1\le u'^2+v'^2\le x/d^2 \\ (u',v')=1}} 1 \qquad \text{where } u' = \frac{u}{d},\ v' = \frac{v}{d} \\ &= \sum_{1\le d\le\sqrt{x}} Q\Big(\frac{x}{d^2}\Big) \end{aligned} \tag{4}$$

Applying a certain version of the Möbius Inversion Formula to (4) we have

$$\begin{aligned} Q(x) &= \sum_{1\le d\le\sqrt{x}} \mu(d)B\Big(\frac{x}{d^2}\Big) \\ &= \sum_{1\le d\le\sqrt{x}} \mu(d)\Big(\frac{\pi x}{d^2} + \mathcal{O}\Big(\frac{\sqrt{x}}{d}\Big)\Big) && \text{(by (2))} \\ &= \pi x \sum_{1\le d\le\sqrt{x}} \frac{\mu(d)}{d^2} + \mathcal{O}\Big(\sqrt{x}\sum_{1\le d\le\sqrt{x}} \frac{\mu(d)}{d}\Big) \\ &= \frac{6}{\pi}x + \mathcal{O}\Big(\frac{1}{\sqrt{x}}\Big) + \mathcal{O}(\sqrt{x}\log x) && \Big(\text{since } \Big|\sqrt{x}\sum_{1\le d\le\sqrt{x}}\frac{\mu(d)}{d}\Big| \le \sqrt{x}\log x\Big) \\ &= \frac{6}{\pi}x + \mathcal{O}(\sqrt{x}\log x)\,. \end{aligned}$$

□

We know that $r(n) = 4P(n) = 4\rho(n)$, hence

$$\sum_{n=1}^{x} P(n) = \sum_{n \le x} \rho(n) = \frac{3}{2\pi} x + \mathcal{O}(\sqrt{x} \log x) = \mathcal{O}(x) \tag{5}$$

Corollary 1. *We have*

$$\sum_{n \le x} \frac{\rho(n)}{n} = \frac{3}{2\pi} \log x + \mathcal{O}(1)\,.$$

Proof. First we will prove that

$$\sum_{n=1}^{x} \frac{r(n)}{n} = \frac{6}{\pi} \log x + \mathcal{O}(1)\,.$$

Applying Abel Transformation with $h(x) = \sum_{n=1}^{x} r(n)$ and $g(x) = \frac{1}{x}$ we get

$$\sum_{n=1}^{x} \frac{r(n)}{n} = \frac{1}{x} \sum_{n=1}^{x} r(n) + \int_1^x \frac{1}{t^2} \left(\sum_{n=1}^{t} r(n) \right) dt$$

From Lemma 2 we have

$$\begin{aligned} \sum_{n=1}^{x} \frac{r(n)}{n} &= \frac{6}{\pi} + \mathcal{O}\left(\frac{\log x}{\sqrt{x}} \right) + \int_1^x \left(\frac{6}{\pi t} + \mathcal{O}(t^{-\frac{3}{2}} \log t) \right) dt \\ &= \frac{6}{\pi} \log x + \mathcal{O}(1) \end{aligned}$$

It follows immediately that

$$\sum_{n \le x} \frac{\rho(n)}{n} = \frac{3}{2\pi} \log x + \mathcal{O}(1)\,.$$

□

Finally we have the tools to prove our asymptotic formula. By Corollary 1 and (5) one easily sees that

$$\sum_{n \le x} \tau(n^2 + 1) = \frac{3}{\pi} x \log x + \mathcal{O}(x).$$

1.3 Idea of Daniel and Jürgen

The proofs of Daniel and Jürgen are essentially the same, with the difference that Daniel restrains from the notion for Dirichlet L-series, and rather uses Abel's inequality for a certain estimate. Later this will become clearer.

First consider the function $|\mu(n)|$, where μ is the Möbius function, i.e. this is the square-counting function which is 1 if n is squrefree and 0 otherwise. Also consider the multiplicative function $\chi : \mathbb{N} \to \mathbb{C}$ defined by

$$\chi(n) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod 4 \\ -1 & \text{if } n \equiv 3 \pmod 4 \\ 0 & \text{otherwise.} \end{cases}$$

The basic steps in the analyses of $\rho(d)$ and the sums from Claim 1 involving this function are:

1. Show ρ is multiplicative and compute $\rho(p^\alpha)$ for all primes p.
2. Show $\rho = |\mu| * \chi$, where $*$ denotes Dirichlet convolution of arithmetic functions.

3. Estimate $\sum_{n\le x}|\mu(n)|$, $\sum_{n\le x}\frac{|\mu(n)|}{n}$, $\sum_{n\le x}\chi(n)$ and $\sum_{n\le x}\frac{\chi(n)}{n}$.

4. Estimate $\sum_{n\le x}\frac{\rho(n)}{n}$ and $\sum_{n\le x}\rho(n)$.

First we note that $\rho(1)=1, \rho(2)=1$ and $\rho(2^k)=0$ for $k\ge 2$. By the first supplement for quadratic residues we know for primes p:

$$\rho(p)=\begin{cases}2, & \text{if } p=4n+1\\ 0, & \text{if } p=4n+3\end{cases}$$

By Hensel's Lemma we get for every solution of $a^2\equiv -1 \pmod p$ a solution of $a^2\equiv -1 \pmod{p^k}$ for $k\ge 2$. Therefore we get for prime powers also

$$\rho(p^k)=\begin{cases}2, & \text{if } p=4n+1\\ 0, & \text{if } p=4n+3\end{cases}$$

Claim 2. *Let μ be the Möbius function. Then we have*

$$\rho(d)=\sum_{ab=d}|\mu(a)|\chi(b)=|\mu|*\chi(d)\,.$$

Proof. By the Chinese Remainder Theorem $\rho(d)$ is multiplicative, but not strongly multiplicative. Obviously by definition (1) $\rho(d)\le d$, so the Dirichlet series D_ρ is absolutely convergent for $\mathrm{Re}(s)>2$ (its absolute convergence abscissa is 1 but we will not pursue this here). Therefore for $\mathrm{Re}(s)>2$ we can write

$$\begin{aligned}\sum_{n\ge1}\frac{\rho(n)}{n^s}&=\prod_p\left(1+\frac{\rho(p)}{p^s}+\frac{\rho(p^2)}{p^{2s}}+\dots\right)\\&=\left(1+\frac{\rho(2)}{2^s}\right)\prod_{p>2}\left(1+\frac{\rho(p)}{p^s}+\frac{\rho(p^2)}{p^{2s}}+\dots\right)\\&=\left(1+2^{-s}\right)\prod_{p=4n+1}\left(1+\frac{2}{p^s}+\frac{2}{p^{2s}}+\dots\right)\\&=\left(1+2^{-s}\right)\prod_{p=4n+1}\left(1+2\sum_{k\ge0}\left(\frac{1}{p^s}\right)^k-2\right)\\&=\left(1+2^{-s}\right)\prod_{p=4n+1}\left(-1+\frac{2}{1-p^{-s}}\right)\\&=\left(1+2^{-s}\right)\prod_{p=4n+1}\left(\frac{1+p^{-s}}{1-p^{-s}}\right)\\&=\prod_p\frac{1+p^{-s}}{1-\chi(p)p^{-s}}\\&=\prod_p\left(1+p^{-s}\right)\prod_p\left(1-\chi(p)p^{-s}\right)^{-1}\\&=\sum_{n\ge1}\frac{|\mu(n)|}{n^s}\sum_{n\ge1}\frac{\chi(n)}{n^s}\end{aligned}$$

By multiplication of Dirichlet series we get

$$\rho(d)=\sum_{a|d}|\mu(a)|\chi\left(\tfrac{d}{a}\right)=\sum_{ab=d}|\mu(a)|\chi(b)\,.$$ □

So we see that $\sum_{d\le x}\frac{\rho(d)}{d}=\sum_{ab\le x}\frac{|\mu(a)|}{a}\frac{\chi(b)}{b}$. Therefore we should look closer at the sums $\sum_{n\le x}\frac{|\mu(n)|}{n}$ and $\sum_{n\le x}\frac{\chi(n)}{n}$.

Claim 3. *Let $\zeta(s)$ be the Riemann zeta function. Then we have:*

$$\sum_{n\le x}\frac{|\mu(n)|}{n}=\frac{\log x}{\zeta(2)}+\mathcal{O}(1)$$

Proof. From the lecture (14.11. Claim4) we know that $\sum_{n\le x}|\mu(n)|=\frac{x}{\zeta(2)}+\mathcal{O}(\sqrt{x})$. Using Abel transformation we get

$$\begin{aligned}\sum_{n\le x}|\mu(n)|\frac{1}{n}&=\frac{1}{x}\sum_{n\le x}|\mu(n)|+\int_1^x\left(\sum_{n\le t}|\mu(n)|\frac{1}{t^2}\right)dt\\&=\frac{1}{x}\left(\frac{x}{\zeta(2)}+\mathcal{O}(\sqrt{x})\right)+\int_1^x\left(\frac{t}{\zeta(2)}+\mathcal{O}(\sqrt{t})\right)\frac{1}{t^2}\,dt\\&=\frac{1}{\zeta(2)}+\mathcal{O}\left(\tfrac{1}{\sqrt{x}}\right)+\frac{1}{\zeta(2)}\int_1^x\frac{1}{t}\,dt+\mathcal{O}\left(\int_1^x\frac{1}{t\sqrt{t}}\,dt\right)\\&=\frac{\log x}{\zeta(2)}+\mathcal{O}(1)\,.\end{aligned}$$

□

We define the Dirichlet L-series at 1 as

$$L(1,\chi)=\sum_{n\ge 1}\frac{\chi(n)}{n}\,.$$

So when we want to calculate $\sum_{n\le x}\frac{\chi(n)}{n}$, we can express it as $L(1,\chi)$ plus an error term.

Claim 4. *We have*

$$\sum_{n\le x}\frac{\chi(n)}{n}=L(1,\chi)+\mathcal{O}\left(\frac{1}{x}\right)\,.$$

Proof. Let $A(x):=\sum_{n\le x}\chi(n)$. By definition of χ it follows that $|A(x)|\le 1$, so we have $A(x)=\mathcal{O}(1)$ for every x. We know that

$$L(1,\chi)=\sum_{n\le x}\frac{\chi(n)}{n}+\sum_{n>x}\frac{\chi(n)}{n}$$

So it remains to show that $\sum_{n>x}\frac{\chi(n)}{n}=\mathcal{O}(\frac{1}{x})$. Using Abel transformation we get

$$\begin{aligned}\sum_{x<n\le y}\frac{\chi(n)}{n}&=A(y)\frac{1}{y}-A(x)\frac{1}{x}+\int_x^y A(t)\frac{1}{t^2}\,dt\\&=\mathcal{O}\left(\frac{1}{x}\right)\end{aligned}$$

So we get

$$\lim_{y\to\infty}\sum_{x<n\le y}\frac{\chi(n)}{n}=\sum_{n>x}\frac{\chi(n)}{n}=\mathcal{O}\left(\frac{1}{x}\right),$$

since the limit doesn't depend on y. □

Leibniz showed that

$$L(1,\chi)=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+-\cdots=\frac{\pi}{4},$$

so Claim 4 reads as

$$\sum_{n\le x}\frac{\chi(n)}{n}=\frac{\pi}{4}+\mathcal{O}\left(\frac{1}{x}\right)\,.$$

Daniel's proof differs from Jürgen's only at this last claim, more precisely in its proof, which he does using Abel's inequality. Note that mentioning Dirichlet L-function is not really necessary for the proof, but we rather use it for a comfortable notation.

Further we know that $\zeta(2) = \pi^2/6$. Putting this together we can find now an expression for $\sum_{n\leq x} \frac{\rho(n)}{n}$:

Claim 5. *We have*

$$\sum_{n\leq x} \tfrac{\rho(n)}{n} = \frac{3}{2\pi} \log x + \mathcal{O}(1)\,.$$

Proof. Using the claims before we get

$$\begin{aligned}
\sum_{n\leq x} \frac{\rho(n)}{n} &= \sum_{ab\leq x} \frac{|\mu(a)|}{a} \frac{\chi(b)}{b} \\
&= \sum_{a\leq x} \frac{|\mu(a)|}{a} \sum_{b\leq \frac{x}{a}} \frac{\chi(b)}{b} \\
&= \left(\frac{\log x}{\zeta(2)} + \mathcal{O}(1)\right)\left(\frac{\pi}{4} + \mathcal{O}\left(\frac{a}{x}\right)\right) \\
&= \frac{3}{2\pi} \log x + \mathcal{O}(1)\,.
\end{aligned}$$

□

Claim 1 states that $\sum_{n\leq x} \tau(n^2+1) = 2x \sum_{d\leq x} \frac{\rho(d)}{d} + \mathcal{O}\left(\sum_{d\leq x} \rho(d)\right)$. Finally we need to look at $\sum_{d\leq x} \rho(d)$. This is not hard, because from Claim 2 and the fact that $\left|\sum_{n\leq x} \chi(n)\right| \leq 1$ follows

$$\sum_{d\leq x} \rho(d) = \sum_{a\leq x} |\mu(a)| \sum_{b\leq \frac{x}{a}} \chi(b) = \sum_{a\leq x} |\mu(a)| \mathcal{O}(1) = \mathcal{O}(x)\,.$$

Putting all together we get the result:

$$\begin{aligned}
\sum_{n\leq x} \tau(n^2+1) &= 2x \left(\frac{3}{2\pi} \log x + \mathcal{O}(1)\right) + \mathcal{O}(x) \\
&= \frac{3}{\pi} x \log x + \mathcal{O}(x).
\end{aligned}$$

References

[1] McKee, James *On the average number of divisors of quadratic polynomials*, Math. Proc. of the Cambridge Philosophical Society 117, p.389-392, 1995.

[2] Niven, Ivan ; Zuckerman, Herbert S. ; Montgomery, Hugh L. *An introduction to the theory of numbers.* Fifth edition. John Wiley & Sons, Inc., New York, 1991.