On computing Thom polynomials

Balázs Kőműves

Supervisor: László Fehér

Submitted to
Central European University, Budapest
Department of Mathematics and its Applications

In partial fulfillment of the requirements
for the degree of Doctor of Philosophy

Budapest, Hungary
2010

Abstract

The subject matter of this thesis is the computation of Thom polynomials of singularities of maps, in particular Thom-Boardman singularity classes. A “singularity” means a type of local behaviour of maps between smooth (or analytic) manifolds; the simplest example is the differential being degenerate. It is well known that the cohomology class of the (closure of the) locus in the source manifold where a map has a given singularity can be expressed as a polynomial of the characteristic classes of the map. This multivariate polynomial, which only depends on the singularity and the dimensions, is called the *Thom polynomial* of the singularity. Even though the above phenomenon was observed by Thom more than 50 years ago, there are still only a few examples where we can explicitly calculate these polynomials. In this work, we contribute both new methods of computations, and explicit calculations of some previously unknown Thom polynomials. In particular, we discover a connection between localization formulae for contact singularities and basic hypergeometric series; we present a new geometric construction to compactify some moduli spaces related to Thom-Boardman classes; and we give new formulae for the Thom polynomials of some second order Thom-Boardman singularities.

Table of Contents

Introduction

This thesis reports on the state of the author's ongoing investigation on the subject of computing Thom polynomials of singularities of maps, in particular Thom-Boardman singularity classes.

A "singularity" here means a type of local behaviour of maps between smooth (analytic) spaces; a very simple example is the vanishing of the differential of the map. There is a general local-global principle, which says the global topological properties of (closed) spaces can be read off from local geometric data. Two well-known examples of this phenomenon are: the Gauss-Bonnet theorem, which expresses the Euler characteristic of a manifold from the curvature; and the Poincaré-Hopf theorem, which expresses the Euler characteristic from the singularities of a vector field on the manifold. Thom polynomials, proposed by René Thom around 1950, are also a manifestation of this principle; in a sense, they are a generalization of the latter example. The basic observation is that given a singularity (that is, a type of local behaviour), the (co)homological properties of the locus where a map has the given type of behaviour depends only on the homotopy type of the map, and not on the fine details of the map itself. This can be made quantitative: the cohomology class of the locus can be expressed as a polynomial of the characteristic classes of the map; this polynomial is called the Thom polynomial of the singularity.

Our focus in this work is the problem of computing these polynomials. However, one quickly bumps into the philosophical question of what 'computing' means: First, these polynomials can be expressed in many different forms (eg. polynomials in Chern roots; polynomials in Chern classes; linear combination of Schur functions; variations of the latter two for the difference alphabet; pushforward formulae; iterated residue formulae; localization formulae; etc.) which are often very hard to convert to each other; second, it is easy to write down formulae which are very hard to evaluate in concrete cases (eg. even small cases are intractable by today's home computers).

Our answer (which is by no means final) is that we prefer expressing the Thom polynomials as linear combinations of Schur polynomials in the difference alphabet; the motivation for this form is that it is elegant, unique, relatively compact, the coefficients are *nonnegative* integers, they do not depend on the dimensions of the source and target spaces, and there is some evidence that they have very rich combinatorics. Furthermore, we seek (when possible) computationally effective methods to compute these coefficients; of course, we still prefer (closed) formulae.

In the last 10 years, there was a new wave of activity in the field; thanks to works of G. Bérczi, A. Buch, L. Fehér, M. Kazarian, A. Némethi, P. Pragacz, R. Rimányi, A. Szenes, the author, and others, we now know much more about both concrete examples and the structures behind Thom polynomials. However explicit computations are still hard, even using computers; and that is the subject matter this thesis contributes to, with both new formulae and new methods.

Organization of the material. The first two chapters are introductory: The first chapter introduces informally the notion of Thom polynomials and the basic methods to compute them, using the Thom-Porteous formula as a running example. The next chapter recalls some basic definitions of singularity theory. The third chapter deals with the general localization formula for contact singularities; it introduces a new idea to evaluate them, which can be also used to derive closed formulae. The fourth chapter investigates the specific case of second order Thom-Boardman singularities from multiple viewpoints. In the final, fifth chapter, we show how to modify a geometric idea introduced in the previous chapter to compute the Thom polynomials of the A_3 singularity. The appendices collect together various results we use during the text.

New results and statement of originality. The sections 3.2, 3.3, 3.4 in Chapter 3; sections 4.4, 4.3, 4.5 in Chapter 4, and Section 5.2.1 contain new results. From the above, Theorem 4.2.2, Section 4.4 (except the proof of Theorem 4.4.3), and Section 4.5.1 was published in the article [**FK06**] joint with László Fehér. The rest of the above is my original work. Furthermore, I gave new proofs of some known statements; in particular, the proofs of Lemma 3.1.2, Theorem A.3.7 and Theorem A.4.4 are my own work; Theorem 4.2.1 of Ronga [**Ron72**] is also reproved as a side-effect of this work.

Acknowledgments. The author would like to thank Gergely Bérczi, László Fehér, Frances Kirwan, András Némethi, Richárd Rimányi and András Szenes for various help and discussions; and his family and friends for support. During this work, he was supported financially by the Central European University (2003-2006), the Alfréd Rényi Institute of Mathematics (2007-2009), and by a Marie Curie fellowship during his six-month stay at the University of Oxford (2006-2007).

This work wouldn't exist without computers. In particular we used the following software and resources: The Maple™ computer algebra system; John Stembridge's `SF` package for Maple [**Ste05**] (to compute with symmetric polynomials); Matthias Franz's `Convex` package for Maple [**Fra09**] (to compute with convex polyhedra); the Glasgow Haskell Compiler [**GHC**] for the Haskell programming language [**PJ03**]; Sloane's On-Line Encyclopedia of Integer Sequences [**OEIS**]; and last, but not least, various digital libraries. The thesis was typeset with LaTeX. The figures were produced with several different software, which should remain nameless, as the range varied from the "rather inconvenient" to the "exceptionally painful".

Notations and conventions

Unless specifically stated, we will always work over the field of complex numbers; ie. vector spaces are complex vector spaces, vector bundles are complex vector bundles, algebraic varieties are complex algebraic varieties—in particular, $\mathbb{P}^n$ is the complex projective space—, etc. Cohomology is singular cohomology, the default coefficient ring is the field $\mathbb{Q}$ of rational numbers (just to be on the safe side; however, most results should work, without any modification, with integer coefficients).

1. Partitions. Partitions are by definition finite nonincreasing sequences of positive integers. They will be usually denoted by the greek letters λ, χ, μ and ν. Our convention is that we allow arbitrary number of zeros at the end of partitions, that is, we treat $(\mu_1,\mu_2,\ldots,\mu_k)$ and $(\mu_1,\mu_2,\ldots,\mu_k,0,0,\ldots,0)$ as the same object; this is often useful in formulae. The length of a partition μ, which is the number of positive elements, is denoted by $\ell(\mu)=k$; the weight, or sum, is denoted by $|\mu|=\sum\mu_i$. Repeated numbers are often denoted by exponents, ie. $(2^3,1^4)=(2,2,2,1,1,1,1)$. The *dual partition*[1] of μ, denoted by $\widetilde{\mu}$, is defined by

$$\widetilde{\mu}_i=\max\left\{\, j \;:\; \mu_j\geq i\,\right\}.$$

This is an involution on the set of partitions; note that $\ell(\mu)=\widetilde{\mu}_1$. We denote by $\lambda\pm\mu$ the pointwise addition (resp. subtraction) of the sequences; the latter isn't necessarily a partition.

A partition μ is a subpartition of an other partition λ, denoted by $\mu\subset\lambda$, if $\mu_i\leq\lambda_i$ for all i. If $\mu\subset(n^k)$, then its *complement* $\complement\mu$ is defined by $(\complement\mu)_i=n-\mu_{k+1-i}$; we omit the 'block' (n^k) from the notation, as it will be always clear from the context. The reverse sequence $(\mu_k,\mu_{k-1},\ldots,\mu_1)$ is denoted by $\mathsf{rev}\,\mu$.

The 'stairway' partition $(n,n-1,n-2,\ldots,1)$ is used often enough to deserve a special notation, for which we will use $\lfloor \mathrm{n}\rceil$.

2. Symmetric functions and characteristic classes. Let $c_1,c_2,c_3\ldots$ and $s_1,s_2,s_3,\ldots$ denote two sequences of formal variables related by the equation

$$(1+c_1t+c_2t^2+c_3t^3+\cdots)\cdot(1-s_1t+s_2t^2-s_3t^3+-\cdots)=1.$$

The convention is that $c_0=s_0=1$, and $c_{<0}=s_{<0}=0$. If we have a polinomial ring $\Bbbk[x_1,\ldots,x_n]$, we can specialize these formal variables to the elementary and complete symmetric functions, respectively; the subring generated by either of the two sequences is exactly the ring of symmetric polynomials $\Bbbk[x_1,\ldots,x_n]^{\mathfrak{S}_n}$. The same thing works with the limit $n\to\infty$, the ring

$$\Lambda=\varprojlim_n \Bbbk[x_1,x_2,\ldots,x_n]^{\mathfrak{S}_n}$$

being called *the ring of symmetric functions*.

If we have a complex vector bundle $E\to M$ (or more generally, a formal difference of two vector bundles), we can specialize to the Chern and Segre[2] classes $c_i(E)$ and $s_i(E)$ in $H^{2i}(M)$, respectively (and that's why we use the notations c_i and s_i instead of the e_i and h_i which are the standard in the theory of symmetric functions).

[1] the term *conjugate partition* is also used

[2] the literature has a sign ambiguity in the definition of Segre classes; for example our convention agrees with that of [**FP98**] but differs by $(-1)^i$ from that of [**Ful98**]

The Schur polynomials are symmetric polynomials parameterized by partitions; they are arguably the most important symmetric functions, and they give an additive basis in the ring of symmetric functions. Given a partition $\mu = (\mu_1, \mu_2, \dots, \mu_k)$, we define the Schur polynomial s_μ by the formula

$$s_\mu = \mathsf{det}[s_{\mu_i+j-i}]_{k\times k} = \mathsf{det}[c_{\widetilde{\mu}_i+j-i}]_{\mu_1\times\mu_1};$$

in particular[3], $s_{(i)} = s_i$ and $s_{(1^i)} = c_i$. As before, we can define the *Schur class* $s_\lambda(E) \in H^{2|\lambda|}(M)$ of a (complex) vector bundle E by substituting its Chern (or Segre) classes into the formula above.

3. A list of various notations. Finally we try to make the life of the reader easier by assembling a (necessarily partial) list of the various notations used in this thesis.

$H^*(Y)$	singular cohomology with coefficients in $\mathbb{Q}$
$H^*_G(Y)$	G-equivariant cohomology
$A^G_i(X)$	G-equivariant Chow groups (see [**EG98a**])
f^*	pullback along the map f (of cohomology classes, or vector bundles)
π_*	pushforward (or Gysin map); if $\pi : X \to Y$ then $\pi_* : H^*(X) \to H^{*+\mathsf{codim}\,\pi}(Y)$
$[X \subset Y]$	the (equivariant) cohomology class represented by X in $H^*(Y)$ (or $H^*_G(Y)$); often denoted simply by $[X]$
$e(V)$, $[V]$	(equivariant) Euler class of a vector bundle or representation; $[V]$—as a shorthand for $[\{0\} \subset V]$,—is used only when there is no danger of confusion
$N_Z X$	normal bundle (or bundle of normal cones, if X is singular) of Z in X
$\mathfrak{S}_n$	the symmetric group of order n
$V^\vee$	dual representation
$\mathsf{Sym}^k V$	symmetric tensor powers of V
$\wedge^k V$	antisymmetric tensor powers of V
$\mathbb{S}^\lambda V$	Schur functors; $\mathbb{S}^{(k)}V = \mathsf{Sym}^k V$ and $\mathbb{S}^{(1,1,\dots,1)}V = \wedge^k V$
curry	the natural isomorphism $\mathsf{curry} : \mathsf{Hom}(U \otimes V, W) \to \mathsf{Hom}(U, \mathsf{Hom}(V, W))$[4]
$U \odot V$	if $V \le U$, this is the image of $U \otimes V$ under the quotient map $U \otimes U \to \mathsf{Sym}^2 U$; isomorphic to $((U/V) \otimes V) \oplus (\mathsf{Sym}^2 V)$. Similarly for $U \le V$.
$i \odot j$	the dimension analogue of the previous entry: $\mathsf{dim}(U^i \odot V^j) = i \odot j$. If $j \le i$, we have $i \odot j = j(i-j) + j(j+1)/2$
$\mathcal{J}_d(V, W)$	the space of d-jets from $(V, 0)$ to $(W, 0)$; that is, $\mathcal{J}_d(V, W) = \oplus_{k=1}^d \mathsf{Hom}(\mathsf{Sym}^k V, W)$
$\mathcal{J}^\circ_d(V, W)$	jets with injective linear part
$\mathcal{J}(V, W)$	shorthand for $\mathcal{J}_d(V, W)$ where d (possibly infinity) is clear from the context, or not important
$\mathcal{J}_d(n, m)$	shorthand for $\mathcal{J}_d(\mathbb{C}^n, \mathbb{C}^m)$
$\mathcal{J}_d(V)$	shorthand for $\mathcal{J}_d(V, \mathbb{C})$, which is a (nilpotent) ring
$\mathcal{E}_d(V)$	$\mathbb{C} \oplus \mathcal{J}_d(V)$; space of jets of *functions*, a local ring with unit
$\mathsf{Diff}_d(V)$	jets of diffeomorphisms; shorthand for $\mathcal{J}^\circ_d(V, V)$

[3] the equation $s_{(i)} = s_i$ motivates our choice of conventions. $\mathbb{S}^{(k)} = \mathsf{Sym}^k$ is another lucky corollary.

[4] 'currying' is a standard terminology for this in computer science and logic, named after the logician Haskell Brooks Curry.

Chapter 1. First order - The Thom-Porteous formula

The aim of this part of the thesis is to introduce the methods, phenomena and notations on the simplest and most widely studied case, which we denote by Σ^i. Correspondingly, we claim no originality for the results presented in this chapter. See also [**FR06**] for a short overview of different methods.

The sentence

"the map $f : N^n \to M^m$ has Σ^i singularity at the point $x \in N$"

means simply that the rank of the differential $\mathrm{d}_x f$ drops by i, that is (assuming $m \geq n$), $\mathsf{dim}\,\mathsf{ker}(\mathrm{d}_x f) = i$. The cohomological properties of the locus

$$\overline{\Sigma^i(f)} = \{\, x \in N \ :\ \mathsf{dim}\,\mathsf{ker}(\mathrm{d}_x f) \geq i \,\} \tag{1}$$

are widely studied in different contexts, including topology, algebraic geometry, etc. The class[1] $[\Sigma^i(f)]$ in the cohomology ring (also in the Chow ring, etc.) is given by the *Thom-Porteous formula* [**Por71**]:

$$\big[\Sigma^i(f)\big] = s_{(i^{m-n+i})}(f^*TM - TN) = \mathsf{det}[c_{m-n+i+j-k}(f^*TM - TN)]_{k,j\in i\times i} \in H^*(N;\mathbb{Z})$$

assuming some mild transversality conditions (or in the complex analytic case, that the locus has the expected codimension $i(m-n+i)$). Chapter 9 of [**FP98**] lists some applications of this formula in algebraic geometry.

1.1. Existence of the Thom polynomial and stability

Thom was interested in the set (1), and more generally, its analogue for other singularities. He proposed the following theorem:

Theorem 1.1.1 ([**Tho56, HK57**]). *Let N^n and M^m be two smooth (real) manifolds, and Σ be a singularity, that is, a $\mathsf{Diff}(n) \times \mathsf{Diff}(m)$ invariant subvariety of $\mathcal{J}_d(n,m)$. Σ defines a subset (which we also denote by Σ) of the global jet space $\mathcal{J}_d(N,M)$. There exists a polynomial P in two set of variables $c_1, c_2, \ldots$ and $d_1, d_2, \ldots$ such that for a map $f : N \to M$, whose jet is transversal to the singularity subset Σ, the cohomology class $[\Sigma(f)] \in H^*(N;\mathbb{Z}_2)$ of the locus*

$$\Sigma(f) = \{\, x \in N \ :\ \textit{the jet of } f \textit{ at } x \textit{ belongs to } \Sigma \,\}$$

is given by substituting the Stiefel-Whitney classes of TN and f^*TM into the polynomial P:

$$[\Sigma(f)] = P(w_1(TN), w_2(TN), \ldots; f^*w_1(TM), f^*w_2(TM), \ldots)$$

This polynomial is called the Thom polynomial of the singularity.

Remark. The theorem remains true if we replace $\mathbb{R}$ with $\mathbb{C}$, 'smooth' by 'analytic', $\mathbb{Z}_2$ with $\mathbb{Z}$ and Stiefel-Whitney classes with Chern classes. In the following, we will focus on the complex case.

[1] With some abuse of notation, we will always write $[X]$ instead of $[\overline{X}]$, as only closed subvarieties represent classes anyway.

Let us demonstrate this theorem for the Σ^i singularity defined above. The derivative $\mathrm{d}_x f$ of f at $x \in N$ is a linear map in $\mathsf{Hom}(T_xN, T_{f(x)}M)$. Assembling these maps for all $x \in N$, we get a section $\mathrm{d}f$ of the vector bundle $\xi = \mathsf{Hom}(TN, f^*TM)$. The structure group of this bundle is $\mathsf{GL}_n \times \mathsf{GL}_m$; since Σ^i (the set of corank i linear maps) is invariant for the action of this structure group, we can define the "smeared" version $\Sigma^i(TN, f^*TM)$ by

$$\Sigma^i(TN, f^*TM) = \left\{ \ (\varphi, x) \in \mathsf{Hom}(TN, f^*TM) \ : \ x \in N, \ \varphi \in \Sigma^i(T_xN, T_{f(x)}M) \ \right\}$$

Clearly, we have

$$\Sigma^i(f) = (\mathrm{d}f)^{-1}\, \Sigma^i(TN, f^*TM),$$

and if the section $\mathrm{d}f$ is transversal to the stratification given by the $\Sigma^i(TN, f^*TM)$ sets, we also have

$$\big[\Sigma^i(f) \subset N\big] = (\mathrm{d}f)^* \big[\Sigma^i(TN, f^*TM) \subset \mathsf{Hom}(TN, f^*TM)\big].$$

Note that at this point, we could have any two vector bundles A^n, B^m and a (nice, transversal) section $\sigma \in \Gamma\mathsf{Hom}(A, B)$ instead of TN and f^*TM and $\mathrm{d}f$. In particular, we can apply the construction to the *universal* bundles

$$U_n = \mathsf{pr}_1^*\big(\mathbb{C}^n \ \times_{\mathsf{GL}_n} \ E\mathsf{GL}_n\,\big)$$
$$U_m = \mathsf{pr}_2^*\big(\mathbb{C}^m \times_{\mathsf{GL}_m} E\mathsf{GL}_m\big)$$

on the classifying space $B\mathsf{GL}_n \times B\mathsf{GL}_m$; and get a *universal class*

$$[\Sigma^i] = [\Sigma^i \subset \mathsf{Hom}(U_n, U_m)] \in H^*(\mathsf{Hom}(U_n, U_m)) \cong H^*(B\mathsf{GL}_n \times B\mathsf{GL}_m) = H^*_{\mathsf{GL}_n\times\mathsf{GL}_m}(\mathsf{pt}).$$

This construction is compatible with pull-backs, and for any bundles A^n and B^m over a (paracompact) manifold $\mathcal{M}$ there is a map $\Phi : \mathcal{M} \to B\mathsf{GL}_n \times B\mathsf{GL}_m$ such that $\Phi^*U_n = A$ and $\Phi^*U_m = B$; putting these together, we get that $\Sigma^i(A, B) = \Phi^*\Sigma^i(U_n, U_m)$, and thus

$$[\Sigma^i(A, B)] = \Phi^*[\Sigma^i(U_n, U_m)].$$

Finally, let us remark that $H^*(B\mathsf{GL}_n \times B\mathsf{GL}_m)$ is a (graded) polynomial ring, generated by the Stiefel-Whitney (or Chern, in the complex case) classes of U_n and U_m, and the pullback Φ^* is given by substituting the Stiefel-Whitney (Chern) classes of A and B into these generators. Thus the universal class $[\Sigma^i]$ *is* the Thom polynomial P.

Next, let us show that this polynomial can be expressed as a polynomial in the (virtual) Stiefel-Whitney or Chern classes of the (formal) difference bundle $B - A$, which are defined (in the Chern case) by the equation

$$\sum_{k\geq 0} c_k(B - A) \cdot t^k = \frac{\sum_{j=0}^m c_j(B) \cdot t^j}{\sum_{i=0}^n c_i(A) \cdot t^i} \in H^*(\mathcal{M}; \mathbb{Z})[\![t]\!],$$

where t is a formal variable. Observe that if E is a third vector bundle, then for the section $\sigma \otimes \mathsf{id} \in \Gamma\mathsf{Hom}(A \oplus E, B \oplus E)$, defined simply by

$$(\sigma \otimes \mathsf{id})_x(a, e) = (\sigma_x(a), e),$$

we have $\Sigma^i(\sigma) = \Sigma^i(\sigma\otimes\mathsf{id})$ already as a *set* (the transversality conditions are also equivalent). Choosing E to be an orthogonal complement[2] of A in some large trivial bundle $\mathbb{C}^K$, we get

[2]Orthogonal complements do not exist in the category of *holomorphic* vector bundles, but here we are dealing simply with complex vector bundles.

that

$$[\Sigma^i(A,B)] = [\Sigma^i(A\oplus A^\perp, B\oplus A^\perp)] = [\Sigma^i(\mathbb{C}^K, B\oplus A^\perp)],$$

but $c(\mathbb{C}^K) = 1$ and $c(B\oplus A^\perp) = c(B-A)$, from which it follows that $P(c(A),c(B)) = P(1, c(B-A))$. Since this works for any compact manifold $\mathcal{M}$, it must be true for the universal polynomial, too.

Rephrasing in the language of equivariant cohomology, we demonstrated that the Thom polynomial is a universal class (characteristic class), in particular, it is the $\mathsf{GL}_n \times \mathsf{GL}_m$-equivariant cohomology class represented by the closed subvariety $\overline{\Sigma^i} \subset \mathsf{Hom}(A^n, B^m)$, where A and B are the standard GL_n resp. GL_m representations (optionally thought as equivariant vector bundles over the point). Also, this class cannot be arbitrary: It lies in the subring generated by $\{c_k(B-A)\}$.

1.2. Porteous' embedded resolution

The "classical" method for calculating the cohomology class $[\Sigma] \in H^*(X)$ represented by a singular subvariety Σ of a smooth ambient variety X is to find an *embedded resolution* of the pair (X,Σ), that is, smooth varieties $\widetilde{\Sigma} \subset Y$ and a map $\pi : Y \to X$ such that

- $\pi^{-1}(\Sigma) = \widetilde{\Sigma}$,
- $\widetilde{\Sigma}$ is a resolution of Σ,
- and the following diagram commutes:

$$\begin{array}{ccc} \widetilde{\Sigma} & \xrightarrow{\ j\ } & Y \\ \pi\downarrow & & \downarrow\pi \\ \Sigma & \xrightarrow{\ i\ } & X \end{array}$$

In this situation we have

$$[\Sigma \subset X] = i_*[1] = i_*\pi_*[1] = \pi_* j_*[1] = \pi_*[\widetilde{\Sigma} \subset Y]$$

This is useful if it is easy to compute $[\widetilde{\Sigma} \subset Y]$; typically Y will be a vector bundle and $\widetilde{\Sigma}$ a subbundle, in which case $[\widetilde{\Sigma} \subset Y]$ is the Euler class of the quotient bundle. The same construction works in the equivariant setting, where a Lie group G acts on X and Y such that Σ and $\widetilde{\Sigma}$ are invariant and π is equivariant. Y is choosed such that the push-forward map $\pi_* : H^*_G(Y) \to H^*_G(X)$ can be computed, for example using the formulae in A.4.

Porteous' construction [**Por71**] uses a straightforward generalization of the usual blow-up construction in algebraic geometry. Using the notation above, the rôles are played as follows:

$$\begin{array}{rcl} X & : & \mathsf{Hom}(V^n, W^m) \\ \Sigma & : & \overline{\Sigma}^i = \{\, \varphi \in \mathsf{Hom}(V,W) \;:\; \dim\ker(\varphi) \geq i \,\} \\ Y & : & \mathsf{Gr}_i(V) \times \mathsf{Hom}(V,W) \\ \widetilde{\Sigma} & : & \widetilde{\Sigma}^i = \{\, (R,\varphi) \in \mathsf{Gr}_i(V) \times \mathsf{Hom}(V,W) \;:\; R \subset \ker(\varphi) \,\} \\ \pi & : & \mathsf{pr}_2 = \text{projection to the second coordinate} \end{array}$$

and the whole diagram is $\mathsf{GL}(V) \times \mathsf{GL}(W)$-equivariant. In this situtation, $\widetilde{\Sigma}$ is a linear subbundle of Y, and the quotient bundle is $\mathsf{Hom}(R, \pi^*W)$, where $R \subset \pi^*V$ is the tautological rank i vector bundle over $\mathsf{Gr}_i(V)$; thus, using the pushforward formula (38), Theorem A.4.1,

$$\begin{aligned}[\Sigma^i(V,W)] &= \pi_* c_{\mathrm{top}}\big(\mathsf{Hom}(R,\pi^*W)\big)\\ &= \pi_* s_{(i^m)}(\pi^*W - R) = (-1)^{im}\pi_* s_{(m^i)}(R-\pi^*W)\\ &= (-1)^{im+i(n-i)} s_{(m^i-(n-i)^i)}(V-W)\\ &= s_{(i^{m-n+i})}(W-V)\end{aligned}$$

Remark. In this toy case, the weaker version of the pushforward formula (36) would be also sufficient, though resulting in a more involved derivation.

1.3. Equivariant localization

This is a variation and generalization of the previous method, which, in the context of computing Thom polynomials, first appeared in [**BSz06**]. The first idea is that we can use equivariant localization to compute the pushforward, see Corollary A.3.3: This works in the general case, where we may not have such a nice formula as in the case of the Grassmannian; of course, then it may be not easy to evaluate the resulting localization formula either. Second, we may not need a full resolution: Since localization works quite well with singular varieties, a partial resolution is often enough.

The typical situation is that we want to compute the (torus-equivariant) class of an invariant affine variety $Z \subset V$, and we can present the closure $\overline{Z}$ as an union of (infinitely many) linear subspaces; that is, we have an (equivariant) vector bundle $Y \subset \mathcal{M}\times V$ over a compact variety $\mathcal{M}$, such that $\mathsf{pr}_2(Y) = \overline{Z}$. Then we can apply Theorem A.3.7, and localize on $\mathcal{M}$:

$$[Z\subset V]_{\mathbb{T}} = \sum_{p\in\mathcal{M}^{\mathbb{T}}} \frac{[Y_p\subset V]_{\mathbb{T}}}{e_{\mathbb{T}}(T_p\mathcal{M})}$$

assuming (for simplicity) that $\mathcal{M}$ is smooth and has isolated fixed point set $\mathcal{M}^{\mathbb{T}}$. Here Y_p denotes the fiber $\mathsf{pr}_1^{-1}(p)$ over $p\in\mathcal{M}$.

In the case of $Z=\Sigma^i$, the situation is the same as described in the previous section:

$$\begin{aligned} V &= \mathsf{Hom}(V,W)\\ \mathcal{M} &= \mathsf{Gr}_i(V)\\ Y &= \big\{\,(R,\varphi)\in \mathsf{Gr}_i(V)\times\mathsf{Hom}(V,W) \;:\; R\subset \mathsf{ker}(\varphi)\,\big\}\end{aligned}$$

which results in the formula

$$\begin{aligned}[\Sigma^i(V,W)] &= \sum_{I\in\binom{n}{i}} \frac{e_{\mathbb{T}}(\mathsf{Hom}(I,W)}{e_{\mathbb{T}}(\mathsf{Hom}(I,n-I))} =\\ &= \sum_{I\in\binom{n}{i}} \frac{\prod_{l=1}^m\prod_{j\in I}(\theta_l-\alpha_j)}{\prod_{k\notin I}\prod_{j\in I}(\alpha_k-\alpha_j)} \in \mathbb{Z}[\alpha_1,\dots,\alpha_n,\theta_1,\dots,\theta_m]^{\mathfrak{S}_n\times\mathfrak{S}_m}.\end{aligned}$$

1.4. Restriction equations

This method was introduced by Richárd Rimányi, [**Rim01**]; see also [**FR04**]. It is based on the geometry of orbits: It works best when the symmetries are large, and there are only finitely many orbits. This is probably the most efficient method for small cases; on the other hand, its scope is limited. While we don't use this method directly in this thesis, it was the original motivation for Section 4.4.

Let V be a representation of a Lie group G, and X be an orbit; as usual, we want to compute the G-equivariant class $[X]_G$ represented by the closure of X. The basic idea is very simple: Take other G-orbits, and restrict the class $[X]$ to them; if we can compute these restrictions, we get equations on $[X]$, and if we have enough equations, maybe they determine $[X]$ completely.

Lemma 1.4.1. *Let Z be any G-orbit, and denote by j_Z the embedding $j_Z : Z \to V$. Then*

$$j_Z^*[X]_G = \begin{cases} 0 & Z \cap \overline{X} = \emptyset \\ e_G(N_Z V) & Z = X \\ [N_Z\overline{X} \subset N_Z V]_G & Z \subset \overline{X} \end{cases}$$

Note that the third case actually contains the other two. For a sketch of proof, see Lemma A.3.5 in the Appendix.

Remark. There is a different interpretation of this result. Let $p \in Z$ be any point, and $G_p \subset G$ be its stabilizer subgroup. Then $Z \cong G/G_p$,

$$j_Z^* : H_G^*(V) = H_G^*(\mathsf{pt}) \;\to\; H_G^*(Z) = H_{G_p}^*(\mathsf{pt})$$

and j_Z^* is also the map induced by $i_p : G_p \to G$ (we could also take any subgroup $H < G_p$, or more generally, a Lie group morphism $H \to G_p$, and restrict further). This viewpoint gives the following interpretation of the lemma: For $h : H \to G_p$

$$(i_p \circ h)^*[X]_G = \begin{cases} 0 & Z \cap \overline{X} = \emptyset \\ e_H\big((N_Z V)|_p\big) & Z = X \\ \big[(N_Z\overline{X})|_p \subset (N_Z V)|_p\big]_H & Z \subset \overline{X} \end{cases}$$

While the original version is geometrically more natural, this version is effectively computable: We can take H to be a subgroup of G_p so that there exists a H-invariant complementary subspace S to T_pZ in T_pV, then

$$\big[(N_Z\overline{X})|_p \subset (N_Z V)|_p\big]_H = \big[N_p(S \cap \overline{X}) \subset S\big]_H.$$

The basis of the theory is the following theorem:

Theorem 1.4.2 ([**FR04**]). *Suppose there are finitely many G-orbits, and for any orbit Z the Euler class of the normal bundle $e_G(N_Z V)$ is not a zero divisor in $H_G^*(Z)$. Then the following set of equations for the class of a G-orbit X*

$$j_Z^*[X] = \begin{cases} e_G(N_X V) & Z = X & \text{'principal equation'} \\ 0 & Z \neq X,\ \mathsf{codim}(Z) \leq \mathsf{codim}(X) & \text{'homogeneous equations'} \end{cases}$$

has a unique solution.

Let's apply this method to our running example: The group $G = \mathsf{GL}_n \times \mathsf{GL}_m$ acts on the space of matrices $\mathsf{Hom}(\mathbb{C}^n, \mathbb{C}^m) = \mathsf{Mat}_{n\times m}$ from the left[3] by

$$(L, R) \cdot A = LAR^{-1}.$$

Let us suppose for simplicity that $m \geq n$. The classification of the orbits is well-known: They are exactly the rank varieties Σ_k for $0 \leq k \leq n$. For such an orbit Σ_k, we can choose a representative matrix of rank k

$$A_k = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} \in \Sigma_k \subset \mathsf{Mat}_{n\times m}$$

with a $k \times k$ identity matrix on the top-left corner, and zero elsewhere. It is easy to compute the tangent space $T_{A_k}\Sigma_k$ of the orbit Σ_k at the A_k, by just applying (a basis of) the Lie algebra of infinitesimal actions $\mathfrak{g} = \mathfrak{gl}_n \times \mathfrak{gl}_m$ to A_k; it turns out that tangent space is

$$T_{A_k}\Sigma_k = \left\{ \begin{bmatrix} * & * \\ * & 0 \end{bmatrix}, \text{ where } * \text{ is anything} \right\} < T_{A_k}\mathsf{Mat}_{n\times m} \cong \mathsf{Mat}_{n\times m}$$

As a side effect of this computation, we get the codimension formula

$$\mathsf{codim}(\Sigma_k) = (n-k)(m-k).$$

A reasonably big subgroup (meaning that it is homotopy equivalent to it) H_k of the stabilizer of A_k is

$$H_k = \left\{ \left(\begin{bmatrix} C & 0 \\ 0 & A \end{bmatrix}, \begin{bmatrix} C & 0 \\ 0 & B \end{bmatrix} \right) \; : \; C \in \mathsf{GL}_k,\ A \in \mathsf{GL}_{n-k},\ B \in \mathsf{GL}_{m-k} \right\}.$$

To do the computation however, it is better to restrict ourselves to the maximal tori. Let us denote by $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_m$ the generators of $H^*_{\mathbb{T}^n\times\mathbb{T}^m}(\mathsf{pt})$; and by $\widehat{\gamma}_1, \ldots, \widehat{\gamma}_k$, $\widehat{\alpha}_{k+1}, \ldots, \widehat{\alpha}_n$ and $\widehat{\beta}_{k+1}, \ldots, \widehat{\beta}_m$ the corresponding generators of the the maximal torus $\mathbb{T}_k$ of H_k. Then the restriction map $j_k^* : H^*_G(\mathsf{pt}) \to H^*_{H_k}(\mathsf{pt})$ is given by

$$\alpha_i \mapsto \begin{cases} \widehat{\gamma}_i & i \leq k \\ \widehat{\alpha}_i & i > k \end{cases} \qquad \text{and} \qquad \beta_i \mapsto \begin{cases} \widehat{\gamma}_i & i \leq k \\ \widehat{\beta}_i & i > k \end{cases}$$

A H_k-invariant normal space to Σ_k at A_k is

$$S_k = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix}, \text{ where } * \text{ is anything} \right\} < T_{A_k}\mathsf{Mat}_{n\times m}$$

with the Euler class

$$e_{\mathbb{T}_k}(S_k) = \prod_{i=k+1}^{n} \prod_{j=k+1}^{m} (\widehat{\beta}_j - \widehat{\alpha}_i).$$

Gathering all this gives us a bunch of *linear* equations for the coefficients of the Thom polynomial (expressed as a polynomial in *two* set variables, one for GL_n and one for GL_m), which we can then solve using a computer algebra software, for example. In fact, in this concrete case the 'principal equation' imply all the others, since clearly the stabilizer of Σ_k contains the stabilizers of Σ_i for all $i < k$. However, it is not easy in general to derive a *formula* which works for any k, n and m; this approach is algorithmic in nature. Of course, when we can guess the result, it is possible to prove it (in this case, for example by using Sylvester's determinantal formula for the resultant); we omit this last step here.

[3]The usual convention to write linear maps as matrices is the transpose of what we use here; by that convention, GL_n would be the *right* group, etc.

1.5. Gröbner degeneration

For the sake of completeness, we have to mention the method of Gröbner degeneration, which is well known among algebraic geometers. The basic fact here is that for an ideal $I \lhd \mathbb{C}[V]$ in a polynomial ring, there is a flat deformation to its *initial ideal* (see eg. [**Eis95**], Section 15.8). Many properties are invariant under flat deformation, in particular, the cohomology class, too (in the equivariant case, of course we need an invariant deformation). Since there are algorithms to compute the Gröbner basis, and thus the initial ideal, this gives us an algorithm to compute the (torus-equivariant) cohomology class represented by a (torus-invariant) affine variety given its ideal, since the geometry corresponding to the initial ideal is just a bunch of coordinate subspaces with multiplicities.

For a very simple example, consider the plane $\mathbb{C}^2$ with the linear action of the multiplicative group $U = \mathbb{C}^\times\langle\omega\rangle$ defined by

$$\omega \cdot (x, y) = (\omega^2 x, \omega^3 y),$$

with weights $(2\alpha, 3\alpha)$. The subvariety Z defined by the equation $y^2 = x^3$ is invariant to this action; let us compute it equivariant class $[Z] \in H^2_U(\mathbb{C}^2) = \mathbb{Z}[\alpha]$. For this, consider the following two U-invariant one-parameter deformations:

$$\begin{aligned} Z_s &= \{y^2 = sx^3\} \subset \mathbb{C}^2 \quad \text{and} \\ Z'_t &= \{ty^2 = x^3\} \subset \mathbb{C}^2, \quad s, t \in \mathbb{C}. \end{aligned}$$

We have $Z_1 = Z'_1 = Z$, and since both deformations are actually flat, $[Z] = [Z_s] = [Z'_t]$; but Z_0 is simply the line $\{y = 0\}$ with multiplicity 2, and Z'_0 is the other coordinate line $\{x = 0\}$, but with multiplicity 3. Thus both give the result $[Z] = 6\alpha$. In general, instead of finding explicit flat deformations, we can just compute the initial ideal using a Gröbner basis algorithm.

However, in the situations studied in this thesis, more often that not we have no idea about the ideals, and even if we knew them, the resulting Gröbner basis computations would be too big (even for computers). Nevertheless, this method can be useful for sub-computations; for example if we want to localize using Theorem A.3.7 over a base which is a singular toric variety, we could in principle compute the virtual tangent Euler classes using Gröbner bases.

Chapter 2. Primer on singularity theory

Singularities, in our context, are types of local behaviour of (smooth or holomorphic) maps. We briefly collect the necessary definitions and facts here while referring to the literature ([**AVGL98**] and the references therein) for the details, as our focus is on the computations.

2.1. Singularities

Probably the most natural definition is to consider germs of maps up to reparameterization of the source and the target: The "left-right" group

$$\mathcal{A} = \mathsf{Diff}(\mathbb{C}^n, 0) \times \mathsf{Diff}(\mathbb{C}^m, 0)$$

acts on the space of holomorphic germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^m, 0)$. The equivalence classes (orbits) are called *left-right singularity classes*. An analogous definition can be given for smooth real germs. A map $f : M \to N$ between manifolds has singularity type η at a point $x \in M$ if the germ of f at x in some (and thus, in any) local coordinate system belongs to the $\mathcal{A}$-orbit of η. A singularity η is *stable* if for any map f having η singularity at x, and any small perturbation f', there exist an x' close to x having the same singularity (in words: the singularity cannot be eliminated by a small perturbation).

While $\mathcal{A}$-equivalence is a certainly a natural notion, a better behaved classification is the so-called *contact equivalence* or $\mathcal{K}$-equivalence, introduced by John Mather. Two germs f and f' are contact equivalent if there is a diffeomorphism germ $h \in \mathsf{Diff}(\mathbb{C}^n)$ and a map germ $\varphi \in \mathcal{J}(\mathbb{C}^n, \mathsf{GL}_m)$ such that

$$f'(x) = \varphi(x) f(h(x)).$$

This can be also thought as a group action: the group $\mathcal{K}$ is

$$\mathcal{K} = \mathsf{Diff}(\mathbb{C}^n) \times \mathcal{J}(\mathbb{C}^n, \mathsf{GL}_m)$$

is acting (from the left) on $\mathcal{J}(n, m)$ by

$$((h, \varphi) f)(x) = \varphi(x) f(h^{-1}(x)).$$

We will need some fundamental definitions.

Definition 2.1.1. The *ideal of a singularity* $f = (f_1, \ldots, f_m) \in \mathcal{J}(n, m)$ is the ideal the generated by the component functions

$$\mathcal{I}_f = (f_1, \ldots, f_m) \lhd \mathcal{E}(n)$$

where $\mathcal{E}(n) = \mathbb{C}[\![x_1, \ldots, x_n]\!]$ is the ring of formal power series on $\mathbb{C}^n$ (similarly for truncated polynomial rings and other function rings). Clearly $\mathcal{I}_f$ is also an ideal in $\mathcal{J}(n) \subset \mathcal{E}(n)$. The *local algebra* of the singularity f is the quotient $\mathcal{E}(n)/\mathcal{I}_f$; we will call the nilpotent quotient

$$\mathcal{Q}_f = \mathcal{J}(n)/\mathcal{I}_f$$

the *quotient algebra*. The dimension of the local algebra is called the *algebraic multiplicity* of the singularity; however, what actually is the important object for us is the quotient algebra and its dimension, which we will denote by μ:

$$\mu_f = \mathsf{dim}(\mathcal{Q}_f) = \mathsf{dim}(\mathcal{E}(n)/\mathcal{I}_f) - 1.$$

Remark. The difference between $\mathcal{E}(n)$ and $\mathcal{J}(n)$ is that the former ring has a unit, while the latter is nilpotent: $\mathcal{E}(n) = \mathbb{C} \oplus \mathcal{J}(n)$. Actually $\mathcal{J}(n)$ is the unique maximal ideal in $\mathcal{E}(n)$. There are versions of our main objects for both rings. The singularity theory literature usually works with the ring of functions (or power series) and the local algebra, however for us it is more natural to work with the nilpotent objects. Note that the literature somtimes use the symbols $\mathcal{Q}$ and μ for the local algebra and the algebraic multiplicity; but it would be very inconvenient for us to follow this convention.

The most important results for us are the following:

Theorem 2.1.2 (Mather). *$\mathcal{K}$-equivalent stable germs are $\mathcal{A}$-equivalent.*

This shows that $\mathcal{K}$-equivalence is a reasonably natural object.

Theorem 2.1.3 (Mather). *Two map germs are $\mathcal{K}$-equivalent if and only if their ideals are taken into each other by a map induced by a germ of diffeomorphism in* $\mathsf{Diff}(n)$.

Corollary 2.1.4. *Two finitely determined map germs are $\mathcal{K}$-equivalent if and only if their local algebras (or equivalently, their quotient algebras) are isomorphic.*

Remark. In this thesis, we are only dealing with *finitely determined* singularities: These are the singularities for which it is possible to determine for any jet whether it belongs to the given singularity by looking at only finitely many (depending on the singularity) derivatives. In this case, it is possible to truncate our rings at a given order; thus we can work with finite dimensional objects. In our viewpoint, this is not a real restriction.

2.2. Thom-Boardman classes

Since the complete classification of (say, contact) singularities is hopeless, it is clearly useful to have more coarse but better behaved classification schemes. The Thom-Boardman classification, introduced by Thom [**Tho56**] and clarified by Boardman [**Boa67**] (see also [**Mat73**]), is probably the most well-known and useful such scheme. It has the clear advantage that the classes are indexed by discrete objects, namely, partitions (non-increasing finite sequences of integers).

Definition 2.2.1 (Thom). For a (nice enough) map $f : N \to M$, define the locus

$$\Sigma^i(f) = \big\{\, x \in N \;:\; \mathsf{dim}(\mathsf{ker}(\mathrm{d}_x f)) = i \,\big\}.$$

Suppose that $\Sigma^i(f) \subset N$ is smooth; then we can define $\Sigma^{ij}(f)$ to be $\Sigma^j(f|_{\Sigma^i f})$, and similarly, for any index set $I = \{i_1, \ldots, i_{k-1}\}$, let

$$\Sigma^{I,i_k}(f) = \Sigma^{i_k}(f|_{\Sigma^I f}).$$

This definition is intuitive enough, but in the definition of $\Sigma^{i_1 i_2 \cdots i_k}$ we have to assume that the loci Σ^{i_1}, $\Sigma^{i_1 i_2}$, etc. are all smooth. Boardman gave a definition which cures this problem, but is much less intuitive.

Definition 2.2.2. Let $U \lhd \mathcal{E}(n) = \{f : \mathbb{C}^n \to \mathbb{C}\}$ be an ideal of functions (or formal power series, etc.; and also similarly for the real case). The kth *Jacobian extension* $\Delta_k(U)$ is the ideal generated by U and all the $k \times k$ determinants $\mathsf{det}[\partial\varphi_i/\partial x_j]$, where x_j is a (fixed) local coordinate system and $\varphi_i \in U$. It will be convenient to also define $\Delta^k(U) = \Delta_{n-k+1}(U)$.

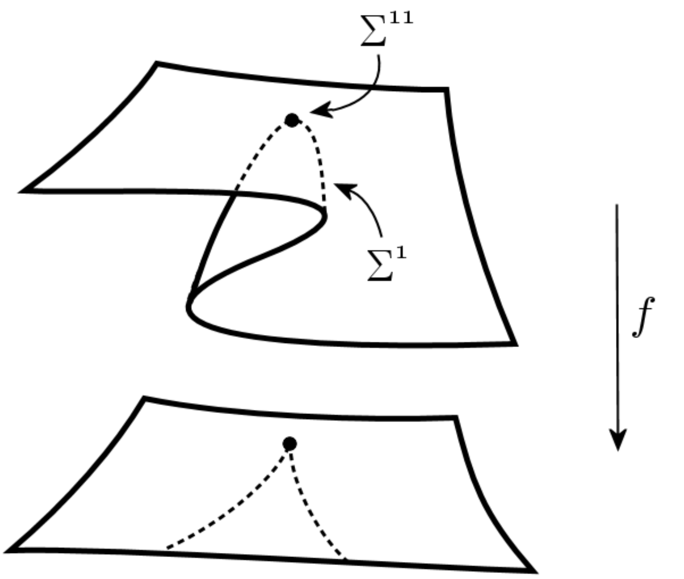


FIGURE 1. A Σ^{11} singularity; $f : \mathbb{R}^2 \to \mathbb{R}^2$ is the vertical projection.

DEFINITION 2.2.3. It is easy to see that

$$U = \Delta^0(U) \subseteq \cdots \subseteq \Delta^k(U) \subseteq \Delta^{k+1}(U) \subseteq \cdots \subseteq \Delta^{n+1}(U) = \mathcal{E}(n).$$

The largest Δ^k such that $\Delta^k(U) \subsetneq \mathcal{E}(n)$ is called the *critical Jacobian extension.*

REMARK. The critical extension of an ideal U is $\Delta^{n-r} = \Delta_{r+1}$, where $r = \mathsf{rank}\, U$ is the *rank* of the ideal U, which is defined as $\mathsf{rank}\, U = \mathsf{dim}_{\mathbb{C}}\, (\mathfrak{m}^2 + U)/\mathfrak{m}^2$.

DEFINITION 2.2.4 (Boardman). The germ $f = (f_1, \ldots, f_m)$, $f(0) = 0$ belongs to Σ^I if the ideal $U = (f_1, \ldots, f_m) \lhd \mathcal{E}(n)$ has successive critical extensions

$$\Delta^{i_1}U, \quad \Delta^{i_2}\Delta^{i_1}U, \quad \Delta^{i_3}\Delta^{i_2}\Delta^{i_1}U, \quad \ldots$$

For a map $g : N \to M$ between manifolds, take a point $x \in N$ and let $(f_1, \ldots, f_m)$ be the coordinate functions of g in some local coordinate system around x and $g(x)$; the definition then tells us the Boardman type of g at the point x.

Boardman proved that the singularity subsets defined this way are smooth submanifolds of the appropriate jet spaces, and that they coincide with Thom's definition when the latter applies, by which we mean that $\Sigma^I_{\text{Thom}}(f) = (\mathcal{J}f)^{-1}(\Sigma^I_{\text{Boardman}})$.

Porteous proposed a third definition in [**Por83**], based on his theory of *intrinsic derivatives* (see Section 4.3.1).

REMARK. Thom-Boardman singularities of order d are d-determined, that is, it's enough to look at the first d differentials of a map to decide whether it belongs to the given Thom-Boardman class (this should be clear from Boardman's definition). They are also stable in the sense that if $f : \mathbb{C}^n \to \mathbb{C}^m$ belongs to Σ^I, then so does $f \oplus \mathsf{id}_{\mathbb{C}} : \mathbb{C}^{n+1} \to \mathbb{C}^{m+1}$.

2.3. Thom polynomials

Singularities describe *local* behaviour of maps, however, Thom polynomials are *global* invariants of singularities, describing (in cohomological terms) the "shape" of the singular locus; they are an instance of the local-global principle.

Let us recall Theorem 1.1.1 from Chapter 1:

Theorem 2.3.1 ([**Tho56, HK57**]). *Let N^n and M^m be two smooth, real (resp. complex analytic) manifolds, and Σ be a singularity, that is, a $\mathsf{Diff}_n \times \mathsf{Diff}_m$ invariant subvariety of $\mathcal{J}_d(n,m)$. Σ defines a subset (which we also denote by Σ) of the global jet space $\mathcal{J}_d(N,M)$. There exists a universal polynomial P in two set of variables $c_1,\dots,c_n$ and $d_1,\dots,d_m$, depending only on n, m and Σ, such that for a map $f : N \to M$ whose jet is transversal to the singularity subset Σ, the cohomology class $[\Sigma(f)] \in H^{\mathsf{codim}(\Sigma)}(N;\mathbb{Z}_2)$ (resp. $H^{2\mathsf{codim}(\Sigma)}(N;\mathbb{Z})$) of the locus*

$$\Sigma(f) = \left\{\ x \in N\ :\ \textit{the jet of } f \textit{ at } x \textit{ belongs to } \Sigma\ \right\}$$

is given by substituting the Stiefel-Whitney (resp. Chern) classes of TN and f^*TM into the polynomial P:

$$[\Sigma(f)] = P(w_1(TN), w_2(TN), \dots; f^*w_1(TM), f^*w_2(TM), \dots) \tag{2}$$

This polynomial is called the Thom polynomial of the singularity.

Sketch of proof. Consider the universal jet bundle $\mathcal{J}_U \to B_U$, and the $\mathsf{Diff}_n \times \mathsf{Diff}_m$-equivariant cohomology class of the corresponding singularity set $\Sigma_U \subset \mathcal{J}_U$. For any concrete case $f : N \to M$, we can pull back from the universal case along the classifying map $\Phi : N \to B_U$:

$$[\Sigma \subset \mathcal{J}(N,M)] = [\Phi^{-1}(\Sigma_U) \subset \mathcal{J}(N,M)] = \Phi^*[\Sigma_U \subset \mathcal{J}_U];$$

furthermore, if f is transversal to Σ, then

$$[\Sigma(f) \subset N] = [(\mathcal{J}f)^{-1}(\Sigma) \subset N] = (\mathcal{J}f)^*[\Sigma \subset \mathcal{J}(N,M)] = (\mathcal{J}f)^*\Phi^*[\Sigma_U \subset \mathcal{J}_U].$$

Since $\mathsf{Diff}_n \times \mathsf{Diff}_m$ is homotopy equivalent to $\mathsf{GL}_n \times \mathsf{GL}_m$, the cohomology ring $H^*(\mathcal{J}_U) = H^*(B_U)$ in the universal situation will be a polynomial ring, and the pullback Φ^* is given by substituting the appropriate characteristic classes; consequently $P = [\Sigma_U \subset \mathcal{J}_U]$. Finally, $(\mathcal{J}f)^*$ is simply an isomorphism. ☐

Remark. In the real case, the set of maps which are transversal to a given singularity are dense and open among all smooth maps; thus (2) is satisfied for almost any map f, and even if f is "bad", we can approximate it with "nice" maps to arbitrary precision. In the complex case, this is no longer true. However, complex analytic maps are rigid, and we expect the formula to hold if the locus $\Sigma(f)$ is a subvariety with the expected dimension (cf. [**Ful98**]).

In fact, the polynomial P cannot be arbitrary:

Theorem 2.3.2 ([**Dam72**]). *For contact singularities, the polynomial P can be written as*

$$P(c_1, c_2, \dots, c_n; d_1, d_2, \dots, d_m) = Q(h_1, h_2, h_3, \dots)$$

where Q is again a polynomial, and h_i is defined by the following identity of formal power series:

$$1 + \sum_{k=1}^{\infty} h_k t^k = \frac{1 + \sum_{j=1}^{m} d_j t^j}{1 + \sum_{i=1}^{n} c_i t^i}.$$

The interpretation of h_i in the above theorem is that they are the Chern classes of the 'virtual difference bundle' $f^*TM \ominus TN$.

Remark. This theorem also holds for the Thom-Boardman classes.

Corollary 2.3.3. *The polynomials P depend only on the relative codimension $r = m - n$, in the following sense:*

$$P_{n,m}(c_1, \dots, c_n; d_1, \dots, d_n) = P_{n+1,m+1}(c_1, \dots, c_n, 0; d_1, \dots, d_n, 0).$$

Furthermore, the sequence $k \mapsto P_{n+k,m+k}$ eventually stabilizes.

We will call both P and Q *the Thom polynomial of* Σ (though we are more interested in computing Q), and use the notation $\mathsf{Tp}_\Sigma(n, m)$ or just Tp_Σ for them. We will frequently write Q as a linear combination of Schur polynomials (cf. Appendix A.2):

$$Q_r(h_1, h_2, \dots) = \sum_\lambda e_r^\lambda \cdot s_\lambda(h_1, h_2, \dots);$$

one observation motivating such a rendition is that the coefficients e^λ are *nonnegative* integers (this was recently proven in [**PW07a, PW07b**], motivated by numerical evidence).

Another such observation is that the coefficients (when they appear) do not actually depend on the dimensions n and m at all; more precisely

$$e_r^\lambda = e_{r+1}^{(\mu,\lambda)}$$

where $\mu = \mu(\Sigma) \in \mathbb{N}$ is the dimension of the quotient algebra of the singularity. This means that if we "shift" the Thom polynomials by $-r$, they fit into an infinite series, called the *Thom series* of the singularity; see [**FR07**], and Sections 3.1, 3.2 for the details.

2.3.1. Known Thom polynomials. To place our results into a context, we tried to collect the list of previously known Thom polynomials here. We will (ab)use the notation $\Sigma(r)$ for the Thom polynomial of Σ in relative codimension r.

- $\Sigma^i(r)$ was calculated by Porteous [**Por71**] (but was already known to Giambelli).
- $\Sigma^{ij}(r)$: A pushforward formula was given by Ronga [**Ron72**]. Some concrete cases, eg. $\Sigma^{2,1}(0)$ and $\Sigma^{2,2}(-1)$ were computed. A simpler version (and a computer program) was given by Kazarian [**Kaz06**].
- $A_2(r)$ was computed by Ronga as a special case of Σ^{ij}.
- $A_4(0)$ was computed by Gaffney [**Gaf83**].
- $A_{\leq 8}(0)$, $A_{\leq 4}(1)$, $I_{a,b}(0)$ for $a + b \leq 8$ and some other examples were computed by Rimányi [**Rim01**], using the restriction equations method.
- $A_3(r)$ was computed in [**BFR02**], [**LP09**].
- $I_{2,2}(r)$: Kazarian gave a pushforward formula; the Thom series was computed in [**FR07, FR08**], [**Pra07**].
- $A_{\leq 6}(r)$: An iterated residue formula was given in [**BSz06**].
- For $I_{2,3}(r)$, $III_{a,b}(r)$, $a + b \leq 6$, $A_{\leq 4}(r)$, $\Sigma^{21}(r)$, and some other cases, localization formulae were derived in [**FR08**], via "extrapolation" from previous results for small r-s. They also computed the coefficients for an unnamed family which includes $I_{2,2}(r)$ and $III_{2,3}(r)$.
- We computed $\Sigma^{i1}(r)$ and $\Sigma^{ij}(-i+1)$ in [**FK06**].

Chapter 3. Localization of Thom polynomials

In this chapter we study the general properties of localization formulae for singularities, which first appeared in [**BSz06**] in the context of A_d singularities, and were then generalized in [**FR08**]. While the localization principle is very powerful in the sense that we can write down formulae in cases which are not accessible to other methods, the resulting formulae are notoriously hard to evaluate, since the terms are *rational functions* instead of polynomials. Summing rational functions in many variables with large denominators is pretty much impossible even using computers; while bringing the terms to a common denominator, the number of temporary terms suffer an exponential explosion, quickly exhausting the memory of the computer. This happens for relatively small examples already: For example suppose that the denominators are products of binoms; today's personal computers cannot handle the case when the number of different factors (binoms) is about 30 or more.

However, Thom polynomials of singularities have some special properties (as opposed to general systems of polynomials in two sets of variables); in particular, we know *a priori* that they depend only on the (Chern classes of the) formal difference bundle $c(f^*TM \ominus TN)$; and we can exploit this fact to remedy the situation described above. It turns out that following this program leads quite naturally into the world of basic hypergeometric series; localization formulae for singularities become q-hypergeometric identities.

Remark. Here we will work within the theory of contact singularities; while Thom-Boardman classes are not, in general, contact classes, everything holds for them too (see [**Mat73**]), as it is also easy to check in each concrete situation we will deal with in the thesis.

3.1. Localization for contact singularities

We present the basic ideas of [**FR08**], which give us insight into the structure of the (Thom polynomials) of contact singularities.

Recall the following notations:

$F = (f_1, f_2, \ldots, f_m) \in \mathcal{J}(n, m)$	the jet of the singularity
$\mathcal{I}_F = (f_1, f_2, \ldots, f_m) \lhd \mathcal{J}(n)$	the ideal of the singularity
$\mathcal{Q}_F = \mathcal{J}(n)/\mathcal{I}_F$	the quotient algebra
$\mu_F = \mathsf{dim}_{\mathbb{C}}(\mathcal{Q}_F)$	the algebraic multiplicity, shifted by -1
$k_{\mathcal{Q}} = \mathsf{corank}(\mathcal{I}_F)$	minimal number of algebra generators of $\mathcal{Q}$
$\mathsf{rank}(\mathcal{I}_F) = \mathsf{dim}_{\mathbb{C}}(\mathfrak{m}^2 + \mathcal{I}_F)/\mathfrak{m}^2$	rank of the ideal

The basic construction is the following: For a (contact) singularity class $Z \subset \mathcal{J}_d(n, m)$ we want to find a vector bundle $E \to \mathcal{M}$ which is an embedded partial resolution of (the closure of) Z; we can then use equivariant localization on a compactification $\bar{\mathcal{M}}$ to compute $[Z]$ (the localization basically computes a pushforward). We will see that there exists a very natural partial resolution satisfying our needs; that construction actually dates back to the seventies (Damon, Mather).

Consider the map $p : Z \to \mathsf{Hilb}^\mu(\mathcal{J}_d(n)) \subset \mathsf{Gr}^\mu(\mathcal{J}_d(n))$, where $\mathsf{Hilb}^\mu(\mathcal{J}_d(n))$ is the (reduced) Hilbert scheme of ideals of codimension μ in $\mathcal{J}_d(n)$, defined by mapping a jet F into its ideal $\mathcal{I}_F$.

PROPOSITION 3.1.1 ([**FR08**], Lemma 4.3). *For any ideal $\mathcal{I} \lhd \mathcal{J}_d(n)$, the closure of $p^{-1}(\mathcal{I})$ is*

$$\overline{p^{-1}(\mathcal{I})} = \mathcal{I} \otimes \mathbb{C}^m \ \subset\ \mathcal{J}_d(n,m) = \mathcal{J}_d(n) \otimes \mathbb{C}^m.$$

Proof. Clearly $p^{-1}(\mathcal{I}) \subset \mathcal{I} \otimes \mathbb{C}^m$. On the other hand it is open in $\mathcal{I} \otimes \mathbb{C}^m$: Suppose $f_1, \dots, f_m$ generates $\mathcal{I}$, and $b_1, \dots, b_m \in \mathcal{I}$ are arbitrary elements of the ideal; we want to show that

$$f_1 + \varepsilon b_1, \dots, f_m + \varepsilon b_m$$

also generates $\mathcal{I}$ if $\varepsilon \in \mathbb{C}$ is small enough. For this, write $b_i = \sum r_{ij} f_j$ for some $r_{ij} \in R = \mathcal{J}_d(n)$ and take any $c = \sum s_i f_i \in \mathcal{I}$ ($s_i \in R$). We have to present c as $c = \sum t_i(f_i + \varepsilon b_i)$ for some $t_i \in R$. But

$$\sum_i t_i(f_i + \varepsilon b_i) = \sum_i t_i\Big(f_i + \varepsilon \sum_j r_{ij} f_j\Big) = \sum_j f_j \Big(\sum_i t_i(\delta_{ij} + \varepsilon r_{ij})\Big);$$

thus we have to solve the system of equations

$$s_j = \sum_i t_i(\delta_{ij} + \varepsilon r_{ij}), \qquad j \in \{1, \dots, m\}$$

for t_i, but the coefficient matrix $[\delta_{ij} + \varepsilon r_{ij}]$ is clearly invertible for $|\varepsilon|$ small enough.

Actually, the same reasoning proves the Zariski-openness: The locus where the coefficient matrix $[\delta_{ij} + r_{ij}]$ is not invertible is closed. □

Basically the object $\mathcal{M} = p(Z) \subset \mathsf{Hilb}^\mu$ encodes everything about the singularity class Z; and unlike Z, at least for large enough m it is independent of m, which shows that the parameter m is "not that important" in this theory.

The following $\mathsf{GL}_n \times \mathsf{GL}_m$-equivariant diagram summarizes the situation:

$$\begin{array}{ccccccccc} E & \to & \bar{E} & \to & \mathcal{I} \otimes \mathbb{C}^m & \to & R \otimes \mathbb{C}^m & \xrightarrow{\pi} & \mathcal{J}_d(n,m) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{M} & \subset & \bar{\mathcal{M}} & \subset & \mathsf{Hilb}^\mu \mathcal{J}_d(n) & \subset & \mathsf{Gr}^\mu \mathcal{J}_d(n) & \xrightarrow{\pi} & \mathsf{pt} \end{array}$$

where R and $\mathcal{I}$ denotes the tautological codimension μ bundles over Gr^μ and Hilb^μ, respectively; $Z = \pi(E)$ and $\bar{Z} = \pi(\bar{E})$. The group GL_m acts on the bottom row trivially. We can restrict the action to the maximal torus $\mathbb{T} = \mathbb{T}^n \times \mathbb{T}^n \subset \mathsf{GL}_n \times \mathsf{GL}_m$, and apply Theorem A.3.7 to compute $[Z]$ by localizing on $\bar{\mathcal{M}}$:

$$[Z] = \sum_{I \in \mathsf{Fix}} [I \otimes \mathbb{C}^m \subset \mathcal{J}_d(n,m)] \cdot \frac{[N_I \bar{\mathcal{M}} \subset T_I \mathsf{Gr}]}{e(T_I \mathsf{Gr})} \tag{3}$$

where Fix is the set of torus-fixed ideals in $\bar{\mathcal{M}}$, and $N_I \bar{\mathcal{M}}$ is the tangent cone of $\bar{\mathcal{M}}$ at I. The quotient can be thought as the (inverse) "tangent Euler class" at the possibly singular point I (also called equivariant multiplicity).

To move forward, we have to understand the (tangent cones of the) fixed points of $\bar{\mathcal{M}} \subset \mathsf{Hilb}^\mu(\mathcal{J}_d(n))$. The fixed points of Hilb are easy to list: they are just the *monomial ideals*, that is, ideals generated by monomials. A good way to visualise them is to consider the case $n = 2$: Then the monomial ideals of codimension μ are in bijection with the partitions

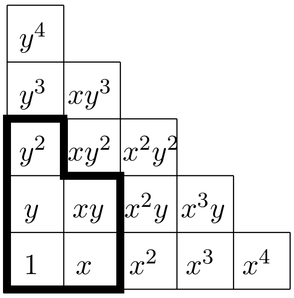


FIGURE 2. The codimension 4 monomial ideal $(y^3, xy^2, x^2) \lhd \mathcal{J}_4(2)$

of weight $(\mu+1)$, see Figure 2. In the $n > 2$ case, monomial ideals correspond to "higher-dimensional partitions" (eg. for $n = 3$, the so called *plane partitions*). We can now write (3) as

$$(4) \qquad [Z] = \mathsf{Tp}(\alpha,\theta) = \mathsf{Tp}(\alpha_1,\dots,\alpha_n;\theta_1,\dots,\theta_m) = \sum_{I\in\mathsf{Fix}} \frac{\prod_{j=1}^m \prod_{i=1}^\mu \left(\theta_j - w_i^{\mathcal{Q}}(\alpha)\right)}{E_I(\alpha)}$$

where I runs over the monomial ideals of codimension μ in $\mathcal{J}(n)$; α_i and θ_j are (-1 times) the weights of the tori $\mathbb{T}^n$ and $\mathbb{T}^m$, respectively; $w_i^{\mathcal{Q}}$ are the weights of the quotient algebra $\mathcal{Q} = \mathcal{J}(n)/I$; and $E_I(\alpha) = \frac{e(T_I\mathsf{Gr})}{[N_I\bar{\mathcal{M}}\subset T_I\mathsf{Gr}]}$ is the "virtual" tangent Euler class of $\bar{\mathcal{M}}$ at I (which is ∞ if $I \notin \bar{\mathcal{M}}$, thus those terms do not contribute to the sum). Note that the tangent Euler class is a rational function in the variables α_i.

It is very important to understand what happens when we increase n. Let us denote the "operation" $n \mapsto n+1$ by $\#$, motivated by the musical notation. Then clearly

$$\begin{aligned} &\mathsf{Fix} \subset \#\mathsf{Fix} \\ &\#I = (I, x_{n+1}) = I \oplus (x_{n+1}\cdot\mathcal{E}_d(n+1)) \lhd \mathcal{J}_d(n+1) \\ &\#\mathcal{Q} = \mathcal{Q} \end{aligned}$$

The important fact here is

LEMMA 3.1.2 ([**FR08**], Lemma 8.2).

$$\#E_I = E_{\#I} = E_I \prod_{i=1}^{\mu} (\alpha_{n+1} - w_i^{\mathcal{Q}}).$$

In an earlier version of [**FR08**] this lemma was proved somewhat indirectly, and was noted that "it would be interesting to find a direct proof of it". Here we give such a direct, geometric proof. In the latest version of that paper, a sketch of a similar proof appeared.

Proof. Consider any ideal $I \lhd \mathcal{J}(n)$ of codimension μ. First, we will show how the tangent spaces of the orbits $\mathcal{O} = \mathsf{Diff}(n)\cdot I$ and $\#\mathcal{O} = \mathsf{Diff}(n{+}1)\cdot(\#I)$ relate to each other. In general, if we have a Lie group action, it is relatively easy to compute the tangent space of an orbit at a point, by applying the infinitesimal action of the correspoding Lie algebra to the point.

The Lie algebra $\mathfrak{diff}(n{+}1)$ is generated by the infinitesimal actions

$$\begin{aligned} x_i &\mapsto x_i + \varepsilon x_j \\ x_i &\mapsto x_i + \varepsilon x_j x_k \\ x_i &\mapsto x_i + \varepsilon x_j x_k x_l \\ &\vdots \end{aligned}$$

Clearly $T_I\mathcal{O} \subset T_{\#I}(\#\mathcal{O})$, so what we really want to calculate is the factor

$$\begin{aligned} T_{\#I}(\#\mathcal{O}) \,/\, T_I\mathcal{O} \;\le\; & T_{\#I}\mathsf{Gr}^{\mu}\mathcal{J}(n{+}1) \,/\, T_I\mathsf{Gr}^{\mu}\mathcal{J}(n) \;= \\ =\; & \mathsf{Hom}(\#I, \mathcal{Q}) \,/\, \mathsf{Hom}(I, \mathcal{Q}) \;=\; \mathsf{Hom}(x_{n+1}\mathcal{E}(n{+}1), \mathcal{Q}) \end{aligned}$$

which we can do by computing the action of the factor $\mathfrak{diff}(n{+}1)/\mathfrak{diff}(n)$; this is generated by (the classes of) two types of infinitesimal transformations

(a) $\qquad x_i \mapsto x_i + \varepsilon(x_{k_1} \cdots x_{k_e})$

(b) $\qquad x_{n+1} \mapsto x_{n+1} + \varepsilon(x_{j_1} \cdots x_{j_e})$

where $1 \le e \le d$ and $(n+1) \in \{k_i\}$. Applying such an infinitesimal action gives us a linear map in $\mathsf{Hom}(\#I, \mathcal{J}(n{+}1))$ (by taking the derivative wrt. ε at $\varepsilon = 0$ for all $v \in \#I$), and we can get a tangent vector in $T_{\#I}\mathsf{Gr}^{\mu}\mathcal{J}(n{+}1)$ via the natural factor map

$$\mathsf{Hom}(\#I, \mathcal{J}(n{+}1)) \to \mathsf{Hom}(\#I, \mathcal{Q}).$$

It is easy to see that only case (b) can lead to a nonzero tangent vector, which identifies $T_{\#I}(\#\mathcal{O})$ as a product

$$\begin{array}{ccccc} \mathsf{Hom}(I, \mathcal{Q}) & \times & \mathcal{Q} & \subset & \mathsf{Hom}(\#I, \mathcal{Q}) \\ \cup & & \| & & \cup \\ T_I\mathcal{O} & \times & \mathcal{Q} & \cong & T_{\#I}(\#\mathcal{O}) \end{array}$$

via the isomorphism

$$(\varphi, u + I) \mapsto \left[(x_{n+1})^k v \mapsto \left\{ \begin{array}{ll} \varphi(v) & k = 0 \\ uv + I & k = 1 \\ 0 & k > 1 \end{array} \right\} \right] \tag{5}$$

Since this is true for *any* orbit, it is true for the union of orbits, that is, for any invariant subset X; and since this isomorphism varies continously as we move around on Hilb^{μ}, it follows that the tangent cone of $\#X$ at $\#I$ is also a product, exactly the same way (actually the bundle of tangent cones of $\#X$, restricted to X, is a vector bundle over the bundle of tangent cones of X).

The only thing remains is to compute the weights of the new directions at a monomial ideal, which is easy using (5): The new directions are μ-dimensional subspace A of

$$\mathsf{Hom}(x_{n+1}\mathcal{E}_{d-1}(n), \mathcal{Q}) \subset \mathsf{Hom}(x_{n+1}\mathcal{E}_{d-1}(n+1), \mathcal{Q})$$

and given a basis $\{u_i + I\}$ of Q, we can construct a basis $\{\psi_i\}$ of A, based on (5), by setting $\psi_i(x_{n+1}v) = u_i v + I$. Observe that if the line $\langle u_i \rangle$ is $\mathbb{T}$-invariant with weight w_i, so is $\langle \psi_i \rangle$ with weight $w_i - \alpha_{n+1}$, which completes the proof (the apparent sign discrepancy comes from our notation system, which gives *negative* weights to $\mathcal{J}(n) = \mathsf{Hom}(\oplus_k \mathsf{Sym}^k \mathbb{C}^n, \mathbb{C})$, because the torus acts on the source side). $\square$

Note that we can specify a monomial ideal I by first specifying *the ismorphism type of* its quotient algebra $\mathcal{Q}$, then choosing the $k = k_{\mathcal{Q}}$ generators $x_{i_1}, \dots, x_{i_k}$ of $\mathcal{Q}$, where $\{i_1, \dots, i_k\} = K \subset \binom{n}{k_{\mathcal{Q}}}$, and finally specifying an order of these generators by choosing a permutation $\sigma \in \mathfrak{S}_{k_{\mathcal{Q}}}/\mathsf{Aut}_{\mathcal{Q}}$. Here, $\mathsf{Aut}_{\mathcal{Q}}$ denotes the group of symmetries of (any) "higher-dimensional partition" corresponding to $\mathcal{Q}$. Thus, allowing some permutation of the variables $x_1, x_2, \dots, x_n$, for every monomial ideal $I \lhd \mathcal{J}(n)$ there is an $I_0 \lhd \mathcal{J}(k_{\mathcal{Q}})$ such that $I = \#^{(n-k_{\mathcal{Q}})} I_0$ (where $\mathcal{Q} = \mathcal{J}(n)/I = \mathcal{J}(k_{\mathcal{Q}})/I_0$, as usual). Denoting E_{I_0} by $P_{\mathcal{Q}}$, all this boils down to the following corollary of Lemma 3.1.2:

$$E_I = P_{\mathcal{Q}}(\alpha_{\sigma(i_1)}, \dots, \alpha_{\sigma(i_k)}) \cdot \prod_{j \notin K} \prod_{l=1}^{\mu} \big(\alpha_j - w_l^{\mathcal{Q}}(\alpha_{\sigma(i_1)}, \dots, \alpha_{\sigma(i_k)})\big) \tag{6}$$

3.2. The substitution trick

It is well known that for reasonably nice singularities (all contact singularities fall into this class, [**Dam72**]), the Thom polynomial can be written as a polynomial in the formal difference $\theta - \alpha$; thus the formula (4) above is redundant. Our idea is the exploit this redundancy to enable actual computations.

Let us start with the equation (compare with (4) above)

$$[Z] = \sum_{\lambda} d_\lambda \cdot s_\lambda(\theta - \alpha) = \sum_{y \in \mathsf{Fix}} \frac{\prod_{i=1}^{\mu} \prod_{j=1}^{m} \big(\theta_j - w_i^y(\alpha)\big)}{E_y(\alpha)}$$

where λ runs over the partitions with weight $|\lambda|$ equalling to the codimension $\mathsf{codim}(Z)$ of the singularity; $d_\lambda \in \mathbb{Z}$ are the unknown coefficients of the Thom polynomial we are interested in. Rewriting in Schur polynomials of α and θ (see Appendix A.2) we get

$$\sum_{\lambda} d_\lambda \sum_{\varphi, \chi} (-1)^{|\chi|} c^{\lambda}_{\varphi\chi} s_\varphi(\theta) s_{\widetilde{\chi}}(\alpha) = (-1)^{m\mu} \sum_{y \in \mathsf{Fix}} \frac{\sum_{\varphi \subset (\mu^m)} (-1)^{|\varphi|} s_\varphi(\theta) s_{\mathsf{C}\widetilde{\varphi}}(W_y(\alpha))}{E_y(\alpha)}$$

(the $c^{\lambda}_{\varphi\chi}$ are the Littlewood-Richardson coefficients). Consider the coefficient of $s_\lambda(\theta)$ in both sides, with $|\lambda| = \mathsf{codim}$: on the LHS, it is just d_λ, which gives the following:

Theorem 3.2.1. *With the notations above, we have*

$$d_\lambda = \sum_{y \in \mathsf{Fix}} \frac{s_{\mathsf{C}\widetilde{\lambda}}(W_y(-\alpha))}{E_y(\alpha)} = (-1)^{m\mu - \mathsf{codim}} \sum_{y \in \mathsf{Fix}} \frac{s_{\mathsf{C}\widetilde{\lambda}}(W_y(\alpha))}{E_y(\alpha)}. \tag{7}$$

An immediate corollary is that $d_\lambda = 0$ unless $\lambda_1 \le \mu$.

Note that $d_\lambda \in \mathbb{Z}$, while the RHS is a rational function in the variables α_i; which boils down the fact that we can substitute basically anything into the α_i-s, as long as $E_y(\alpha)$ does not became zero (which is very easy to guarantee in practice), and (7) still holds. That means that for example a computer can substitute either randomly or deterministically chosen integers or rational numbers into the α_i-s, and compute the coefficients of the Thom polynomial from the localization data; this was more-or-less impossible before, except for very small cases. The reason we can do it is that summing (rational) numbers is a *much* easier task

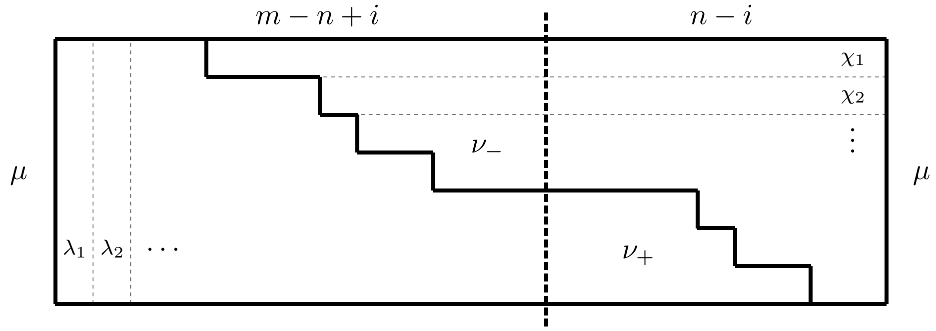


FIGURE 3. The relation between λ, $\chi = \complement\widetilde{\lambda}$, n, m and the pair (ν_+, ν_-)

than summing rational functions. Also we can compute the coefficients d_λ independently of each other.

An important corollary of Theorem 3.2.1 is that the coefficients of the Thom polynomials do not depend on m (and since they are a polynomial in the difference $\theta - \alpha$, they don't depend on n either). While there is a shifting, in the sense that when we replace m by $m+1$, d_λ becomes $d_{(\mu,\lambda)}$, we can relabel the coefficients, indexing them with the pair $\nu_\pm = (\nu_+, \nu_-)$ (see Figure 3). This way we get an infinite series (as a linear combination of "renormalized Schur polynomials" in the difference alphabet $\theta - \alpha$), called *Thom series* of the singularity, from which the Thom polynomial for any n, m can be read off.

REMARK. This stability property was noticed only recently: first in [**BFR02**] in the single case of the A_3 singularity (expressed in Chern monomials, instead of Schur polynomials), then by the author in the one-parameter family of Thom-Boardman singularities $\Sigma^{i,1}$ (see [**FK06**] and Section 4.4), and then, motivated by these examples, proved in [**FR07**].

The coefficients d_λ are also known to be nonnegative; this was conjectured by the author (based on numerical evidence), and also independently by Pragacz, and finally proved in [**PW07a**], [**PW07b**].

Let us explain Figure 3 in more detail. The vertical dotted line in the middle is our 'base line': relative to this line are things stable. n and m are the dimensions of the source and target, as usual; the big box has width m and height μ. The exact placement of the base line is not very important, in the sense that it could be shifted by a fixed finite amount; however, the natural choice seems to be $(m - n + i, n - i)$, where the positive integer i is defined by letting Σ^i to be the unique first-order Thom-Boardman class our singularity belongs to. The bottom-left partition (ignoring the base line) is λ; the terms $d_\lambda \cdot s_\lambda(\theta - \alpha)$ appear in the Thom polynomial. The top-right partition is the complement $\chi = \complement\widetilde{\lambda}$; these appear in the RHS of formula (7) for the coefficients d_λ. ν_- is the portion of χ lying on the left of the base line; similarly, ν_+ is the portion of $\widetilde{\lambda}$ lying on the right of the base line. We will denote the pair (ν_+, ν_-), or more specifically, the "signed partition" (which is just a non-increasing sequence of integers) $(\nu_+, -\mathsf{rev}\,\nu_-) \in \mathbb{Z}^\mu$ by $\nu_\pm$; analogously, $\nu_\mp = (\nu_-, -\mathsf{rev}\,\nu_+)$. Clearly, $\ell(\nu_+) + \ell(\nu_-) = \mu$; and, since $|\lambda| = \mathsf{codim}(Z)$, and the codim changes by μ when we increase m or decrease n (this follows for example from Lemma 3.1.2), it is also true that $\mathsf{ofs} = |\nu_+| - |\nu_-|$ is also a constant, depending only on the singularity (and the choice of the base line).

For an example, consider the singularity A_2. The Thom polynomials and the Thom series are

$$\mathsf{Tp}_{n,m}(A_2) = \sum_{k=0}^{m-n+1} 2^k \cdot s_{(m-n+1+k,m-n+1-k)^\sim}(\theta - \alpha) =$$
$$= \sum_{k=0}^{m-n+1} 2^k \cdot s_{(2^{m-n+1-k},1^{2k})}(\theta - \alpha)$$
$$\mathsf{Ts}(A_2) = \sum_{k=0}^{\infty} 2^k \cdot \mathsf{rs}_{(k,-k)}$$

where $\mathsf{rs}_{\nu_\pm}$ denotes the "renormalized Schur polynomials": we can recover $\mathsf{Tp}_{n,m}$ from Ts by the substitution

$$\mathsf{rs}_{\nu_\pm} \;\mapsto\; s_{((m-n+i)^\mu+\nu_\pm)^\sim}(\theta - \alpha).$$

(in this case, $i = 1$, since $A_2 = \Sigma^{11} \subset \Sigma^1$).

3.3. Principal specialization

Theorem 3.2.1 works pretty well for computer calculations, however it does not allow any insight into the structure behind the scenes. What we will do now is to substitute $1, q, q^2, q^3, \ldots$ (where q is a formal variable) into the variables α_i, and let n tend to infinity. This is called the *(stable) principal specialization* in the symmetric polynomial literature [**Sta99**].

Remark. The reader could ask why we singled out this substitution instead of some others, especially since it breaks the symmetry of the variables? The answer is first of all that we couldn't find any other substitution which looks at least somewhat natural in any way and works in the limit $n \to \infty$; the only other standard specialization is the so-called *exponential specialization*, but to use that we would need our expressions to contain symmetric polynomials instead of roots. We shouldn't worry about the breaking of the symmetry: As the literature shows, this is a rather natural specialization, and taking the limit $n \to \infty$ restores some of the symmetry. Finally, note that since our formulae are homogeneous of degree 0, shifting the exponents to $q^k, q^{k+1}, q^{k+2}, \ldots$ would not change the result.

The idea is to expand the terms of (7) into Laurent series (after the specialization); since we know that the sum is an integer, we only need to extract the constant terms of the individual Laurent series, and sum them. We will use the following notation:

$$G_{\nu_\pm,y,n}(q) = \frac{s_\chi(W_y(-1,-q,-q^2,\ldots,-q^{n-1}))}{E_y(1,q,q^2,\ldots,q^{n-1})} = \sum_{j\in\mathbb{Z}} g_{\nu_\pm,y,n,j} \cdot q^j \in \mathbb{Q}[\![q]\!][q^{-1}]$$

where $W_y(\alpha) = \{w_1^y, \ldots, w_\mu^y\}$ is the set of weights at the fixed point y, and $\chi = \complement\widetilde{\lambda} = (n-1)^\mu + \mu_\mp$ as usual. Thus (for any m)

$$\mathsf{Tp}_{n,m} = \sum_{\nu_\pm} s_\lambda(\theta - \alpha) \left[\sum_{y\in\mathsf{Fix}_n} G_{\nu_\pm,y,n}(q)\right] = \sum_{\nu_\pm} s_\lambda(\theta - \alpha) \left[\sum_{y\in\mathsf{Fix}_n} g_{\nu_\pm,y,n,0}\right]$$

We already understood what happens with the denominator E_y when we increase n, and it is very easy to see what happens with the numerator:

$$W_{\#y} = W_y$$
$$\#\chi = \chi + 1^\mu = (\chi_1 + 1, \chi_2 + 1, \dots, \chi_\mu + 1)$$
$$\#[s_\chi(W_y)] = s_{(\chi+1^\mu)}(W_y) = s_\chi(W_y) \cdot \prod_{i=1}^{\mu} w_i$$

From this, we have the

COROLLARY 3.3.1.

$$G_{\nu_\pm,y,n+1} = G_{\nu_\pm,y,n} \prod_{i=1}^{\mu} \frac{-w_i(q)}{q^n - w_i(q)} = G_{\nu_\pm,y,n} \prod_{i=1}^{\mu} \left(1 - \frac{q^n}{q^n - w_i(q)}\right) \tag{8}$$

THEOREM 3.3.2. *For any fixed $\nu_\pm$, $y \in \mathsf{Fix}$, and $j \in \mathbb{Z}$ the series of rational numbers $n \mapsto g_{\nu_\pm,y,n,j}$ eventually stabilizes. We will denote the stable limit by $g^{\mathrm{stab}}_{\nu_\pm,y,j} \in \mathbb{Q}$.*

Proof. Let us fix an n_0. According to Corollary 3.3.1

$$G_{\nu_\pm,y,n} = G_{\nu_\pm,y,n_0} \cdot \prod_{k=n_0+1}^{n} \prod_{i=1}^{\mu} \left(1 - \frac{q^k}{q^k - w_i(q)}\right).$$

Denote by $c_i^{\min} q^{e_i^{\min}}$ the leading (smallest) term of $w_i(q)$; expanding the multiplier $1-q^n/(q^n-w_i(q))$ into Taylor series, the expansion starts with

$$1 - \frac{q^n}{q^n - w_i(q)} = 1 + \frac{q^{n-e_i^{\min}}}{c_i^{\min}} + \dots;$$

from which it follows that $g_{\nu_\pm,y,n,j} = g_{\nu_\pm,y,n+1,j}$ if $n > j - f_{n_0} + \max\{e_i^{\min}\}$, where f_{n_0} denotes the leading degree of the (the Laurent series of) $G_{\nu_\pm,y,n_0}$. □

THEOREM 3.3.3. *For any fixed $\nu_\pm$ and j, there are only finitely many $y \in \mathsf{Fix}_\infty$ such that $g^{\mathrm{stab}}_{\nu_\pm,y,j}$ is not zero.*

Proof. We will estabilish a lower bound for the leading degree of (the Laurent series of) $G_{\nu_\pm,y,n}$. Since there are only finitely many types of quotient algebras, it is enough to consider a single one, denoted by $\mathcal{Q}$; similarly, we can fix a permutation $\sigma \in \mathfrak{S}_{k_\mathcal{Q}}/\mathsf{Aut}_\mathcal{Q}$. Fixed points (monomial ideals) of this fixed type correspond to the choice of $k = k_\mathcal{Q}$ integers

$$K = \big\{0 \le i_1 < i_2 < \cdots < i_k < n\big\}.$$

It is instructive to look at the example of Figure 2 (page 24): There $\mathcal{Q} = \mathcal{J}(2)/(y^3, xy^2, x^2)$ with an (additive) basis $\{y^2, y, xy, x\}$, the corresponding weights are $\{2\beta, \beta, \alpha+\beta, \alpha\}$; we have to choices for the permutation: For $0 \le i_1 < i_2 < n$ either $\alpha \mapsto q^{i_1}$ and $\beta \mapsto q^{i_2}$ or vice versa. For each weight w_i, let $e_i^{\min}$ denote the leading (meaning smallest) degree of of $w_i(q)$; this is one of the i_j-s (in our running example, either $\{i_2, i_2, i_1, i_1\}$ or $\{i_1, i_1, i_1, i_2\}$ depending on the permutation).

It is easy to determine the leading degree of the numerator $s_{((n-1)^\mu + \nu_\mp)}(-W(q))$: Schur polynomials are sums of monomials determined by semistandard Young tableaux, from which it is immediate that the smallest degree is $\sum_{i=1}^{\mu} e_i^{\min}(n-1+(\nu_\mp)_i)$ if we order the weights such that $e_1^{\min} \le e_2^{\min} \le \cdots \le e_\mu^{\min}$ (similarly the largest degree can be obtained by reversing the ordering of the $e_i^{\min}$-s). Now consider the denominator

$$P_Q(q^{\sigma(i_1)}, \ldots, q^{\sigma(i_k)}) \prod_{l\notin K} \prod_{i=1}^{\mu} (q^l - w_i(q)).$$

The leading degree of the P_Q is a linear function of the i_j-s, since P_Q is a rational function. The leading degree of a product $\prod_{l\notin K}(q^l - w_i(q))$ is

$$\sum_{l<e_i^{\min}, l\notin K} l + \sum_{l>e_i^{\min}, l\notin K} e_i^{\min} = (n-1)e_i^{\min} - \binom{e_i^{\min}+1}{2} - \sum_{l\in K} \min(l, e_i^{\min})$$

To sum it up, the leading degree of G is

$$L + \sum_{i=1}^{\mu} \binom{e_i^{\min}+1}{2}$$

where L is linear in the i_j-s. If any i_j is big, then there must be at least one corresponding $e_i^{\min}$ which equals to it (since we have a monomial *ideal*), and then the second order binomial term will dominate the degree. For any concrete case it is easy to convert this argument into an explicit lower bound, but writing down a general formula is somewhat cumbersome. □

Corollary 3.3.4. *The stable limit*

$$G^{\mathrm{stab}}_{\nu_\pm, y} = \lim_{n\to\infty} G_{\nu_\pm, y, n} \in \mathbb{Q}[\![q]\!][q^{-1}]$$

is well-defined by its Laurent series.

Corollary 3.3.5. *We have the following formulae for the Thom series:*

$$\mathsf{Ts} = \sum_{\nu_\pm} \mathsf{rs}_{\nu_\pm} \sum_{y\in \mathsf{Fix}_\infty} G^{\mathrm{stab}}_{\nu_\pm, y}(q) \tag{9}$$

$$= \sum_{\nu_\pm} \mathsf{rs}_{\nu_\pm} \sum_{y\in \mathsf{Fix}_\infty} g^{\mathrm{stab}}_{\nu_\pm, y, 0} \tag{10}$$

Note that the first formula is a priori *a Laurent series in q; however since it expresses the Thom series, it must be independent of q.*

3.3.1. An algorithmic approach. Formula (10) leads to a new algorithm to compute the coefficients of the Thom series: For each $\nu_\pm$ and each fixed point y, we have a (sharp) lower bound for the n where $g_{\nu_\pm, y, n, 0}$ stabilizes, and for each $\nu_\pm$ we have an upper bound for the fixed points whose contribution is nonzero; furthermore, computing the Laurent series expansions can be done fast, since the coefficients satisfy simple recursions.

The following small computer program, written in the Haskell programming language [**PJ03**], uses these recursions to efficiently compute the Taylor series of the reciprocal of a product of univariate polynomials (with constant terms 1, which is not a real restriction); it is very easy to extend it to work for arbitrary (univariate) rational functions (the fact that the denominator is factored into a product is important only for performance; but in our situation it is typically presented in that form anyway).

Our key function convolves an arbitrary formal power series with the Taylor expansion of the inverse of a polynomial. The polynomial is given in the first argument, encoded as a list of (*coefficient,exponent*) pairs. The second argument is an infinite list, representing a power series. It is assumed that the polynomial has constant term 1 (which is not included in the list); also the coefficients are negated.

```
convolveWith :: Num a => [(a,Int)] -> [a] -> [a]
convolveWith terms series = ys where
  ys = worker terms ys
  worker [] _ = series
  worker ((coeff,exp):rest) xs =
    zipWith (+)
      (replicate exp 0 ++ map (*coeff) xs)
      (worker rest xs)
```

Our other function use the previous one to convolve several such Taylor series series:

```
productSeries :: Num a => [[(a,Int)]] -> [a]
productSeries = foldl (flip convolveWith) unit
```

starting with the multiplicative unit in the ring of formal power series:

```
unit :: Num a => [a]
unit = 1 : repeat 0
```

As an example, consider the function

$$F(q) = \frac{1}{(1-15q^2+17q^3)\cdot(1-14q^5)\cdot(1-29q^2+37q^7+11q^9)}.$$

Its Taylor series around $q=0$ can be computed with the function call

```
productSeries
  [ [ (15,2) , (-17,3) ]
  , [ (14,5) ]
  , [ (29,2) , (-37,7) , (-11,9) ] ]
```

The result is infinite list of integers, which starts with

```
[1,0,44,-17,1501,-989,47193,-39983,1431989,-1392409,42670891, ... ]
```

In addition, if we have a polynomial in the numerator of F, we just have to replace `unit` in the above code by (the power series representation of) that polynomial.

3.4. Some analytic computations

In this section, we evaluate Formula (9) analytically for some simple cases. As we will see, these computations fit very well with the theory of basic hypergeometric series; we refer to [**GR90**] for the background on this theory. The notations and the fundamental results we use are summarized in Appendix A.5.

The necessary input data for the localization formula, that is, the virtual tangent Euler classes $P_{\mathcal{Q}}$, are computed in [**FR08**] for small singularities ($\mu = 2$: A_2; $\mu = 3$: A_3, $I_{2,2}$, $III_{2,3}$; $\mu = 4 : A_4$, $I_{2,3}$, $III_{2,4}$, $III_{3,3}$, $\Sigma^{2,1}$) by "reverse engineering", using earlier computations of Thom polynomials for these singularities. We will rederive a few of these (Σ^{ij}, A_3) from first principles, understanding the geometry of the moduli spaces $\mathcal{M}$, in the next chapters.

3.4.1. Σ^1, A_1. A_1 is open in Σ^1, thus their Thom polynomials are the same. This is the simplest possible case; it serves as an introduction before we dive into more complicated computations. Since we have $|\nu_+| = |\nu_-|$ and $\ell(\nu_+) + \ell(\nu_-) = \mu = 1$, there is only a single possibility for $\nu_\pm$, namely, $\nu_\pm = (0)$. So the Thom "series" consists of a single term in this case. There is also a single type of quotient algebra of codimension 1: $\mathcal{Q} = (x\mathbb{C}[x])/(x^2)$, so all the fixed points are of the same type. The fact is that $P_{\mathcal{Q}} = 1$, so (after substituting $\alpha_i = q^{i-1}$) Theorem 3.2.1 with Lemma 3.1.2 gives $\mathsf{Ts}(A_1) = \{\lim_n d_0^{(n)}\}\mathsf{rs}_{(0)}$, with

$$\begin{aligned} d_0^{(n)} &= \sum_{i=0}^{n-1} \frac{s_{(n-1)}(-q^i)}{\prod_{l=0}^{i-1}(q^l - q^i) \cdot \prod_{l=i+1}^{n-1}(q^l - q^i)} \\ &= \sum_{i=0}^{n-1} \frac{(-1)^{n-1} q^{i(n-1)}}{q^{\binom{i}{2}} \prod_{l=0}^{i-1}(1 - q^{i-l}) \cdot (-1)^{n-i-1} q^{i(n-i-1)} \prod_{l=i+1}^{n-1}(1 - q^{l-i})} \\ &= \sum_{i=0}^{n-1} \frac{(-1)^i q^{\binom{i+1}{2}}}{(q;q)_i (q;q)_{n-i-1}} \end{aligned}$$

At this form, it is clear that we can take the limit $n \to \infty$.

$$d_0 = \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{(-1)^i q^{\binom{i+1}{2}}}{(q;q)_i (q;q)_{n-i-1}} = \frac{1}{(q,q)_\infty} \sum_{i=0}^{\infty} \frac{(-1)^i q^{\binom{i+1}{2}}}{(q;q)_i} = 1$$

by the limit case A.5.6 of the q-binomial theorem.

3.4.2. Σ^2, $III_{2,2}$. Again, the singularity $III_{2,2}$ is open in Σ^2; their Thom polynomials are the same. In this case, $\mu = 2$, $\mathsf{codim} = 2(m - n + 2)$, $\mathsf{ofs} = 0$; the possible (ν_+, ν_-) pairs are (a, a) for $a \in \mathbb{N}$; that is, $\nu_\pm = (a, -a)$. There are two types of quotient algebras of codimension 2, but only $\mathcal{Q} = \mathcal{J}(2)/(x^2, xy, y^2)$ contributes to the Thom polynomial; and

sing.	ideal	μ	ofs	type	$\mathsf{codim} = \mu(m-n+i) + \mathsf{ofs}$
A_d	(x^{d+1})	d	0	Σ^1	$d(m-n+1)$
$I_{a,b}$	$(xy, x^a + y^b)$	$a+b-1$	$2-a-b$	$\Sigma^{2,0}$	$(a+b-1)(m-n+1)+1$
$III_{a,b}$	(xy, x^a, y^b)	$a+b-2$	$4-a-b$	$\Sigma^{2,0}$	$(a+b-2)(m-n+1)+2$

TABLE 2. Table of singularities of Boardman type Σ^1 and $\Sigma^{2,0}$, in Mather's notation

$P_{\mathcal{Q}} = 1$. Thus $\mathsf{Ts}(\Sigma^2) = \sum_{a=0}^{\infty} \big\{ \lim_n d_a^{(n)} \big\} \mathsf{rs}_{(+a,-a)}$, where

$$
\begin{aligned}
d_a^{(n)} &= \sum_{0 \le i < j < n} \frac{s_{(n-2+a,n-2-a)}(-q^i, -q^j)}{\prod_{l \ne i,j} (q^l - q^i)(q^l - q^j)} \\
&= \sum_{0 \le i < j < n} \frac{q^{(i+j)(n-2)} (-1)^{i+j} q^{\binom{i+1}{2} + \binom{j+1}{2}} (q^i - q^j)(q^j - q^i) \sum_{s=-a}^{+a} q^{s(j-i)}}{q^{(i+j)(n-1)}\, (q;q)_i\, (q;q)_{n-1-i}\, (q;q)_j\, (q;q)_{n-1-j}} \\
&= \sum_{0 \le i < j < n} \frac{(-1)^{i+j} q^{\binom{i+1}{2} + \binom{j+1}{2}} (1 - q^{j-i})(1 - q^{i-j}) \sum_{s=-a}^{+a} q^{s(j-i)}}{(q;q)_i\, (q;q)_{n-1-i}\, (q;q)_j\, (q;q)_{n-1-j}}
\end{aligned}
$$

Note how we multiplied both the numerator and the denominator by $(q^i - q^j)(q^j - q^i)$, so that we can have the nice denominator in the last line.

Introduce the function

$$
F(z) = (1 - z)(1 - z^{-1}) \sum_{s=-a}^{+a} z^s = -z^{-a-1} + z^{-a} + z^a - z^{a+1}
$$

so that

$$
\begin{aligned}
d_a^{(n)} &= \sum_{0 \le i < j < n} \frac{(-1)^{i+j} q^{\binom{i+1}{2} + \binom{j+1}{2}} F(q^{j-i})}{(q;q)_i\, (q;q)_{n-1-i}\, (q;q)_j\, (q;q)_{n-1-j}} \\
&= \frac{1}{2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \frac{(-1)^{i+j} q^{\binom{i+1}{2} + \binom{j+1}{2}} F(q^{j-i})}{(q;q)_i\, (q;q)_{n-1-i}\, (q;q)_j\, (q;q)_{n-1-j}}
\end{aligned}
$$

since $F(1) = 0$ and $F(z) = F(z^{-1})$. At this point the naïve idea is to expand F into Laurent series, and exchange the order of the summation; that in fact works in this particular case, since F is a *Laurent polynomial*, but has a subtle problem when F is an actual series with a convergence annulus $R_1 < |z| < R_2$ strictly smaller than $0 < |z| < \infty$: When we take the limit $n \to \infty$, the difference $j - i$ can be an arbitrarily large positive or negative integer, and thus $R_1 < |q^{j-i}| < R_2$ implies $|q| = 1$; on the other hand, the rest of the formula requires $|q| < 1$ to work.

So let's take a step back, and consider the following sum in *two independent* variables q and u

$$
\mathcal{Z}_q^{(n)}(F, u) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \frac{(-1)^{i+j} q^{\binom{i+1}{2} + \binom{j+1}{2}} F(u^{j-i})}{(q;q)_i (q;q)_{n-1-i} (q;q)_j (q;q)_{n-1-j}} \tag{11}
$$

Lemma 3.4.1. *Suppose F is meromorphic on $\mathbb{P}^1$, and it has no poles on $|z| = 1$. Then $\mathcal{Z}_q^{(n)}(F, u)$ holomorphic in u for $|q| < 1$ and $u \in \Omega^{(n)} = \mathbb{C} - \Delta^{(n)}$ where $\Delta^{(n)}$ is the (finite) set of at most $(n-1)$-th roots of the poles of F (including the 'negative roots' $p^{-1/k}$). Furthermore, $\mathcal{Z}_q^{(n)}(F, u)$ converges as $n \to \infty$ for $|q| < 1$ and $u \in \mathbb{C} - \Delta$, and the limit $\mathcal{Z}_q(F, u)$ is holomorphic on the domain $\Omega_{<1} = \{|u| < 1\} - \Delta$ (and also on $\Omega_{>1} = \{|u| > 1\} - \Delta$, but we won't need that), where $\Delta = \cup_n \Delta^{(n)}$ is the (countable) set of all roots of the poles of F.*

Proof. The sequence $\mathcal{Z}_q^{(n)}(F, u)$ converges because for large $(j - i)$, $|F(u^{j-i})|$ is asymptotically $|u|^{r_0(j-i)}$, bounded, or $|u|^{r_\infty(j-i)}$ for $|u| < 1$, $|u| = 1$ and $|u| > 1$, respectively, where r_0 and r_∞ are the orders of the poles of F at 0 and ∞ (cf. Corollary 3.4.4 below).

The only other thing not immediately clear is the holomorphicity of the limit. To see this, consider the smaller domain $U_\varepsilon = \{|u| < 1-\varepsilon\} - \Delta_\varepsilon$, where $\Delta_\varepsilon = \cup_{p\in\Delta,|p|<1-\varepsilon}\{|u-p| < \varepsilon\}$ is the ε-neighbourhood of $\Delta \cap \{|u| < 1-\varepsilon\}$. Note that the latter is a finite set. The closure $\overline{U}_\varepsilon \subset \Omega_{<1}$ is a compact set, thus $\mathcal{Z}_q^{(n)}$ converges uniformly on it, and then the limit $\mathcal{Z}_q$ must be holomorphic on U_ε. Since this works for any small $\varepsilon > 0$, $\mathcal{Z}_q$ is holomorphic on $\cup_{\varepsilon>0} U_\varepsilon = \Omega_{<1}$. □

PROPOSITION 3.4.2. *For $u \neq 0$, $z \in \mathbb{C}$*

$$\sum_{i=0}^{n} \frac{(-1)^i q^{\binom{i+1}{2}}}{(q;q)_i (q;q)_{n-i}} (zu^{\pm 1})^i = \frac{(zqu^{\pm 1};q)_n}{(q;q)_n}$$

Proof. Set $b = zqu^{\pm 1}/a$ in Theorem A.5.7 and let a tend to infinity. □

COROLLARY 3.4.3. *For $u \neq 0$, $z \in \mathbb{C}$*

$$\sum_{i=0}^{n}\sum_{j=0}^{n} \frac{(-1)^{i+j} q^{\binom{i+1}{2}+\binom{j+1}{2}}}{(q;q)_i (q;q)_{n-i} (q;q)_j (q;q)_{n-j}} z^{(i+j)} u^{(j-i)} = \frac{(zqu^{-1};q)_n (zqu;q)_n}{(q;q)_n (q;q)_n}.$$

COROLLARY 3.4.4. *For $u \neq 0$, $z \in \mathbb{C}$*

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{i+j} q^{\binom{i+1}{2}+\binom{j+1}{2}}}{(q;q)_i (q;q)_j} z^{(i+j)} u^{(j-i)} = (zqu^{-1};q)_\infty (zqu;q)_\infty.$$

Now consider the Laurent series expansion $F(z) = \sum_{m\in\mathbb{Z}} c_m z^m$ on an annulus containing $|z| = 1$. Substituting this back into (11) and using Corollary 3.4.3, we get that for a small (depending on n) neighbourhood of $|u| = 1$

$$\begin{aligned} \mathcal{Z}_q^{(n)}(F,u) &= \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \frac{(-1)^{i+j} q^{\binom{i+1}{2}+\binom{j+1}{2}} \left(\sum_{m\in\mathbb{Z}} c_m u^{m(j-i)}\right)}{(q;q)_i (q;q)_{n-1-i} (q;q)_j (q;q)_{n-1-j}} \\ &= \frac{1}{(q;q)_{n-1}(q;q)_{n-1}} \sum_{m\in\mathbb{Z}} c_m \cdot (qu^{-m}, qu^m; q)_{n-1} \end{aligned} \tag{12}$$

We will use the finite version of Jaboci's triple product formula A.5.11.

$$\begin{aligned} (qz, qz^{-1};q)_n &= \begin{cases} \frac{1-z^{-1}q^n}{1-z^{-1}} (qz, z^{-1};q)_n, & \text{if} \quad z \neq 1 \\ (q,q;q)_n, & \text{if} \quad z = 1 \end{cases} \\ &= \begin{cases} \frac{1-z^{-1}q^n}{1-z^{-1}} \sum_{k=-n}^{n} (-1)^k \left[{2n \atop n+k}\right]_q q^{\binom{k+1}{2}} z^k, & \text{if} \quad z \neq 1 \\ (q,q;q)_n, & \text{if} \quad z = 1 \end{cases} \end{aligned}$$

Since this formula has a special case for $|z| = |u^m| = 1$, we have to separate the $m = 0$ case; we can do that by introducing $F_0(z) = F(z) - c_0$. Using the expansions

$$\frac{1-z^{-1}q^n}{1-z^{-1}} = \begin{cases} 1 - (1-q^n)\sum_{l=0}^{\infty} z^l & \text{if} \quad |z| < 1 \\ 1 + (1-q^n)\sum_{k=1}^{\infty} z^{-l} & \text{if} \quad |z| > 1 \end{cases}$$

we get that for $1-\varepsilon<|u|<1$

$$\begin{aligned}
\mathcal{Z}_q^{(n)}(F,u) &= c_0 + \frac{1}{(q,q;q)_{n-1}} \sum_{m\neq 0} c_m \frac{1-u^{-m}q^{n-1}}{1-u^{-m}} \sum_{k=1-n}^{n-1} (-1)^k \begin{bmatrix} 2n-2 \\ n-1+k \end{bmatrix}_q q^{\binom{k+1}{2}} u^{mk} \\
&= c_0 + \frac{1}{(q,q;q)_{n-1}} \sum_{k=1-n}^{n-1} (-1)^k \begin{bmatrix} 2n-2 \\ n-1+k \end{bmatrix}_q q^{\binom{k+1}{2}} \cdot \\
&\quad \cdot \left\{ \sum_{m>0} c_m u^{mk} \Big[1-(1-q^{n-1}) \sum_{l\geq 0} u^{ml} \Big] + \sum_{m<0} c_m u^{mk} \Big[1+(1-q^{n-1}) \sum_{l<0} u^{ml} \Big] \right\} \\
&= c_0 + \frac{1}{(q,q;q)_{n-1}} \sum_{k=1-n}^{n-1} (-1)^k \begin{bmatrix} 2n-2 \\ n-1+k \end{bmatrix}_q q^{\binom{k+1}{2}} \cdot \\
&\quad \cdot \left\{ F_0(u^k) - (1-q^{n-1}) \Big[\sum_{l\geq 0} F_+(u^{l+k}) - \sum_{l>0} F_-(u^{l-k}) \Big] \right\}
\end{aligned}$$

where $F_0(z)=F_+(z)+F_-(z^{-1})$ is the decomposition of F_0 to its principal part and the rest. The important observation is that though our derivation works only for a limited set of u-s, both $\mathcal{Z}_q^{(n)}(F,u)$ and the function defined by the last formula are holomorphic on $\Omega_{<1}$, so if they agree on a small set, they must be equal on on the whole domain. Now we can take the limit of both sides as $n\to\infty$:

$$\begin{aligned}
\mathcal{Z}_q(F,u) &= c_0 + \frac{1}{(q,q,q;q)_\infty} \sum_{k=-\infty}^{\infty} (-1)^k q^{\binom{k+1}{2}} \Big\{ -\sum_{l>0} F_+(u^{l+k}) + \sum_{l\geq 0} F_-(u^{l-k}) \Big\} \\
&= c_0 + \frac{1}{(q,q,q;q)_\infty} \sum_{k=0}^{\infty} (-1)^k q^{\binom{k+1}{2}} \sum_{l=0}^{\infty} \Big\{ \big[F_+ + F_-\big](u^{l-k}) - \big[F_+ + F_-\big](u^{l+k+1}) \Big\} \\
&= c_0 + \frac{1}{(q,q,q;q)_\infty} \sum_{k=0}^{\infty} (-1)^k q^{\binom{k+1}{2}} \sum_{j=-k}^{+k} \big[F_+ + F_-\big](u^j) \qquad (13)
\end{aligned}$$

and substitute $u=q$. While all this complexity was unnecessary for this particular case, because our F was simple enough (namely, a Laurent polynomial), it will be needed in the next subsection for the computation of $\mathsf{Ts}(A_2)$.

Lemma 3.4.5. *When F is a Laurent polynomial, $\mathcal{Z}_q(F,q)=c_0$.*

Proof. Since a Laurent polynomial is a *finite* sum of monomials, it is enough to consider the case $F_0(z)=z^b$, $b\neq 0$. Then the statement follows from (12), observing that already for $n>b$ it is true that $\mathcal{Z}_q^{(n)}(F,q)=c_0$. Alternatively we can also start from (13), prove the cases $b=1$ and $b=2$ "by hand", then proceed by the induction step $b\to b+2$. □

In the Σ^2 case we started with, we have

$$F_+(z)=F_-(z)=\left\{\begin{array}{ll} -z, & a=0 \\ z^a(1-z), & a>0 \end{array}\right\} \qquad \text{and} \qquad c_0=\left\{\begin{array}{ll} 2, & a=0 \\ 0, & a>0 \end{array}\right\}$$

thus

$$d_a = \frac{1}{2}\mathcal{Z}_q(q) = \begin{cases} 1, & a = 0 \\ 0, & a > 0 \end{cases}$$

$$\mathsf{Ts}(\Sigma^2) = \mathsf{rs}_{(0,0)}$$

3.4.3. A_2**,** Σ^{11}**.** This is the first really interesting computation. In this case, $\mu = 2$, $\mathsf{codim} = 2(m - n + 1)$, $\mathsf{ofs} = 0$; the possible $(\nu_\pm)$-s are again $(a, -a)$ for $a \in \mathbb{N}$. There are two types of quotient algebras of codimension 2, and thus two types of fixed points:

$$\mathcal{Q}_1 = \mathcal{J}(1)/(x^3) \qquad P_{\mathcal{Q}_1} = 1$$

$$\mathcal{Q}_2 = \mathcal{J}(2)/(x^2, xy, y^2) \qquad P_{\mathcal{Q}_2} = \frac{1}{3}(\alpha - 2\beta)(\beta - 2\alpha)$$

So the Thom series is $\mathsf{Ts}(A_2) = \sum_{a=0}^{\infty} \left\{ \lim_n (d_a^{(n)} + e_a^{(n)}) \right\} \mathsf{rs}_{(a,-a)}$, where

$$\begin{aligned} d_a^{(n)} &= \sum_{i=0}^{n-1} \frac{s_{(n-1+a,n-1-a)}(-q^i, -2q^i)}{\prod_{l \neq i}[(q^l - q^i)(q^l - 2q^i)]} \\ &= \left(\sum_{s=-a}^{+a} 2^s\right) \cdot \sum_{i=0}^{n-1} \frac{2^i \left[(-1)^i q^{\binom{i+1}{2}}\right]^2}{(q;q)_i\,(2q;q)_i\,(q;q)_{n-1-i}\,(q/2;q)_{n-1-i}} \end{aligned}$$

and

$$\begin{aligned} e_a^{(n)} &= \sum_{0 \le i < j < n} \frac{3 \cdot s_{(n-1+a,n-1-a)}(-q^i, -q^j)}{(q^i - 2q^j)(q^j - 2q^i) \cdot \prod_{l \neq i,j}(q^l - q^i)(q^l - q^j)} \\ &= \sum_{0 \le i < j < n} \left(\sum_{s=-a}^{+a} q^{s(j-i)}\right) \frac{3(q^i - q^j)(q^j - q^i)}{(q^i - 2q^j)(q^j - 2q^i)} \cdot \frac{(-1)^{i+j} q^{\binom{i+1}{2} + \binom{j+1}{2}}}{(q;q)_i\,(q;q)_{n-1-i}\,(q;q)_j\,(q;q)_{n-1-j}} \end{aligned}$$

The first fixed point type is pretty straightforward. We can simply take the limit $n \to \infty$:

$$\begin{aligned} \frac{d_a}{(2^{a+1} - 2^{-a})} &= \frac{1}{(q, q/2; q)_\infty} \sum_{i=0}^{\infty} \frac{2^i \left[(-1)^i q^{\binom{i+1}{2}}\right]^2}{(q, 2q; q)_i} \\ &= \frac{1}{(q, q/2; q)_\infty} \lim_{a \to \infty} {}_2\Phi_1\left[\begin{matrix} a, a \\ 2q \end{matrix} \,\middle|\, q, \frac{2q^2}{a^2}\right] \\ &= \frac{1}{(q, q/2; q)_\infty} \lim_{a \to \infty} \frac{(2q/a, 2q^2/a; q)_\infty}{(2q, 2q^2/a^2; q)_\infty} \, {}_2\Phi_1\left[\begin{matrix} q, a \\ 2q^2/a \end{matrix} \,\middle|\, q, \frac{2q}{a}\right] \\ &= \frac{1}{(q, 2q, q/2; q)_\infty} \sum_{i=0}^{\infty} (-1)^i q^{\binom{i+1}{2}} 2^i \\ &= 1 + \frac{3}{2}q + \frac{21}{4}q^2 + \frac{117}{8}q^3 + \frac{633}{16}q^4 + \frac{3129}{32}q^5 + \cdots \end{aligned}$$

using Theorem A.5.9, version (40) in the middle. As far as we know, there is no closed formula for the type of sum appearing in the last formula.

The second fixpoint type is more involved. We start with the machinery built in the previous computation: Note that we have exactly the same type of sum $\mathcal{Z}_q^{(n)}(F, q)$, but here our F is

$$F(z) = \frac{3(1-z)^2}{2(1-2z)(1-z/2)} \sum_{s=-a}^{+a} z^s = H(z) \cdot \sum_{s=-a}^{+a} z^s$$

The Laurent series around $|z| = 1$ is convergent for $\frac{1}{2} < |z| < 2$, and it is relatively straightforward to compute

$$\begin{aligned} F(z) &= \left[H_+(z) + 1 + H_-(z^{-1})\right] \sum_{s=-a}^{+a} z^s \\ &= (2^{a+1} - 2^{-a})H(z) + \left[X_+(z) + x_0 + X_-(z^{-1})\right] \end{aligned}$$

where

$$\begin{aligned} H_+(z) = H_-(z) &= \frac{z}{2(z-2)} = -\frac{1}{4}z - \frac{1}{8}z^2 - \frac{1}{16}z^3 - \cdots \\ H_+(z^{-1}) = H_-(z^{-1}) &= \frac{1}{2(1-2z)} \\ x_0 &= -(2^{a+1} - 2^{-a+1}) \\ X_+ = X_- &= \sum_{i=1}^{a} (2^{i-1} + 2^{-i}) \cdot z^{a+1-i}. \end{aligned}$$

But we proved in the previous section (Lemma 3.4.5) that Laurent *polynomials* without a constant term give a zero sum, thus we can discard the $X_\pm$ part, and calculate just with

$$x_0 + (2^{a+1} - 2^{-a})H(z) = 2^{-a} + (2^{a+1} - 2^{-a})\left[H_+(z) + H_-(z^{-1})\right]$$

Now, lets compute $\mathcal{Z}_q(H_0, q)$; starting from (13):

$$\begin{aligned} \mathcal{Z}_q(H_0, q) &= \frac{1}{(q;q)_\infty^3} \sum_{k=0}^{\infty} (-1)^k q^{\binom{k+1}{2}} \sum_{j=-k}^{+k} \left[H_+ + H_-\right](q^j) \\ &= \frac{1}{(q;q)_\infty^3} \sum_{j=-\infty}^{+\infty} \frac{1}{1-2q^j} \sum_{k=|j|}^{\infty} (-1)^k q^{\binom{k+1}{2}} \\ &= \frac{1}{(q;q)_\infty^3} \sum_{j=-\infty}^{+\infty} \frac{1}{1-2q^j} \sum_{k=j}^{\infty} (-1)^k q^{\binom{k+1}{2}} \\ &= \frac{1}{(q;q)_\infty^3} \sum_{j=-\infty}^{+\infty} (-1)^j q^{\binom{j+1}{2}} \frac{1}{1-2q^j} \sum_{l=0}^{\infty} (-1)^l q^{\binom{l+1}{2}} q^{lj} \\ &= \frac{1}{(q;q)_\infty^3} \sum_{l=0}^{\infty} (-1)^l q^{\binom{l+1}{2}} \sum_{j=-\infty}^{+\infty} (-1)^j q^{\binom{j+1}{2}} \frac{q^{lj}}{1-2q^j} \end{aligned}$$

where, at first we used $H_\pm(z^{-1})$ instead of $H_\pm(z)$, which is fine since the sum is symmetric for $z \mapsto z^{-1}$ anyway; then we used the trivial identity $\sum_{k=-n}^{n-1} (-1)^k q^{\binom{k+1}{2}} = 0$; finally, substituted $k = l + j$. Now let us concentrate on the inner sum. Observe that

$$\frac{1-x}{1-xq^j} = \frac{(x;q)_j}{(xq;q)_j};$$

writing $y = q^l$ and briefly $x = 2$ (just for the symmetry), the inner sum is $(1-x)^{-1}$ times

$$\begin{aligned}
\sum_{j=-\infty}^{+\infty} (-1)^j q^{\binom{j+1}{2}} \frac{(x;q)_j}{(xq;q)_j} y^j &= \lim_{a\to\infty} {}_2\Psi_2\left[\begin{matrix} a & ,x \\ a^{-1} & ,xq \end{matrix} \,\middle|\, q, \frac{yq}{a}\right] \\
&= \lim_{a\to\infty} \frac{(qy,\, qx/a,\, 1/ax,\, q/y;q)_\infty}{(qy/a,\, qx,\, q/x,\, 1/ay;q)_\infty} {}_2\Psi_2\left[\begin{matrix} a & ,y \\ a^{-1} & ,yq \end{matrix} \,\middle|\, q, \frac{xq}{a}\right] \\
&= \frac{(qy, qy^{-1};q)_\infty}{(qx, qx^{-1};q)_\infty} \sum_{j=-\infty}^{+\infty} (-1)^j q^{\binom{j+1}{2}} \frac{(y;q)_j}{(qy;q)_j} x^j \\
&= \frac{(qy, qy^{-1};q)_\infty}{(qx, qx^{-1};q)_\infty} \sum_{j=-\infty}^{+\infty} (-1)^j q^{\binom{j+1}{2}} \frac{1-y}{1-q^j y} x^j
\end{aligned}$$

using the bilateral transformation formula A.5.14. Now we want to substitute back $x = 2$ and $y = q^l$; however, the latter is a bit tricky: If we do it naively, we get zeros both in the numerator and the denominator for $l > 0$, $j = -l$. Fortunately, they just cancel out, and the rest of terms ($j \neq -l$) becomes simply zero when multiplied by $(q^{1-l};q)_\infty = 0$. To see what happens with the critical term, set $y = q^l + \varepsilon$, and take the limit $\varepsilon \to 0$:

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\left[\frac{(qy, qy^{-1};q)_\infty}{(2q, q/2^{-1};q)_\infty} \sum_{j=-\infty}^{+\infty} (-1)^j q^{\binom{j+1}{2}} \frac{1-y}{1-q^j y} 2^j\right]_{y=q^l+\varepsilon} \\
&= \lim_{\varepsilon\to 0}\left[\frac{(q(q^l+\varepsilon);q)_\infty}{(2q, q/2;q)_\infty} (-1)^{-l} q^{\binom{-l+1}{2}} 2^{-l} \cdot \frac{(1-(q^l+\varepsilon))}{(1-q^{-l}(q^l+\varepsilon))} \cdot \right. \\
&\qquad \left. \cdot \underbrace{\left[\left(\frac{q}{q^l+\varepsilon};q\right)_{l-1} \cdot \left(1-\frac{q^l}{q^l+\varepsilon}\right) \cdot \left(\frac{q^{l+1}}{q^l+\varepsilon};q\right)_\infty\right]}_{(qy^{-1};q)_\infty}\right] \\
&= \frac{(q^{l+1};q)_\infty}{(2q, q/2;q)_\infty} (-1)^{-l} q^{\binom{-l+1}{2}} 2^{-l} \cdot (1-q^l)(q^{-l+1};q)_{l-1}(q;q)_\infty \cdot \underbrace{\lim_{\varepsilon\to 0}\left[\frac{1-\frac{q^l}{q^l+\varepsilon}}{(1-q^{-l}(q^l+\varepsilon))}\right]}_{=-1} \\
&= \frac{(q;q)_\infty^2}{(2q, q/2;q)_\infty} 2^{-l}
\end{aligned}$$

using that $(q^{-l+1};q)_{l-1} = (q;q)_{l-1}(-1)^{l-1} q^{-\binom{l}{2}}$. Note that though we handled the $l = 0$ case separately, this last formula is valid for $l = 0$, too. Thus

$$\mathcal{Z}_q(H_0, q) = \frac{-1}{(q, 2q, q/2;q)_\infty} \sum_{l=0}^{\infty} (-1)^l q^{\binom{l+1}{2}} 2^{-l}$$

The sign comes from the $(1-x)^{-1}$ factor ($x=2$). This is *almost* the same as the formula for the other fixpoint type! The two are connected by Jacobi's triple product identity A.5.10:

$$\begin{aligned}
-(q,2q,q/2;q)_\infty &= \sum_{n=-\infty}^{\infty}(-1)^n q^{\binom{n+1}{2}}\frac{1}{2^n} \\
&= \sum_{l\geq 0}(-1)^l q^{\binom{l+1}{2}}\frac{1}{2^l} \;+\; \sum_{i<0}(-1)^i q^{\binom{i+1}{2}}\frac{1}{2^i} \\
&= \sum_{l\geq 0}(-1)^l q^{\binom{l+1}{2}}\frac{1}{2^l} \;+\; \sum_{i>0}(-1)^i q^{\binom{i}{2}}2^i \\
&= \sum_{l\geq 0}(-1)^l q^{\binom{l+1}{2}}\frac{1}{2^l} \;-\; 2\sum_{j\geq 0}(-1)^j q^{\binom{j+1}{2}}2^j
\end{aligned}$$

by substituting $i=j+1$. Reorganizing, we get

$$\mathscr{Z}_q(H_0,q) = -\frac{\sum_{l=0}^{\infty}(-1)^l q^{\binom{l+1}{2}}2^{-l}}{(q,2q,q/2;q)_\infty} = 1-2\frac{\sum_{j=0}^{\infty}(-1)^j q^{\binom{j+1}{2}}2^j}{(q,2q,q/2;q)_\infty}$$

thus

$$e_a = \frac{1}{2}\mathscr{Z}_q(F,q) = \frac{1}{2}\left[2^{-a}+(2^{a+1}-2^{-a})\mathscr{Z}_q(H_0,q)\right] = 2^a + \left(2^{-a}-2^{a+1}\right)\frac{\sum_{j=0}^{\infty}(-1)^j q^{\binom{j+1}{2}}2^j}{(q,2q,q/2;q)_\infty}.$$

Finally, combining with the other fixed point type, $d_a+e_a=2^a$, and the Thom series is

$$\mathsf{Ts}(A_2) = \sum_{a\geq 0} 2^a\,\mathsf{rs}_{(a,-a)}$$

Remark. We could also turn the whole argument upside-down, and say that starting from the Thom polynomial theory, we proved an interesting q-hypergeometric identity (note that even if we didn't know the Thom polynomial, the general theory guarantees that the sum of the functions appearing is a constant, thus we get a hypergeometric identity up to an unknown constant).

In fact, to compute the Thom series, we need to compute only the constant term of the series appearing, which is (in this case) much easier than proving that the series actually cancel each other (as we did). However, for more complicated singularities, these series may contain negative powers of q, so similar computations will be necessary.

Concluding remarks. It is easy to make mistakes in such a long computation; however, we can be reasonably confident in its correctness: In addition of being careful, we cross-checked each step using computer algebra software (Maple), typically by examining the first 30-40 coefficients of the Taylor (or Laurent) series expansion (with respect to q). Indeed, a subtle sign error was discovered this way.

The reader probably noticed that, in spite of the complexity of the computation, the result—the Thom polynomials of the A_2 singularity—, is nothing new: They were first calculated by Ronga in [**Ron72**]. However, we discovered a surprising connection with the theory of basic hypergeometric series; indeed we find it quite astonishing how well the basic results of this theory fit the needs of our computation. While computing the next cases ($\mu=3$) this way is inherently more difficult, we still hope that the connection can be generalized; there is also some (very) light evidence suggesting that the $\mu=3$ case already contains all the essential complexity. Note that we are not aware of any other method for computing the Thom series directly from the localization data.

Chapter 4. Second order - Σ^{ij}

The Thom polynomials of second order Thom-Boardman singularities Σ^{ij} are well-studied, by Porteous [**Por71**], Ronga [**Ron72**], Kazarian [**Kaz06**]. Some would say that this question is solved; however, we argue that this is not the case. While it is true that there are many different formulae for these Thom polynomials, the coefficients in the Schur (or Chern-monomial) expansion are not known; in fact, as we will show (see Theorem 4.4.4), for the particular case Σ^{ii}, these coefficients has a very nice combinatorial interpretation, and the resulting combinatorial problem of finding some kind of formula, or positive enumeration for these numbers is unsolved. We regard this fact as a solid evidence for the richness of combinatorics of the coefficients of Thom polynomials expressed as linear combination of (supersymmetric) Schur polynomials.

In this chapter, we will first derive a localization formula for the Thom polynomials of Σ^{ij} singularities, which leads to a new proof of Ronga's theorem (Theorem 4.2.1); then (by completely different methods) we derive closed formulae for the coefficients of the Σ^{i1}, Σ^{22} singularities, and also for all Σ^{ij} in the smallest codimension they appear. With the exception of the Σ^{22} case, these were first presented in [**FK06**].

4.1. Equivalence of the different definitions

We are using different definitions of the algebraic sets Σ^{ij}; here we collect them in one place and show that they are equivalent.

Recall that we are working in the second jet space

$$\mathcal{J}_2(n,m) = \mathsf{Hom}(\mathbb{C}^n,\mathbb{C}^m) \oplus \mathsf{Hom}(\mathsf{Sym}^2\mathbb{C}^n,\mathbb{C}^m);$$

Σ^{ij} is a $\mathsf{Diff}_2(n) \times \mathsf{Diff}_2(m)$-invariant smooth quasi-affine subvariety of this space. Usually we are interested in the closure $\bar{\Sigma}^{ij}$, which is singular; the boundary $\bar{\Sigma}^{ij} - \Sigma^{ij}$ contains (jets of) more complicated singularities.

We will need some definitions, which we recall from Chapter 2.

Definition 4.1.1. Let $U \lhd \mathcal{E}(n)$ be an ideal of functions. The kth *Jacobian extension* $\Delta_k(U)$ is the ideal generated by U and all the $k\times k$ determinants $\mathsf{det}[\partial\varphi_i/\partial x_j]$, where x_j is a (fixed) local coordinate system and $\varphi_i \in U$. It will be convenient to also define $\Delta^k(U) = \Delta_{n-k+1}(U)$.

Lemma 4.1.2 (Boardman, [**Boa67**]). *Let $(\varphi_1,\dots,\varphi_N)$ be a generating system for U; then $\Delta_k(U)$ is generated by these functions together with the determinants with entries being partial derivatives of functions belonging to this particular generating set.*

Definition 4.1.3. It is easy to see that

$$U = \Delta^0(U) \subseteq \cdots \subseteq \Delta^k(U) \subseteq \Delta^{k+1}(U) \subseteq \cdots \subseteq \Delta^{n+1}(U) = \mathcal{E}(n).$$

The largest Δ^k such that $\Delta^k(U) \subsetneq \mathcal{E}(n)$ is called the *critical Jacobian extension.*

Remark. The critical extension of an ideal U is $\Delta^{n-r} = \Delta_{r+1}$, where $r = \mathsf{rank}\, U$ is the *rank* of the ideal U, which is defined as $\mathsf{rank}\, U = \mathsf{dim}_{\mathbb{C}}\,(\mathfrak{m}^2 + U)/\mathfrak{m}^2$.

Now we are ready to state

Theorem-Definition 4.1.4. The following 4 definitions of Σ^{ij} agree (at least on a Zariski-open set):

(1) (Thom) Let $f : \mathbb{C}^n \to \mathbb{C}^m$ be a smooth map. We say $0 \in \Sigma^{ij}(f)$ if $0 \in \Sigma^i(f)$, $\Sigma^i(f)$ is a smooth submanifold with the expected codimension, and $0 \in \Sigma^j(f|_{\Sigma^i(f)})$. Note: This works for "nice" maps $f : N^n \to M^m$; the relationship with the other definitions is that $\Sigma^{ij}(f) = (\mathcal{J}_2 f)^{-1}(\Sigma^{ij} \subset \mathcal{J}_2(n,m))$.

(2) (Boardman) Let $F = (f_1, \ldots, f_m) \in \mathcal{J}(n,m)$ be a jet of a map. $F \in \Sigma^{ij}$ if the ideal $U = (f_1, \ldots, f_m) \lhd \mathcal{E}(n)$ has successive critical extensions $\Delta^i(U)$ and $\Delta^j(\Delta^i(U))$.

(3) (Porteous, Boardman, Ronga) Let

$$F = (F_1, F_2) \in \mathcal{J}_2(n,m) = \mathsf{Hom}(\mathbb{C}^n, \mathbb{C}^m) \oplus \mathsf{Hom}(\mathsf{Sym}^2\mathbb{C}^n, \mathbb{C}^m).$$

Then $F \in \Sigma^{ij}$ if $\mathsf{dim}(\mathsf{ker}\, F_1) = i$ and $\mathsf{dim}(\mathsf{ker}(\mathsf{curry}\, \widehat{F}_2)) = j$, where

$$\begin{aligned} \widehat{F}_2 \ &: \ \mathsf{Sym}^2(\mathsf{ker}\, F_1) \longrightarrow \mathsf{Sym}^2\mathbb{C}^n \xrightarrow{F_2} \mathbb{C}^m \longrightarrow \mathsf{coker}\, F_1 \\ \mathsf{curry}\, \widehat{F}_2 \ &: \ \mathsf{ker}\, F_1 \longrightarrow \mathsf{Hom}(\mathsf{ker}\, F_1,\, \mathsf{coker}\, F_1). \end{aligned}$$

(4) (Porteous) With $F = (F_1, F_2)$ as before, $F \in \Sigma^{ij}$ if there exists $(\alpha_1, \alpha_2) \in \mathcal{J}_2^\circ(i,n)$, $\beta_1 \in \mathcal{J}_1^\circ(j,i)$ such that $F_1 \circ \alpha_1 = 0$ and

$$F_2 \circ (\alpha_1 \otimes (\alpha_1 \circ \beta_1)) + F_1 \circ \alpha_2 \circ (\mathsf{id} \otimes \beta_1) = 0 \ : \ \mathbb{C}^i \otimes \mathbb{C}^j \to \mathbb{C}^m,$$

and no such α, β exists with higher indices. See Appendix A.1, in particular Figure 12, to gain some intuition about such expressions. Note: the second equation can be rewritten as

$$F_2\big(\alpha_1(x), \alpha_1(\beta_1(y))\big) + F_1\big(\alpha_2(x, \beta_1(y))\big) = 0 \quad \forall\, x \in \mathbb{C}^i,\, y \in \mathbb{C}^j.$$

Proof.

(1) ⇔ (2). See [**Boa67**], Section 6.

(2) ⇔ (3). See [**Boa67**], Section 7.

(4) ⇒ (3). The correspondence between the two definitions will be $\mathsf{im}(\alpha_1) = \mathsf{ker}(F_1)$ and $\mathsf{im}(\alpha_1 \circ \beta) = \mathsf{ker}(\mathsf{curry}\widehat{F}_2)$. Factoring out by $\mathsf{coker}(F_1)$ in the target of the second equation of (4), the second term vanishes by definition, and the first term becomes equivalent to (3).

(3) ⇒ (4). Choose α_1, β_1 such that $\mathsf{im}(\alpha_1) = \mathsf{ker}(F_1)$ and $\mathsf{im}(\alpha_1 \circ \beta) = \mathsf{ker}(\mathsf{curry}\widehat{F}_2)$, and choose α_2 to be

$$\alpha_2 = -\Big(F_1\big|_{\mathsf{coim}\, F_1}^{\mathsf{im}\, F_1}\Big)^{-1} \circ \Big(F_2\big|^{\mathsf{im}(F_1)}\Big) \circ \big(\alpha_1 \otimes \alpha_1\big)$$

(note that F_1 has rank $n - i$). □

Remark. All these definitions generalize for higher order Thom-Boardman singularities (eg. Σ^{ijk}).

4.2. Ronga's formula

Ronga was the first to study in detail the Thom polynomials of Σ^{ij} singularities in [**Ron72**]. By constructing a resolution of the closure of Σ^{ij}, he derived a pushforward formula, which we present here (transcribed into more modern language).

Let V^n and W^m be representations of GL_n and GL_m, respectively; $\mathsf{Gr}_i(V)$ the Grassmannian of i-planes, and $0 \to I \to V \to Q \to 0$ the tautological exact sequence over it. The (total space of the) Grassmannian bundle $\mathsf{Gr}_j(I)$ is nothing else but the partial flag variety $\mathsf{Fl}_{ij}(V)$; we will denote its tautological bundle with J. Denote by p_1, p_2 and $\pi = p_1 \circ p_2$ the collapse maps $p_1 : \mathsf{Gr}_i(V) \to \mathsf{pt}$, $p_2 : \mathsf{Gr}_j(I) \to \mathsf{Gr}_i(V)$ and $\pi : \mathsf{Fl}_{ij}(V) \to \mathsf{pt}$, respectively. Then

THEOREM 4.2.1 (Ronga).

$$\big[\Sigma^{ij}(V,W)\big] = \pi_*\Big\{e(\mathsf{Hom}(I,W)) \cdot s_{(i\odot j)^{m-n+i}}\big(W - (I\odot J + Q)\big)\Big\}$$

We will re-derive this theorem via equivariant localization in Section 4.3.3.

This theorem gives an algorithm to compute the Thom polynomials of Σ^{ij}, since we can use the pushforward formula (Theorem A.4.1) to compute the pushforwards along p_2 and p_1 after the separation of variables using the formulae in A.2. However, this algorithm is effective only for very small cases, and it's hard to derive general formulae from it (except in the case $i = j = 1$, [**Ron72**]). Nevertheless, we can use it to prove the following theorem, which we will use in section 4.4.

THEOREM 4.2.2. *Write the Thom polynomial of* $\Sigma^{ij}(n,m)$ *as a linear combination of Schur polynomials:* $[\Sigma^{ij}(V^n, W^m)] = \sum e^\lambda s_\lambda(W - V)$*, where* $e^\lambda \in \mathbb{Z}$ *are (nonnegative integer) coefficients. Then* $e^\lambda = 0$ *if* λ *satisfies any of the following three conditions:*

(a) $i^{(m-n+i)} \not\subset \lambda$
(b) $(i+1)^{(m-n+i+1)} \subset \lambda$
(c) $\lambda_1 > \mu = i + i \odot j$

REMARK. (c) follows from the general theory, too; (a) and especially (b) is what is important here. In the language of Chapter 3, this statement means that using the shifted 'base line' $(m-n+i, n-i)$, for all terms appearing in the Thom series, we have $\ell(\nu_+) = i$ and $\ell(\nu_-) = i \odot j$ (cf. Figure 3). Note that (b) is in general false for higher-order singularities: Already $A_3 = \Sigma^{111}$ is a counterexample (that is, the Thom polynomials of A_3 contain nonzero terms $e^\lambda s_\lambda$ with λ satisfying (b)).

Proof of Theorem 4.2.2. We will use the shorthand notations $h = m - n + i$ and $k = i \odot j$. All three claims will be the consequence of the following computation. First, substituting the trivial m dimensional representation for W and using the expansion (see A.2)

$$s_\lambda(A+B) = \sum_{\mu,\nu} c^\lambda_{\mu\nu} \cdot s_\mu(A) s_\nu(B)$$

—which, for the special case $\lambda = h^k$ gives $s_{(h^k)}(A+B) = \sum_{\mu\subset h^k} s_\mu(A) s_{\complement\mu}(B)$— we get

$$\big[\Sigma^{ij}(-V)\big] = \pm \sum_{\lambda\subset h^k} p_{1*}\Big[s_{(m^i)}(I) s_\lambda(V-I) \cdot p_{2*} s_{\complement\lambda}(I\odot J)\Big].$$

We are not interested in the exact result of the inner pushforward; instead we just set

$$\pm\, p_{2*} s_{\complement\lambda}(I\odot J) = \sum_{\ell(\mu)\le i} f^\mu_\lambda \cdot s_\mu(I),$$

where f^μ_λ are some coefficients. Using the above expansion again, now for $s_\lambda(V-I)$ we get:

$$\big[\Sigma^{ij}(r)\big](-V) = \sum_{\lambda\subset h^k} \sum_{\alpha,\beta\subset\lambda} \sum_{\ell(\mu)\le i} c^\lambda_{\alpha\beta} f^\mu_\lambda \cdot s_\alpha(V) \cdot p_{1*}\Big[s_{(m^i)}(I) s_{\widetilde{\beta}}(I^\vee) s_\mu(I)\Big].$$

Using the Littlewood-Richardson rule A.2.5, Theorem A.4.1 and that the rank of I is i, it follows immediately that

$$p_{1*}\Big[s_{(m^i)}(I)s_{\widetilde{\beta}}(I^\vee)s_\mu(I)\Big] = \pm p_{1*}\Big[s_{(m^i)}(I)s_{\widetilde{\beta}}(I)s_\mu(I)\Big] = \sum_{\ell(\gamma)\leq i} g_\gamma \cdot s_{(h^i+\gamma)}(V),$$

where the g_γ's are integer coefficients. Now, we see that $[\Sigma^{ij}](-V)$ is a linear combination of terms of the form $s_\alpha(V)s_{(h^i+\gamma)}(V)$, where $\alpha \subset h^k$ and $\ell(\gamma) \leq i$. From the Littlewood-Richardson rule it follows directly that the expansion of such a term satisfies the *duals* of all three claims of the theorem, that is, the duals of the partitions appearing in the expansions satisfy the three conditions; thus, using the identity $s_\lambda(-V) = s_{\widetilde{\lambda}}(V^\vee) = (-1)^{|\lambda|}s_{\widetilde{\lambda}}(V)$ the theorem follows. □

4.3. Localization

In this section, we will apply equivariant localization to derive a formula for the Thom polynomials of Σ^{ij} singularities. As usual with such efforts, the main difficulty is that the space we want to localize over is not compact, therefore we have to compactify it; but this compactification cannot be arbitrary, since our space has a vector bundle over it which has to extend to the compactified space. In other words, there is a canonical compactification inside a Grassmannian, and we require a dominant map to that. While in this particular case the canonical compactification is simple enough to understand directly, that's not the case in general; thus first we present another, rather convoluted construction, which we hope has some chance to work in some other cases too (eg. it can be adapted to work for the A_3 singularity, see Chapter 5).

4.3.1. The probe model for Thom-Boardman singularities. Porteous proposed in [**Por83**] the following definition of Thom-Boardman singularities (which is the generalization of the fourth definition of Σ^{ij} in 4.1.4 above). As we will see this definition is well-suited for the purposes of localization.

Recall the following notations:

$$\begin{aligned}
\mathcal{J}_d(V,W) &:= \bigoplus_{k=1}^{d} \mathsf{Hom}(\mathsf{Sym}^k V, W) \\
\mathcal{J}_d^\circ(V,W) &:= \big\{\, (\varphi_1,\varphi_2,\ldots,\varphi_d) \in \mathcal{J}_d(V,W) \ \text{ s.t. } \ \mathsf{ker}\,\varphi_1 = \{0\} \,\big\} \\
\mathsf{Diff}_d(V) &:= \mathcal{J}_d^\circ(V,V) \\
\mathcal{J}_d(n,m) &:= \mathcal{J}_d(\mathbb{C}^n,\mathbb{C}^m), \ \text{etc.}
\end{aligned}$$

Let $F \in \mathcal{J}_d(n,m)$ be the d-jet of an analytic map; we would like to decide whether it is in the singularity set Σ^I for a given $I = (i_1 \geq i_2 \geq \cdots \geq i_d)$.

Proposition 4.3.1 ([**Por83**]). *$F \in \Sigma^I$ if and only if there exists a* probe $(\alpha^1,\alpha^2,\ldots,\alpha^d)$, *where $\alpha^k \in \mathcal{J}^\circ_{d+1-k}(i_k,i_{k-1})$ (using the convention that $i_0 = n$), such that the following d equations are satisfied:*

$$\begin{aligned}
0 &= \mathrm{d}(F\circ\alpha^1)|_0 \\
0 &= \mathrm{d}(\mathrm{d}(F\circ\alpha^1)\circ\alpha^2)|_0 \\
0 &= \mathrm{d}(\mathrm{d}(\mathrm{d}(F\circ\alpha^1)\circ\alpha^2)\circ\alpha^3)|_0 \\
&\ \ \vdots
\end{aligned}$$

and no such probe exists for higher Boardman indices.

Remark. An important property of these equations is that they are *linear* in the unknown F; thus for a fixed probe $\{\alpha^{(i)}\}$, the solutions form a linear subspace of $\mathcal{J}_d(V,W)$. This means that they separate the "trivial" part of Σ^I (the linear fibers) from the "essence" (the moduli space of probes).

Another very important observation is that the equations are *filtered*: The solution space of the first k equations can be determined without looking on the remaining equations, for any k.

The main difficulty with the application of this theorem is that such a probe is not at all unique, and in general fails to be unique in complicated ways. To start with, if we are given (jets of) diffeomorphisms $\psi^1,\dots,\psi^d$, where $\psi^k \in \mathsf{Diff}_{d+1-k}(i_k)$, and a probe $(\alpha^1,\dots,\alpha^d)$ for F, then we can define a new probe $(\tilde\alpha^1,\dots,\tilde\alpha^d)$ by the following diagram:

$$\begin{array}{ccccccccccc}
\mathbb{C}^m & \xleftarrow{F} & \mathbb{C}^n & \xleftarrow{\alpha^1} & \mathbb{C}^{i_1} & \xleftarrow{\alpha^2} & \mathbb{C}^{i_2} & \xleftarrow{\alpha^3} & \cdots & \xleftarrow{\alpha^d} & \mathbb{C}^{i_d} \\
 & \nwarrow & \Vert & & \downarrow{\scriptstyle\psi^1} & & \downarrow{\scriptstyle\psi^2} & & & & \downarrow{\scriptstyle\psi^d} \\
 & & \mathbb{C}^n & \xleftarrow{\tilde\alpha^1} & \mathbb{C}^{i_1} & \xleftarrow{\tilde\alpha^2} & \mathbb{C}^{i_2} & \xleftarrow{\tilde\alpha^3} & \cdots & \xleftarrow{\tilde\alpha^d} & \mathbb{C}^{i_d}
\end{array}$$

Thus the group $G_I = \prod_{k=1}^d \mathsf{Diff}_{d+1-k}(i_k)$ acts on the space of probes $\mathcal{P}_I = \prod_{k=1}^d \mathcal{J}^\circ_{d+1-k}(i_k,i_{k-1})$, and we are only interested in the factor space $\mathcal{P}/G$. However, that is unfortunately not all the ambiguity the probes have. Consider for example the case $d=2$, $I=(i,j)$. We will use the symbols $\alpha_1,\alpha_2,\beta_1$, resp. F_1,F_2, for the components of the probe, resp. the map F:

$$\begin{array}{rcccl}
(F_1,F_2) & \in & \mathcal{J}_2(n,m) & = & \mathsf{Hom}(\mathbb{C}^n,\mathbb{C}^m)\oplus\mathsf{Hom}(\mathsf{Sym}^2\mathbb{C}^n,\mathbb{C}^m), \\
(\alpha_1,\alpha_2) & \in & \mathcal{J}^\circ_2(i,n) & = & \mathsf{Hom}^\circ(\mathbb{C}^i,\mathbb{C}^n)\oplus\mathsf{Hom}(\mathsf{Sym}^2\mathbb{C}^i,\mathbb{C}^n), \\
\beta_1 & \in & \mathcal{J}^\circ_1(j,i) & = & \mathsf{Hom}^\circ(\mathbb{C}^j,\mathbb{C}^i).
\end{array}$$

It's not hard to compute $\mathcal{P}_{ij}/G_{ij}$ (see Appendix A.1):

$$\mathcal{P}_{ij}/G_{ij} = \big\{\,(\mathsf{im}(\alpha_1),\mathsf{im}(\alpha_1\circ\beta_1),\bar\alpha_2)\,\big\} \in \mathsf{Fl}_{ij}(n)\times\mathsf{Hom}(\mathsf{Sym}^2\mathbb{C}^i,\ \mathbb{C}^n/\mathsf{im}(\alpha_1)).$$

But the equations for the probe written out in the components are

$$\begin{array}{rcll}
0 & = & F_1(\alpha_1 v) & \forall\, v\in\mathbb{C}^i \\
0 & = & F_2(\alpha_1 v,\alpha_1(\beta_1 w)) + F_1(\alpha_2(v,\beta_1 w)) & \forall\,(v,w)\in\mathbb{C}^i\times\mathbb{C}^j
\end{array}$$

which tells us that what matters is not α_2 itself (which is a symmetric bilinear map), but the restriction of α_2 to $\mathsf{im}(\beta_1)$ in *one of its inputs.*

In general, solving the equations for F gives a map from the space of probes $\mathcal{P}_I$ to a Grassmannian

$$\mathsf{sol}_I : \mathcal{P}_I \to \mathsf{Gr}^\mu(\mathcal{J}_d(V,W)),$$

and the space we are really interested in is the factor space $\mathcal{M}_I = \mathcal{P}_I/\!\sim$, where we call two probes $\alpha^\bullet$ and $\beta^\bullet$ equivalent if $\mathsf{sol}(\alpha^\bullet)=\mathsf{sol}(\beta^\bullet)$. This space is of course isomorphic to the image $\mathsf{im}(\mathsf{sol})$. We will call $\mathcal{M}_I$ the *moduli space of probes.* Note that it also depends on $n=\mathsf{dim}(V)$; however, it does not depend on W: it comes with a canonical embedding $j_{I,W}:\mathcal{M}_I\to\mathsf{Gr}^\mu(J_d(V,W))$ for any W. The natural choice to work with is $W=\mathbb{C}$, because it's the simplest possible, and also because $\mathcal{J}_d(V,\mathbb{C})$ comes with an extra structure: It is a (nilpotent) ring, and in fact, $\mathcal{M}_I$ embeds into the space of *ideals* of this ring.

Example. Consider the simplest possible case of Σ^i. In this case, we have

$$\begin{aligned}
\mathcal{J}_1(\mathbb{C}^n,\mathbb{C}^m) &= \mathsf{Hom}(\mathbb{C}^n,\mathbb{C}^m)\\
\mathcal{P} &= \mathsf{Hom}^\circ(\mathbb{C}^i,\mathbb{C}^n)\\
\mathsf{sol}(\alpha) &= \big\{F\in\mathsf{Hom}(\mathbb{C}^n,\mathbb{C}^m)\ :\ F|_{\mathsf{im}(\alpha)}=0\big\}\\
\mathcal{M} &= \mathsf{Gr}_i(\mathbb{C}^n)\\
j &: \mathsf{Gr}_i(\mathbb{C}^n)\to\mathsf{Gr}_{(n-i)m}\big(\mathsf{Hom}(\mathbb{C}^n,\mathbb{C}^m)\big)\\
&\quad j(\Lambda) = (\mathbb{C}^n/\Lambda)^\vee\otimes\mathbb{C}^m
\end{aligned}$$

In general, the space $\mathcal{M}_I$ can be quite complicated. There are however two cases which we understand pretty well:

Theorem 4.3.2. *For Σ^{ij}, the moduli space of probes $\mathcal{M}_{ij}$ is the vector bundle*

$$\mathcal{M}_{ij}=\mathsf{Hom}(I\odot J,\,\mathbb{C}^n/I)\to\mathsf{Fl}_{ij}(\mathbb{C}^n)$$

where I^i and J^j are the tautological bundles over the partial flag variety $\mathsf{Fl}_{ij}(\mathbb{C}^n)$. The solutions over a fixed probe $(I,J,\widehat{\alpha}_2)\in\mathcal{M}_{ij}$ are the pairs $(F_1,F_2)\in\mathcal{J}_2(n,m)$ such that

$$\begin{aligned}
0&=F_1|_I\\
0&=F_2|_{I\odot J}+\widehat{F}_1\circ\widehat{\alpha}_2
\end{aligned}\tag{14}$$

where $\widehat{F}_1:\mathbb{C}^n/I\to\mathbb{C}^m$ is obtained from F_1 using the first equation: Since $F_1|_I=0$, F_1 factors through the linear quotient $\mathbb{C}^n/I$. The factor map $q:\mathcal{P}_{ij}=\mathcal{J}_2^\circ(i,n)\times\mathcal{J}_1^\circ(j,i)\to\mathcal{M}_{ij}$ is given by

$$\begin{aligned}
I &= \mathsf{im}(\alpha_1)\ \subset\ \mathbb{C}^n\\
J &= \mathsf{im}(\alpha_1\circ\beta_1)\ \subset\ I\ \subset\ \mathbb{C}^n\\
\widehat{\alpha}_2 &= (\mathbb{C}^n\to\mathbb{C}^n/I)\circ\alpha_2\circ(\alpha_1^{-1}\otimes\alpha_1^{-1})|_{I\odot J}\ :\ I\odot J\to\mathbb{C}^n/I
\end{aligned}$$

Theorem 4.3.3 ([**BSz06**], [**Gaf83**]). *For $A_d=\Sigma^{11\dots1}$, the moduli of probes (which in this particular case is also called the moduli of* test curves*) is the quotient*

$$\mathcal{M}_d=\mathcal{J}_d^\circ(1,n)/\mathsf{Diff}_d(1),$$

that is, jets of curves in $\mathbb{C}^n$ up to reparameterization. The solutions in $\mathcal{J}_d(n,m)$ for a fixed test curve $\gamma\in\mathcal{J}_d^\circ(1,n)$ are

$$\mathsf{sol}(\gamma)=\big\{\,F\in\mathcal{J}_d(n,m)\ :\ F\circ\gamma=0\,\big\}.$$

Remark. Note that the group $\mathsf{Diff}_d(1)$ is *not* reductive, thus the usual techniques dealing with reductive group quotients (namely, Geometric Invariant Theory) do not apply. Still, this particular quotient is nice enough to enable us to understand it "by hand".

Proof of Theorem 4.3.2. First, we show that $\mathcal{M}_{ij}$ is the moduli space *set-theoretically*: two probes has the same solution space if and only if their image under the quotient map

$$q:\mathcal{P}_{ij}=\mathcal{J}_2^\circ(i,n)\times\mathcal{J}_1^\circ(j,i)\to\mathcal{M}_{ij}$$

is the same point. To see this, we will apply Gaussian elimination to Equations (14). We can assume without loss of generality that $\beta_1:\mathbb{C}^j\to\mathbb{C}^i$ and $\alpha_1:\mathbb{C}^i\to\mathbb{C}^n$ are just embeddings of the first j resp. i coordinates (since it can be achieved by a change of coordinates). This simplifies the equations considerably:

$$0 = F_1|_{\mathbb{C}^i}$$
$$0 = (F_2 + F_1 \circ \alpha_2)|_{\mathbb{C}^i \odot \mathbb{C}^j}$$

It may be easier to grasp when written in a matrix form

$$M \cdot \big[(F_2|_{\mathbb{C}^i\odot\mathbb{C}^j})\big|F_1\big]^{\mathsf{t}} = 0 \quad \text{where} \quad M = \begin{array}{c|c|c} \mathbb{C}^i \odot \mathbb{C}^j & \mathbb{C}^{n-i} & \mathbb{C}^i \\ \hline 0 & 0 & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \\ \hline \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & \multicolumn{2}{c}{\alpha_2} \end{array}$$

or more formally, with wedge products (μ stands for $\mu = i + i \odot j$ here):

$$\mathsf{sol} : \mathcal{P}_{ij} \to \mathsf{Gr}_\mu(\mathbb{C}^n \oplus \mathsf{Sym}^2\mathbb{C}^n) \subset \mathbb{P}[\wedge^\mu(\mathbb{C}^n \oplus \mathsf{Sym}^2\mathbb{C}^n)]$$
$$(\mathsf{id}, \alpha_2, \mathsf{id}) \mapsto \left[\left(\bigwedge_{i=1}^{i} (\, e_i \,,\, 0 \,)\right) \wedge \left(\bigwedge_{i,j} (\, \alpha_2(e_i \otimes e_j) \,,\, e_i \otimes e_j \,)\right)\right]$$

During the Gaussian elimination, the rightmost region of α_2 in the matrix is eliminated, which shows that $\mathcal{M}_{ij}$ is indeed the set we claimed it to be.

Second, we have to show that the algebraic structure on $\mathcal{M}_{ij}$ is the right one, that is, that map q is algebraic when we put on $\mathcal{M}_{ij}$ the algebraic structure coming from it being a vector bundle over a flag manifold. But that's clear from the description of q given above. ☐

4.3.2. The compactifications. We would like to apply equivariant localization in the following situation:

$$\begin{array}{ccccc} R & \xleftarrow{j} & \mathrm{Sol} & \xrightarrow{\pi} & \mathcal{J}_d(n,m) \\ \downarrow & & \downarrow & & \downarrow \\ \mathsf{Gr}^\mu(\mathcal{J}_d(n,1)) & \xleftarrow{j} & \mathcal{M} & \xrightarrow{\pi} & \mathsf{pt} \end{array}$$

where j is the embedding discussed above, π is the projection, and we are interested in the class $[\Sigma] = [\pi(\mathrm{Sol})] = \pi_*[\mathrm{Sol}] \in H^*_{\mathsf{GL}_n \times \mathsf{GL}_m}(\mathcal{J}_d(n,m))$. However, for this to work, $\mathcal{M}$ has to be compact (since otherwise the pushforward map π_*—which is basically what we want to compute via localization—is not even defined); but in general, our moduli spaces are never compact (the only exception being Σ^i). Thus we have to compactify the moduli spaces, and we also need to extend to bundle Sol to the compactified moduli space $\widehat{\mathcal{M}}$.

There is a canonical compactification to start with, namely the closure of $j(\mathcal{M})$ in the Grassmannian $\mathsf{Gr}^\mu(\mathcal{J}_d(n,1))$. While in this particular case (Σ^{ij}) we can understand this space directly, in general it can be very complicated; so first we show a different (and rather complicated) route, which we hope has some chance to work in other situations as well (eg. for A_3, see Chapter 5). After that, we will show the "direct" route, in Section 4.3.2.2.

4.3.2.1. *The blow-up method.* What we will do instead is to consider a natural but *wrong* compactification, in the sense that the bundle Sol (or equivalently, the embedding j) *does not* extend to it; and then "repair" this problem with repeated blow-ups. (The embedding j will become a rational map, and it is well known (eg. [**Har95**] Theorem 7.21) that any rational map can be resolved by a finite sequence of blow-ups, thus it at least sounds reasonable). Our first candidate compactification will be simply the projective bundle

$$\mathbb{P}\big[1 \oplus \mathsf{Hom}(I \odot J,\, \mathbb{C}^n/I)\big] \to \mathsf{Fl}_{ij}(\mathbb{C}^n).$$

We will denote the new coordinate (on the trivial line bundle 1) by ξ; the torus $\mathbb{T}^n \subset \mathsf{GL}_n$ should act on it trivially so that the compactification is equivariant. To work with projective coordinates, we have to homogenize our equations. This is very straightforward:

$$0 = F_1|_I$$
$$0 = \xi F_2|_{I\odot J} + \widehat{F}_1 \circ \widehat{\alpha}_2$$

or, in matrix form $(\widehat{F}_1, F_2|_{I\odot J})|_{\mathsf{im}A} = 0$, where A is the matrix

$$\begin{array}{c|c|c|c} & I\odot J & \mathbb{C}^n/I & I \\ \hline I & & & \begin{smallmatrix}1 & & \\ & \ddots & \\ & & 1\end{smallmatrix} \\ \hline I\odot J & \begin{smallmatrix}\xi & & \\ & \ddots & \\ & & \xi\end{smallmatrix} & \widehat{\alpha}_2 & \end{array}$$

Note that our convention is that the linear map associated with a matrix A is $x \mapsto xA$ (as opposed to the more popular $x \mapsto Ax$).

It is easy to see which are the "bad points", where the map sol does not extends to: The points where the rank of the matrix above is less than μ, that is, where $\xi = 0$ and the rank of $\widehat{\alpha}_2$ is not maximal. These "bad points" are stratified by the rank of $\widehat{\alpha}_2$:

$$\Sigma_1 \cup \Sigma_2 \cup \cdots \cup \Sigma_{i\odot j} = \mathbb{P}\mathsf{Hom}(I \odot J, \mathbb{C}^n/I) = \big\{\xi = 0\big\} \subset \mathbb{P}\big[1 \oplus \mathsf{Hom}(I \odot J, \mathbb{C}^n/I)\big]$$

Thus our strategy will be the following: First we blow up Σ_1 (it is a smooth subvariety), then we blow up the strict transform of Σ_2, and so on until $\Sigma_{i\odot j-1}$.

Remark. For this to work, we have to assume that $n - i = \mathsf{dim}(\mathbb{C}^n/I) \geq \mathsf{dim}(I \odot J) = i \odot j$, or, rearranging it, $n \geq \mu = i + i \odot j$.

Theorem-Definition 4.3.4. This way we got a tower of GL_n-equivariant blow-ups

$$\widehat{\mathcal{M}}_{ij} := B^{(i\odot j)} \to \cdots \to B^{(3)} \to B^{(2)} \to B^{(1)} = \mathbb{P}\big[1 \oplus \mathsf{Hom}(I \odot J, \mathbb{C}^n/I)\big] \to \mathsf{Fl}_{ij}(\mathbb{C}^n)$$

and each $B^{(k)}$ is stratified (by the rank):

$$B^{(k)} = U \cup E_1^\circ \cup \cdots \cup E_{k-1}^\circ \cup \Sigma_k^{(k)} \cup \Sigma_{k+1}^{(k)} \cup \Sigma_{k+2}^{(k)} \cup \cdots \cup \Sigma_{i\odot j}^{(k)}$$

such that

- $\Sigma_k^{(k)} \subset B^{(k)}$ is a smooth subvariety;
- $B^{(k+1)}$ is the blow-up of $B^{(k)}$ along $\Sigma_k^{(k)}$;
- $\Sigma_l^{(k+1)} \subset B^{(k+1)}$ is the strict transform of $\Sigma_l^{(k)} \subset B^{(k)}$ for $l > k$;
- $E_k^\circ \subset E_k$ is the exceptional divisor of the kth blow-up, minus the strict transforms.

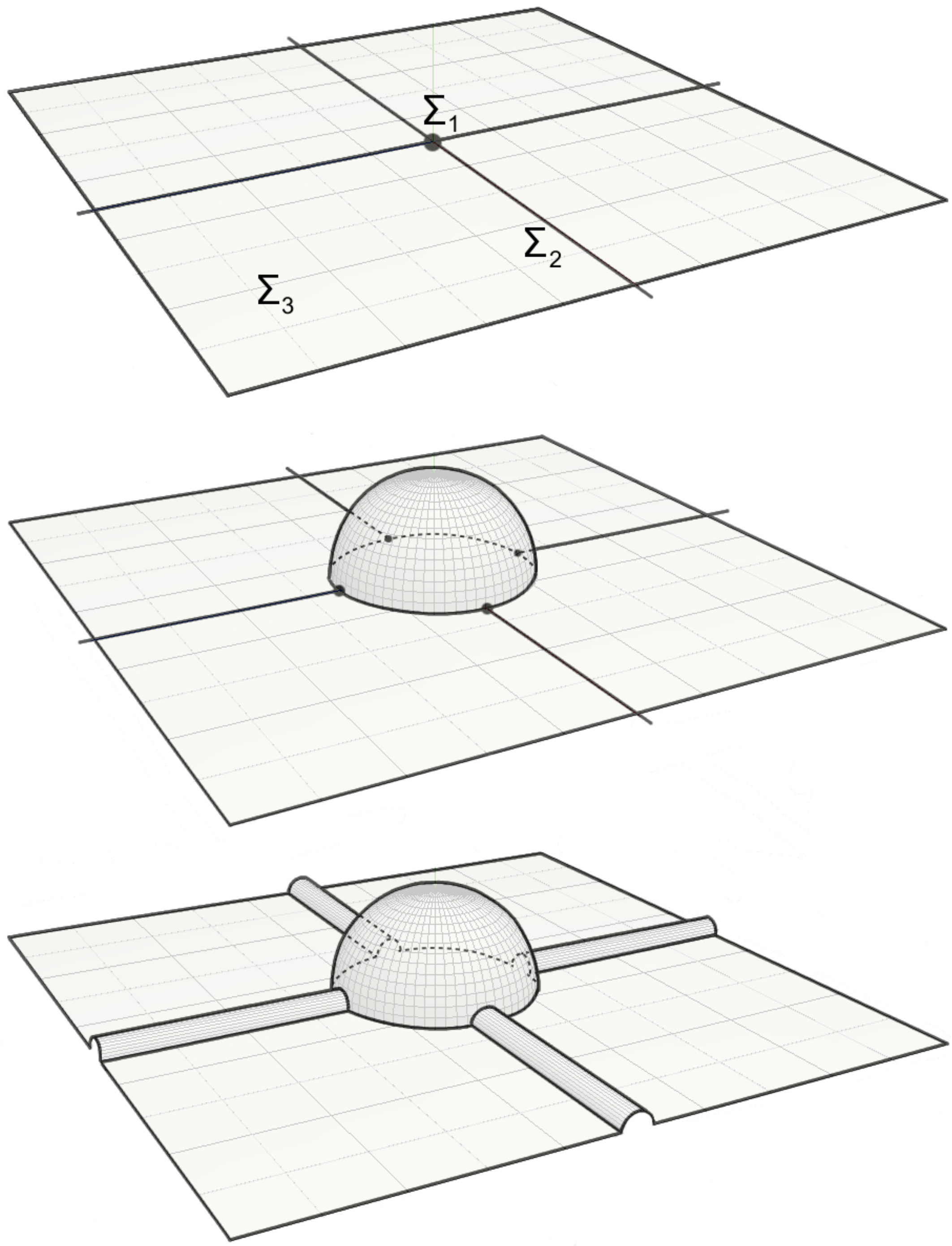


FIGURE 4. Schematic drawing of the blow-up process. The plane symbolizes $\{\xi = 0\}$.

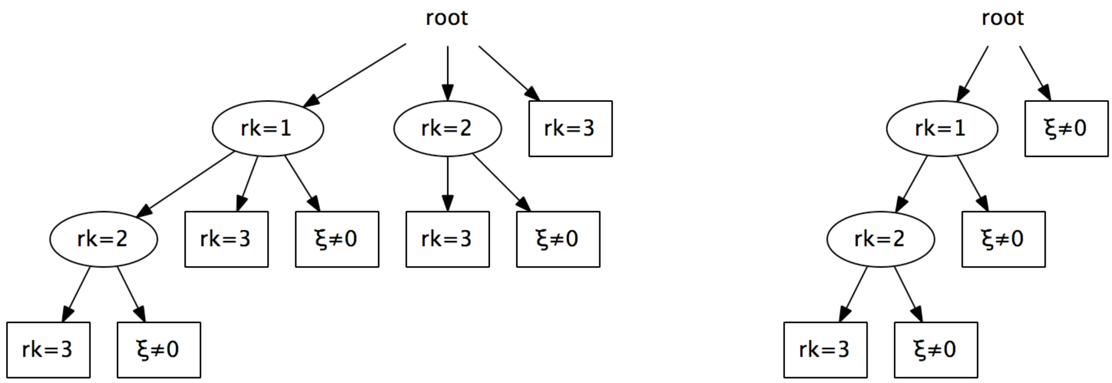


FIGURE 5. The trees indexing the strata (left) and types of fixed points (right) in the $i \odot j = 3$ case.

The open set U is simply $U = \{\xi \neq 0\} \cong \mathsf{Hom}(I \odot J, \mathbb{C}^n/I)$. The blow-ups will be denoted by $\pi_k : B^{(k)} \to B^{(k-1)}$.

This blow-up process is illustrated in the $i \odot j = 3$ case on Figure 4. The three pictures show $B^{(1)}$, $B^{(2)}$ and $B^{(3)}$, respectively; the planes represent the $\xi = 0$ hyperplane.

Sketch of proof. The only thing not clear is that $\Sigma_{k+1}^{(k+1)} \subset B^{(k+1)}$ is smooth; this follows from the fact that normal cone of the (closure of the) rank variety $\Sigma_{k+1} \subset \mathsf{Mat}_{n\times m}$ over Σ_k is the cone of the Segre varieties of the projective normal bundle, and the Segre varieties are smooth, thus the blow-up completely resolves the singularity. $\square$

We can further stratify the sets $\Sigma_l^{(k)}$ (and thus $B^{(k)}$) by distinguishing the points added in the process of strict transform, that is, those which are in the exceptional divisor E_k. This finer stratification, which is the common refinement of the coarser stratifications at the different levels $B^{(\leq k)}$, is best described by pictures. Looking at Figure 4, the first drawing, representing $B^{(1)}$, has 3 strata (not counting U); the second, $B^{(2)}$ has 5; finally, $B^{(3)}$ has 7. The strata are indexed by nodes of trees: Figures 5 (left) and 6 shows these trees in the $i \odot j = 3$ and $i \odot j = 4$ cases, respectively (the $\mathsf{root} \to \{\xi \neq 0\}$ edge is missing from these trees; it would represent the open stratum U). The (boxed) leaves of the trees index the strata of the final stratification $B^{(i\odot j)}$.

The blow-up, by definition, replaces the subvariety X we are blowing up by the projective bundle associated to its normal bundle. We can visualize that by imagining that we are moving away from X by an infinitesimal distance, into different directions. From this point of view, the strata (the nodes of the trees) enumerate the combinatorial possibilities of "travelling" between the subsets of different rank.

The reason why we did this complicated blow-up process is of course the following

PROPOSITION 4.3.5. *The rational map* $\mathsf{sol} : B^{(1)} \dashrightarrow \mathsf{Gr}^{\mu}(\mathcal{J}_2(n))$, *with domain of regularity* $V = U \cup \Sigma_{i\odot j}$, *extends to a regular (and birational) map* $\mathsf{sol} : B^{(i\odot j)} \to \mathsf{Gr}^{\mu}(\mathcal{J}_2(n))$.

Before proving this theorem, let us construct, for all strata (actually for any point in the indeterminacy locus $Z = B^{(i\odot j)} - V$), curves $\gamma(t) : \mathbb{C}^\times \to V$ such that $\lim_{t\to 0} \gamma(t)$ lands in

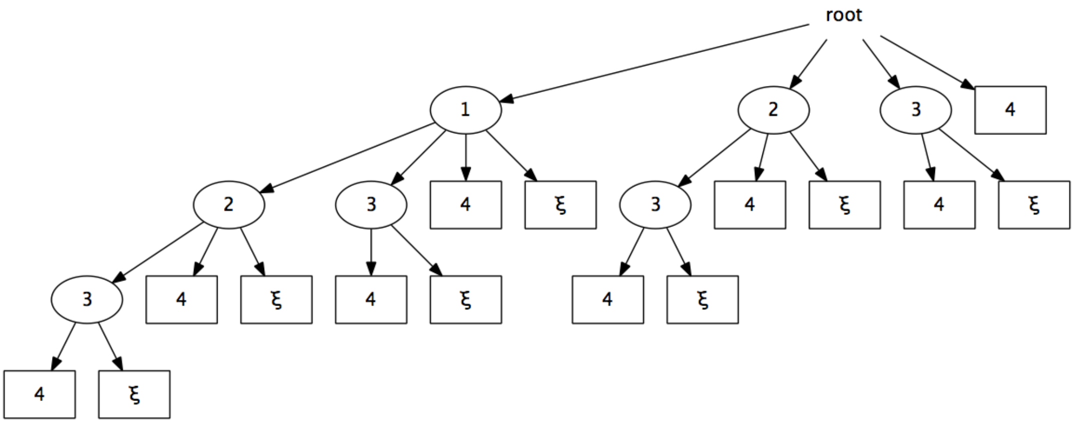


FIGURE 6. The tree indexing the strata in the $i \odot j = 4$ case.

the given strata (is the given point). This is easy to do: Consider the path in the index tree from the root to the leaf corresponding to the given strata. This path, for example

$$\mathsf{root} \to \{\mathsf{rk} = 1\} \to \{\mathsf{rk} = 3\} \to \{\mathsf{rk} = 4\} \to \{\xi \neq 0\},$$

describes how we "travel" between the different rank varieties, and thus can be directly translated into a curve $\gamma(t)$, for example in this case (and $i \odot j = 5$, $n - i = 7$)

$$[\,\xi(t)\cdot\mathsf{id}_{I\odot J}\,|\,\widehat{\alpha}_2(t)\,] = \left[\begin{array}{ccccc|ccccccc} t^3 & & & & & 1 & & & & & & \\ & t^3 & & & & & t & & & & & \\ & & t^3 & & & & & t & & & & \\ & & & t^3 & & & & & t^2 & & & \\ & & & & t^3 & & & & & 0 & 0 & 0 \end{array}\right]$$

works (the whole curve lives over a fixed flag $(I, J) \in \mathsf{Fl}_{ij}(n)$). More formally: If the path describing the strata is

$$\mathsf{root} \to \{\mathsf{rk} = r_1\} \to \{\mathsf{rk} = r_2\} \to \cdots \to \{\mathsf{rk} = r_k\} \to \{\xi \neq 0\},$$

we put

$$\big(\, \underbrace{1, \ldots, 1}_{r_1}, \underbrace{t, \ldots, t}_{r_2 - r_1}, \underbrace{t^2, \ldots, t^2}_{r_3 - r_2}, \ldots, \underbrace{t^{k-1}, \ldots, t^{k-1}}_{r_k - r_{k-1}}, \underbrace{0, \ldots, 0}_{i\odot j - r_k} \,\big)$$

on the diagonal of $\widehat{\alpha}_2(t)$, and set $\xi(t) = t^k$; the other possibility is that last node is $\{\mathsf{rk} = i \odot j\}$, in which case we set $\xi(t) = 0$. For any given point in the indeterminacy locus $Z = B^{(i\odot j)} - V$, we can take a curve which looks like this in a suitable coordinate system. Note that from this form, it is very easy to read off the limit

$$\lim_{t\to 0} \mathsf{sol}(\gamma(t)) \in \mathsf{Gr}^{\mu}(\mathcal{J}_2(n));$$

in our example it is the subspace of linear functions in $\mathcal{J}_2(n) = (\mathsf{Sym}^2\mathbb{C}^n \oplus \mathbb{C}^n)^\vee$ vanishing on $K = (\mathsf{id} \oplus q_I^{-1})(L) \subset \mathsf{Sym}^2\mathbb{C}^n \oplus \mathbb{C}^n$,

$$\begin{aligned} L &= \big\{\, [\,0\,0\,0\,0\,*\,|\,*\,*\,*\,*\,0\,0\,0\,]\,\big\} \subset I \odot J \oplus (\mathbb{C}^n/I), \\ q_I \,&:\, \mathbb{C}^n \to \mathbb{C}^n/I. \end{aligned}$$

Next, we will extend the above construction, so that not only we can approach any point $z \in Z$ on a curve, but we can approach it from *any direction.* That is, given a tangent vector $v \in T_z B^{(i \odot j)} - T_z Z$, we want a curve $\gamma'(t) : \mathbb{C}^\times \to V$ such that

$$\lim_{t \to 0} \gamma'(t) = z \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}t} \gamma'(t)\big|_{t=0} = v.$$

To do that, we modify the existing curves; basically we can "shear" along the directions parallel to $\Sigma_k^{(k)}$. In our running example, $\widehat{\alpha}_2(t)$ will become the modified $\widehat{\alpha}_2'(t)$

$$\widehat{\alpha}_2'(t) = A_\lambda(t) + t \cdot \begin{bmatrix} & * & * & * & * & * & * \\ * & & & & & & \\ * & & & & & & \\ * & & & & & & \\ * & & & & & & \end{bmatrix} + t^2 \cdot \begin{bmatrix} \cdot & & & & & & \\ & & * & * & * & * & * \\ & * & & * & * & * & * \\ & * & * & & & & \\ & * & * & & & & \end{bmatrix} + t^3 \cdot \begin{bmatrix} \cdot & & & & & & \\ & \cdot & & & & & \\ & & \cdot & & & & \\ & & & & * & * & * \\ & & & * & & & \end{bmatrix} + t^4 \cdot \begin{bmatrix} \cdot & & & & & & \\ & \cdot & & & & & \\ & & \cdot & & & & \\ & & & \cdot & & & \\ & & & & * & * & * \end{bmatrix}$$

$$\xi'(t) = \lambda_\infty t^3 + * \cdot t^4 \qquad\qquad A_\lambda(t) = \begin{bmatrix} \lambda_1 & & & & & & \\ & \lambda_2 t & & & & & \\ & & \lambda_3 t & & & & \\ & & & \lambda_4 t^2 & & & \\ & & & & 0 & 0 & 0 \end{bmatrix}$$

where we can write are any (constant) numbers into the place of stars, and any *nonzero* numbers into the place of λ_i-s (the small dots in the matrices are only there to indicate to diagonal). Note that we overspecified the tangent vector by one dimension, but that causes no problems.

We can also describe the neighbourhood relationship between the strata. Given a stratum S by its path from the root to a leaf, we can get all the strata S' for which $S \subset \overline{S}'$, that is, those who can degenerate into it (in another words, the neighbouring strata) by taking all the possible proper subsets of nodes, and closing with a $\xi \neq 0$ node those which do not end correctly (that is, with either maximal rank or $\xi \neq 0$). This can be seen via induction. For example, the neighbours of ① → ② → ③ (which corresponds to the corner points on the semisphere in Figure 4) are: ①→②→Ⓧ, ①→③, ②→③, ①→Ⓧ, ②→Ⓧ, ③ and Ⓧ (the last one is just open open stratum U). The codimension of a stratum is the length (number of nodes) of the corresponding path, not counting the Ⓧ nodes.

Proof of Proposition 4.3.5. We show that the map sol extends from $V = B^{(i \odot j)} - Z$ to $B^{(i \odot j)}$ as a continuous map (in the complex topology). For this, simply apply Gauss elimination (alternatively, compute the wedge product of the rows) to the "sheared" matrix $[\, \xi'(t) \cdot \mathsf{id}_{I \odot J} \,|\, \widehat{\alpha}_2'(t) \,]$, and observe that its image does not depend on the the numbers substituted into the stars (and λ_i-s) in the limit $t \to 0$, that is, on the direction we are approaching from.

We also want to show that this limit depends continuously on the point $z \in Z$. Since we have the GL_n symmetry, and also each fixed fiber over $(I, J) \in \mathsf{Fl}_{ij}$ has the extra symmetry given by the action of $G = \mathsf{GL}(I \odot J) \times \mathsf{GL}(\mathbb{C}^n / I)$ with the strata there being precisely the G-orbits, the only interesting situation is when we degenerate from a larger stratum S' to a smaller one S. Consider a parametric family of curves $\gamma_\varepsilon(t)$ for which $\lim_{t=0}(\gamma_0(t)) = z \in S$

and $\lim_{t=0}(\gamma_{\neq 0}(t)) \in S'$; two representative examples of such families are

$$M_a = \left[\begin{array}{ccccc|cccccc} t^5 &&&&& 1 &&&&& \\ & t^5 &&&&& t &&&& \\ && t^5 &&&&& (t^2+\varepsilon_1 t) &&& \\ &&& t^5 &&&&& (t^3+\varepsilon_2 t) && \\ &&&& t^5 &&&&& t^4 & 0 \end{array}\right]$$

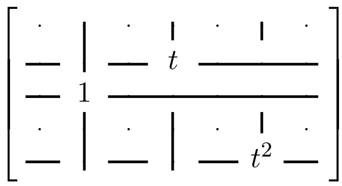
for $S = \textcircled{1} \to \textcircled{2} \to \textcircled{3} \to \textcircled{4} \to \textcircled{5}$ and $S' = \textcircled{1} \to \textcircled{4} \to \textcircled{5}$; and

$$M_b = \left[\begin{array}{ccccc|cccccc} t^3+\varepsilon t^2 &&&&& 1 &&&&& \\ & t^3+\varepsilon t^2 &&&&& t &&&& \\ && t^3+\varepsilon t^2 &&&&& t &&& \\ &&& t^3+\varepsilon t^2 &&&&& t^2 && \\ &&&& t^3+\varepsilon t^2 &&&&& t^2 & 0 \end{array}\right]$$

for $S = \textcircled{1} \to \textcircled{3} \to \textcircled{5}$ and $S' = \textcircled{1} \to \textcircled{3} \to \textcircled{\xi}$. It is a straightforward computation with wedge products to show that in such situtations, $\varepsilon \mapsto \lim_{t=0}(\gamma_\varepsilon(t))$ is continuous (there are three cases, depending on what happens with the last node).

Finally it is an application of the Riemann extension theorem (see eg. [**GH78**]) that the resulting extended map is holomorphic. □

Now we would like to apply Theorem A.3.7 for $\widehat{\mathcal{M}}_{ij} = B^{(i\odot j)}$. For this, we have to understand the fixed points. This is easy, since fixed points comes in a hierarchy: The map $\pi_k : B^{(k)} \to B^{(k-1)}$ maps the fixed point set $\mathsf{Fix}^{(k)}$ to $\mathsf{Fix}^{(k-1)}$. Since $B^{(1)}$ is just a projective bundle, its fixed points are triples $(I, J, [l])$, where $(I, J) \in \mathsf{Fl}_{ij}(\mathbb{C}^n)$ is a coordinate flag, and l is a coordinate axis. The blow-up process replaces fixed points by projectivized normal spaces, and thus introduces new fixed points. So a typical fixed points (over an implicitly specified fixed flag) looks like this:

$\widehat{\alpha}_2 =$  , $\qquad \xi = t^3.$

The lines in the matrix represent the tangent spaces of the $\Sigma_k^{(k)}$. Types of fixed points are indexed by a tree similar to that which indexes the strata, except that we cannot "jump over" ranks (Figure 5, right). More formally, a fixed point is specified by the following data:

- A coordinate flag (I, J), which we identify with subsets $J \subset I \subset n = \{1, 2, \ldots, n\}$, where $|I| = i$ and $|J| = j$;
- a natural number $0 \leq k \leq i \odot j$, specifying the type of the fixed point;
- two subsets $S \subset I \odot J$ and $R \subset n - I$, with $|S| = |R| = k$;
- two permutations $\sigma, \varrho \in \mathfrak{S}_k$.

A curve tending to the fixed point given by this data is

$$[\widehat{\alpha}_2(t)]_{ab} = \begin{cases} t^{i-1} & a = S_{\sigma_i} \text{ and } b = R_{\varrho_i}, \\ 0 & \text{otherwise;} \end{cases} \qquad \xi(t) = \begin{cases} t^k & k < i \odot j, \\ 0 & k = i \odot j; \end{cases}$$

(distinguishing between two cases for ξ is actually not necessary). The solution over that fixed point is the space of jets vanishing on $K = (I \odot J - S) \oplus q^{-1}(R)$.

Remark. The number of fixed points is

$$\#\mathsf{Fix} = \underbrace{\binom{n}{i}\binom{i}{j}}_{\text{the flag}} \cdot \sum_{k=0}^{i\odot j} \binom{i\odot j}{k}\binom{n-i}{k}(k!)^2.$$

An obvious disadvantage of our blow-up method is the combinatorial explosion of the fixed points; however, in this particular case, we can simplify a bit and will remove the $(k!)^2$ factor below.

Finally, we have to understand the tangent Euler class, that is, the tangent weights of a fixed point. As before, this is done inductively, from blow-up to blow-up. We start with the space $\mathsf{Hom}(I\odot J, \mathbb{C}^n/I)$, which has weights

$$\{ \ \alpha_k - \alpha_i - \alpha_j \ : \ (i,j) \in I\odot J, \ k \notin I \ \}.$$

However, to simplify the notation, for a moment consider just any torus representations A^n and B^m, $m \geq n$, with weights $\varphi_1, \varphi_2, \dots$ and $\psi_1, \psi_2, \dots$, and apply the blow-up process to $\mathbb{P}[1 \oplus \mathsf{Hom}(A,B)]$. Also, let us assume that R and S are simply the first k coordinates, and the permutations are trivial. These assumptions are of course not essential, but clarify the presentation significantly.

We will also use the notation $w_{ab} = \psi_b - \varphi_a$ for the weights of $\mathsf{Hom}(A,B)$. The fixed point $z^{(1)} \in \mathbb{P}[1 \oplus \mathsf{Hom}(A,B)]$ has tangent weights

$$N^{(0)} = \{ \ 0 - w_{11} \ \} \cup \{ \ w_{ab} - w_{11} \ : \ a \neq 1 \text{ or } b \neq 1 \ \},$$

which can be partitioned to the tangent and normal spaces $T^{(1)}$ and $N^{(1)}$ of $\Sigma_1^{(1)}$ at $z^{(1)}$:

$$T^{(1)} = \{ \ w_{ab} - w_{11} \ : \ a = 1 \text{ xor } b = 1 \ \},$$
$$N^{(1)} = \{ \ w_{ab} - w_{11} \ : \ a > 1 \text{ and } b > 1 \ \} \cup \{ \ 0 - w_{11} \ \},$$

where we use 'xor' as the standard abbreviation for 'exclusive or'. The blow-up leaves $T^{(1)}$ unchanged, and replaces $N^{(1)}$ by $L^{(1)} \oplus T_{z^{(2)}}\mathbb{P}N^{(1)}$, the line $L^{(1)} \subset N^{(1)} = N^{(0)}/T^{(1)}$ being the tautological line $\langle z^{(2)} \rangle + T^{(1)}$. Thus the tangent weights of $z^{(2)}$ are $T^{(1)} \cup L^{(1)} \cup T^{(2)} \cup N^{(2)}$,

$$L^{(1)} = \{ \ w_{22} - w_{11} \ \}$$
$$T^{(2)} = \{ \ (w_{ab} - w_{11}) - (w_{22} - w_{11}) \ : \ a = 2 \text{ xor } b = 2, \ a \geq 2, \ b \geq 2 \ \} =$$
$$= \{ \ w_{ab} - w_{22} \ : \ a = 2 \text{ xor } b = 2, \ a \geq 2, \ b \geq 2 \ \},$$
$$N^{(2)} = \{ \ (w_{ab} - w_{11}) - (w_{22} - w_{11}) \ : \ a > 2 \text{ and } b > 2 \ \} \cup \{ \ (0 - w_{11}) - (w_{22} - w_{11}) \ \} =$$
$$= \{ \ w_{ab} - w_{22} \ : \ a > 2 \text{ and } b > 2 \ \} \cup \{ \ 0 - w_{22} \ \}.$$

At this point, it is easy to spot the pattern, which is illustrated on Figure 7. On the picture, the matrix $\mathsf{Hom}(A,B)$ is shown; the arrows denote subtractions of weights.

At the end of the day, the weights of our fixed point will be

$$T^{(1)} \cup L^{(1)} \cup T^{(2)} \cup L^{(2)} \cup \dots \cup T^{(k)} \cup L^{(k)} \cup N^{(k)}.$$

Note that since in the last blow-up we always select the $\xi \neq 0$ direction, $L^{(k)} = \{0 - w_{kk}\}$ and $N^{(k)}$ will be simply

$$N^{(k)} = \{ \ w_{ab} \ : \ a > k \text{ and } b > k \ \},$$

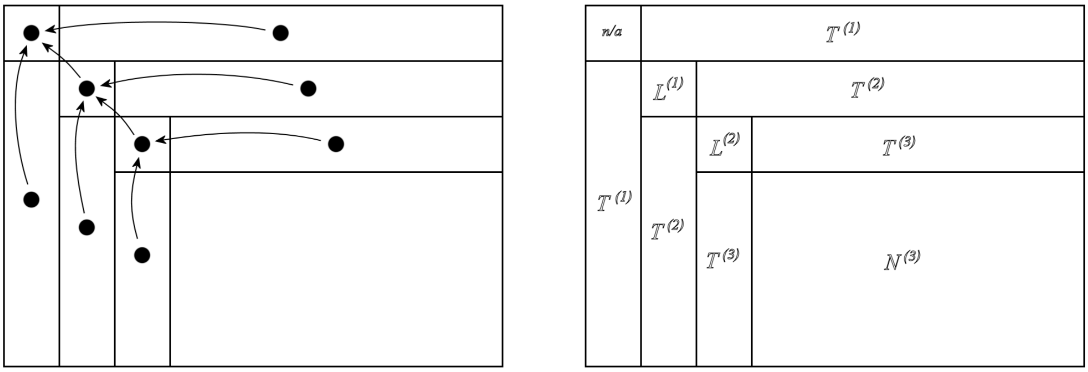


FIGURE 7. The weights of a fixed point. The arrows denote subtractions.

except when $k = \mathsf{dim}(A)$, in which case $L^{(k-1)} = \{w_{kk} - w_{k-1,k-1}\}$ and $N^{(k)} = \{0 - w_{kk}\}$.

We can now write down the tangent Euler class at our fixed point x specified by the quadruple $x = (S, R, \sigma, \varrho)$ (specifying k is superfluous, since $k = |S| = |R| = |\sigma| = |\varrho|$). Note that we use the convention that $\{\sigma(u) : 1 \le u \le k\} = S \subset \{1, 2, \ldots, i \odot j\}$, and similarly for ϱ and R.

$$\begin{aligned} e\big(T_x B^{(n)}\big) = \prod_{u=1}^{k} \Bigg[& \prod_{a=u+1}^{k} \big(w_{\sigma(a),\varrho(u)} - w_{\sigma(u),\varrho(u)}\big) \cdot \prod_{i \notin S} \big(w_{i,\varrho(u)} - w_{\sigma(u),\varrho(u)}\big) \cdot \\ & \cdot \prod_{b=u+1}^{k} \big(w_{\sigma(u),\varrho(b)} - w_{\sigma(u),\varrho(u)}\big) \cdot \prod_{j \notin R} \big(w_{\sigma(u),j} - w_{\sigma(u),\varrho(u)}\big) \Bigg] \cdot \\ & \cdot \big(0 - w_{\sigma(k),\varrho(k)}\big) \cdot \prod_{u=2}^{k} \big(w_{\sigma(u),\varrho(u)} - w_{\sigma(u-1),\varrho(u-1)}\big) \cdot \prod_{i \notin S} \prod_{j \notin R} w_{i,j}. \end{aligned}$$

We can rewrite this as follows. To simplify the notation, let us introduce the convention that $w_{\sigma(k+1),\varrho(k+1)} = w_{0,0} = 0$; also, $(i, a) \in \binom{S}{2}$ means the set of pairs $(i, a) \in S$ such that $i < a$.

$$\begin{aligned} e\big(T_x B^{(n)}\big) = & \prod_{u=1}^{k} \big(w_{\sigma(u+1),\varrho(u+1)} - w_{\sigma(u),\varrho(u)}\big) \cdot \\ & \cdot \mathsf{sgn}(\sigma) \cdot \prod_{(i,a) \in \binom{S}{2}} (-\varphi_a + \varphi_i) \cdot \mathsf{sgn}(\varrho) \cdot \prod_{(j,b) \in \binom{R}{2}} (+\psi_b - \psi_j) \cdot \\ & \cdot \prod_{a \in S} \prod_{i \notin S} (-\varphi_i + \varphi_a) \cdot \prod_{b \in R} \prod_{j \notin R} (+\psi_j - \psi_b) \cdot \prod_{i \notin S} \prod_{j \notin R} (\psi_j - \varphi_i). \end{aligned}$$

Since the solution over a fixpoint depends only on S and R, but not on σ and ϱ, we can try to simplify the sum

$$\sum_{\sigma,\varrho} \frac{1}{e(T_{(S,R,\sigma,\varrho)} B^{(n)})}.$$

Lemma 4.3.6. *We have*

$$\sum_{\sigma\in\mathfrak{S}_k}\sum_{\varrho\in\mathfrak{S}_k}\left[\prod_{u=1}^{k}\left(w_{\sigma(u+1),\varrho(u+1)}-w_{\sigma(u),\varrho(u)}\right)\cdot\mathsf{sgn}(\sigma)\mathsf{sgn}(\varrho)\right]^{-1}=$$

$$=(-1)^k\cdot\frac{\prod_{(i,a)\in\binom{S}{2}}(-\varphi_a+\varphi_i)\cdot\prod_{(j,b)\in\binom{R}{2}}(+\psi_b-\psi_j)}{\prod_{i\in S}\prod_{j\in R}\left(\psi_j-\varphi_i\right)}=(-1)^k\cdot\mathsf{det}\left[\frac{1}{\psi_j-\varphi_i}\right]_{k\times k}.$$

Remark. The second equality is Cauchy's double alternant; we won't actually use that.

Corollary 4.3.7.

$$\sum_{\sigma,\varrho}\frac{1}{e(T_{(S,R,\sigma,\varrho)}B^{(n)})}=$$

$$=\frac{1}{\left[\mathsf{Hom}(R,S)\right]\cdot\left[\mathsf{Hom}(A-S,S)\right]\cdot\left[\mathsf{Hom}(R,B-R)\right]\cdot\left[\mathsf{Hom}(A-S,B-R)\right]}=$$

$$=\frac{1}{\left[\mathsf{Hom}\big(R+(A-S),\,S+(B-R)\big)\right]}.$$

Remark. Note that the last expression is just the (inverse of the) tangent Euler class of the Grassmannian $\mathsf{Gr}_n(A\oplus B)$ at the fixed point $K=S^\perp\oplus R$. The sign $(-1)^k=(-1)^{(k^2)}$is hidden in the term $[\mathsf{Hom}(R,S)]$.

Proof of Lemma 4.3.6. We can apply the idea presented in Section A.3.1, and use localization to prove this identity. Consider the representation $\mathsf{Hom}(A,B)$; first blow up the origin (locus of rank 0 matrices), then blow up the locus the rank 1 matrices, and so on. The construction is very similar what we did before, except that we here started with $\mathsf{Hom}(A,B)$ instead of $\mathbb{P}(1\oplus\mathsf{Hom}(A,B))$. The method of A.3.1 applied to this this geometric situation proves a formula which is, up to sign and ordering of variables (which does not matter, as the *sum* is symmetric), is the same as the statement of the Lemma.

More formally, this sequence of blow-ups gives us a chain of identities

$$E_0:=\frac{1}{\prod_{i\in A}\prod_{j\in B}(\psi_j-\varphi_i)}=E_1=E_2=\cdots=E_k,$$

where

$$E_r=\sum_{|S|=r}\sum_{|R|=r}\sum_{\sigma\in\mathfrak{S}_S}\sum_{\varrho\in\mathfrak{S}_R}\frac{1}{D\cdot V_1\cdot V_2\cdot N}$$

$$D=w_{\sigma(1),\varrho(1)}\cdot\left(w_{\sigma(2),\varrho(2)}-w_{\sigma(1),\varrho(1)}\right)\cdots\left(w_{\sigma(r),\varrho(r)}-w_{\sigma(r-1),\varrho(r-1)}\right)$$

$$V_1=\prod_{s=1}^{r}\left[\prod_{u=s+1}^{r}\left(w_{\sigma(u),\varrho(s)}-w_{\sigma(s),\varrho(s)}\right)\cdot\prod_{i\notin S}\left(w_{i,\varrho(s)}-w_{\sigma(s),\varrho(s)}\right)\right]$$

$$V_2=\prod_{s=1}^{r}\left[\prod_{v=s+1}^{r}\left(w_{\sigma(s),\varrho(v)}-w_{\sigma(s),\varrho(s)}\right)\cdot\prod_{j\notin R}\left(w_{\sigma(s),j}-w_{\sigma(s),\varrho(s)}\right)\right]$$

$$N=\prod_{i\notin S}\prod_{j\notin R}(w_{ij}-w_{\sigma(r),\varrho(r)})$$

$$w_{i,j}=\psi_j-\varphi_i$$

Note that though these A, B, S, R, etc. are very similar to the previous ones, they are unrelated, local to this proof! The two ends of the chain gives $E_0 = E_k$, which for $|A| = |B| = k$ is equivalent to the lemma. □

4.3.2.2. *The canonical compactification.* It turns out that it is actually easy to understand the closure of $j(\mathcal{M}_{ij}) \subset \mathsf{Gr}^\mu \mathcal{J}_2(n)$ directly. For

$$(I, J, \widehat{\alpha}_2) \in \{\mathsf{Hom}(I \odot J, \mathbb{C}^n/I) \to \mathsf{Fl}_{ij}(n)\} = \mathcal{M}_{ij}$$

and $(F_1, F_2) \in \mathcal{J}_2(n, m)$ (in particular, for $m = 1$ too) we can rewrite our equations into the single equation $(F_1 + F_2)|_K = 0$ where

$$K = (\mathsf{id} \oplus q_I^{-1})(\mathsf{im}(\mathsf{id}_{I\odot J}, \widehat{\alpha}_2))$$
$$q_I : \mathbb{C}^n \to \mathbb{C}^n/I$$

In other words, we take the graph $\mathsf{graph}(\widehat{\alpha}_2) \subset (I \odot J) \oplus (\mathbb{C}^n/I)$ of the (linear) map $\widehat{\alpha}_2 : I \odot J \to \mathbb{C}^n/I$. This gives the map $j : \mathcal{M}_{ij} \to \mathsf{Gr}_\mu \mathcal{J}_2(n)^\vee = \mathsf{Gr}^\mu \mathcal{J}_2(n)$. Clearly, j factors through the bundle of Grassmannians $Y = \mathsf{Gr}_{i\odot j}(I \odot J \oplus \mathbb{C}^n/I) \to \mathsf{Fl}_{ij}$, which is compact, and is embedded into $\mathsf{Gr}^\mu \mathcal{J}_2(n)$; thus the the closure of $j(\mathcal{M}_{ij})$ can be constructed by taking the closure in Y, and embedding it to $\mathsf{Gr}^\mu \mathcal{J}_2(n)$.

Now the question is, basically, that which linear subspaces arise as limits of graphs? And the answer is: 'all'. For two vector spaces V^v and W^w, the image of $\mathsf{graph} : \mathsf{Hom}(V, W) \to \mathsf{Gr}_v(V \oplus W)$ is the open Schubert cell in $\mathsf{Gr}_v(V \oplus W)$. So we have

$$\overline{j(\mathcal{M}_{ij})} = Y \subset \mathsf{Gr}^\mu \mathcal{J}(n)$$

4.3.3. The localization formula. We get the same formula out of both methods:

$$[\Sigma^{ij}] = \sum_{I \in \binom{n}{i}} \frac{[\mathsf{Hom}(I, \Theta)]}{[\mathsf{Hom}(I, n - I)]} \sum_{J \in \binom{I}{j}} \frac{1}{[\mathsf{Hom}(J, I - J)]} \sum_{K \in \binom{I\odot J + n - I}{i\odot j}} \frac{[\mathsf{Hom}(K, \Theta)]}{[\mathsf{Hom}(K, (I \odot J + n - I) - K)]}$$

where, with some abuse of notation, n stands for $(\alpha_1, \ldots, \alpha_n)$, $\Theta = (\theta_1, \ldots, \theta_m)$, and the brackets denote the Euler class of the representation inside (as in $[V] = [\{0\} \subset V] \in H^*_G(\mathsf{pt})$). Notice that the inner sum is just the localization formula for $\Sigma^{i\odot j}(I \odot J + n - I, \Theta)$ (see Section 1.3), which we can evaluate as a Schur polynomial, thus

$$[\Sigma^{ij}] = \sum_{I \in \binom{n}{i}} \sum_{J \in \binom{I}{j}} \frac{s_{(i^m)}(\Theta - I) \cdot s_{((i\odot j)^{m-n+i})}(\Theta - (I \odot J + n - I))}{[\mathsf{Hom}(I, n - I)] \cdot [\mathsf{Hom}(J, I - J)]}$$

And this sum is just calculating the pushforward along $\pi : \mathsf{Fl}_{ij}(n) \to \mathsf{pt}$ (see Corollary A.3.3), resulting in

$$[\Sigma^{ij}] = \pi_* \{ s_{(i^m)}(\Theta - I) \cdot s_{((i\odot j)^{m-n+i})}(\Theta - (I \odot J + n - I)) \}$$

which is just Ronga's pushforward formula (Theorem 4.2.1).

We can also evaluate the localization formula using the technique presented in Section 3.2. We implemented this method as a computer program (written in Haskell), substituting rational numbers, and found that it is practical for $n, \mu \leq 9$: Table 3 shows the running times on an average personal computer (note that for the cases not marked 'new', we actually have explicit formulae, see the next two sections). Using floating point arithmetic one can gain several orders of magnitude in speed, which may further extend the range where computer calculations are possible with this method (which has the advantage of computing the $m=\infty$ case directly; see Figure 3 on page 27).

4.4. Explicit formulae for the coefficients

In this (and part of the next) section we present the results of our article [**FK06**] joint with László Fehér. It is based on the idea of restricting Σ^{ij} to Σ^k (in the sense of Section 1.4) for $k \le i$, though this is implicit in the argument we present here.

Recall that

$$\Sigma^{ij}(V,W) = \Big\{ (\alpha_1,\alpha_2) \in \mathcal{J}_2(V,W) = \mathsf{Hom}(V,W) \oplus \mathsf{Hom}(\mathsf{Sym}^2 V, W) \\ : \mathsf{corank}(\alpha_1) = i, \text{ and } \hat\alpha_2 \in \Sigma^{\bullet,j}\big(\mathsf{ker}(\alpha_1), \mathsf{coker}(\alpha_1)\big) \Big\}$$

where $\hat\alpha_2 : \mathsf{Sym}^2(\mathsf{ker}(\alpha_1)) \to \mathsf{coker}(\alpha_1)$ is the natural map induced by α_2, and

$$\Sigma^{\bullet,j}(A,B) = \big\{ \varphi \in \mathsf{Hom}(\mathsf{Sym}^2 A, B) \ : \ \mathsf{corank}(\mathsf{curry}(\varphi)) = j \big\}.$$

singularity	new	$\mu = i + i \odot j$	n	running time	# of terms
$\Sigma^{2,1}$		4	4	negligible	7
			5	0.3 seconds	17
			6	2 seconds	33
			7	10 seconds	57
			8	45 seconds	90
$\Sigma^{2,2}$		5	5	1 second	31
			6	7 seconds	64
			7	40 seconds	117
			8	3.5 minutes	199
$\Sigma^{3,1}$		6	6	31 seconds	110
			7	7 minutes	277
			8	*not measured*	592
			9	*not measured*	1137
			6	5.5 minutes	230
			7	105 minutes	689
$\Sigma^{3,2}$	$\star$	8	8	several hours – one day	1733
			6	2.5 minutes	156
$\Sigma^{3,3}$	$\star$	9	9	one day – few days	3222
			6	34 seconds	107
			7	25 minutes	450
$\Sigma^{4,1}$		8	8	several hours	1393
			6	11 minutes	269
$\Sigma^{4,2}$	$\star$	11	11	*out of reach*	?

Table 3. Running times for computing $\mathsf{Tp}_{\Sigma^{ij}}(n,\infty)$ using the method of Section 3.2. Where times are approximate, the calculations were ran on different (faster) computer, and not exactly measured.

Here curry denotes the (restriction of the) natural isomorphism

$$\mathsf{curry} : \mathsf{Hom}(A \otimes B, C) \xrightarrow{\sim} \mathsf{Hom}(A, \mathsf{Hom}(B, C)).$$

Remark. In the following, we will either assume that $m \geq n$, or define corank as the *source corank*, that is, $\mathsf{corank}(\varphi : \mathbb{C}^n \to \mathbb{C}^m) = n - \mathsf{rank}(\varphi)$, even if $m < n$. Note that though this theory works for $m < n$, from the Thom series point of view it is enough to consider the $m \gg n$ case.

It follows directly from this definition that $\Sigma^{ij}(n, m)$ is empty if $n < i$ and it is particularly simple for $n = i$:

$$\Sigma^{ij}(A^i, B^{m-n+i}) = \{0\} \times \Sigma^{\bullet,j}(A, B) \subset \mathcal{J}_2(A, B);$$

which means that its class is a product:

$$\left[\Sigma^{ij}(A^i, B^{m-n+i})\right] = e(\mathsf{Hom}(A, B)) \cdot \left[\Sigma^{\bullet,j}(A, B)\right].$$

We can exploit the simplicity of these cases because the *stability*: The Thom polynomial actually depends only on the (formal) difference $B - A$, that is, there exists a universal polynomial in the formal variables c_i

$$\mathsf{Tp}(r) \in \mathbb{Z}[c_1, c_2, c_3, \dots]$$

such that the classes $[\Sigma(A^n, B^m)]$ can be obtained by the specialization $\varrho_{A,B}\mathsf{Tp}(m-n)$, where

$$\begin{aligned} \varrho_{A,B} : \mathbb{Z}[c_1, c_2, c_3, \dots] &\to H^*_G(A) \otimes H^*_G(B) \\ c_i &\mapsto c^G_i(B - A) \end{aligned}$$

This notation is a bit strange (the correct interpretation is that A and B are G-equivariant vector bundles over some base manifold M), so let us work with the formal version instead:

$$\varrho_{n,m} : \mathbb{Z}[c_1, c_2, c_3, \dots] \to \mathbb{Z}[a_1, a_2, \dots a_n; b_1, b_2, \dots, b_m]$$

which is defined by the equation

$$\sum_{k=0}^{\infty} \varrho_{n,m}(c_i)t^i = \frac{1 + \sum_{j=1}^{m} b_j t^j}{1 + \sum_{i=1}^{n} a_i t^i}$$

with t being a formal variable. (This corresponds to let A and B be the standard GL_n resp. GL_m representations, viewed as $\mathsf{GL}_n \times \mathsf{GL}_m$-equivariant vector bundles over the point; a_i and b_j are then the equivariant Chern classes of A resp. B).

We will use the following well-known properties of the map $\varrho_{n,m}$.

Proposition 4.4.1.

(i) $\mathsf{ker}(\varrho_{n,m})$ *is spanned (over $\mathbb{Z}$) by the Schur polynomials $s_\lambda(c)$ with $(n+1)^{(m+1)} \subset \lambda$;*

(ii) $\mathsf{im}(\varrho_{n,m})$ *is spanned by the images of the Schur polynomials $s_\lambda(c)$ with $(n+1)^{(m+1)} \not\subset \lambda$;*

(iii) *Suppose that $n^m \subset \lambda$ but $(n+1)^{(m+1)} \not\subset \lambda$; that is, λ has the form $\lambda = (n^m + \beta, \alpha)$ with $\ell(\beta) \leq m$ and $\ell(\widetilde{\alpha}) = \alpha_1 \leq n$ (see Figure 8). Then we have to so called* factorization formula

$$\varrho_{n,m}(s_\lambda(c)) = s_{(n^m)}(b - a)\, s_\alpha(-a)\, s_\beta(b).$$

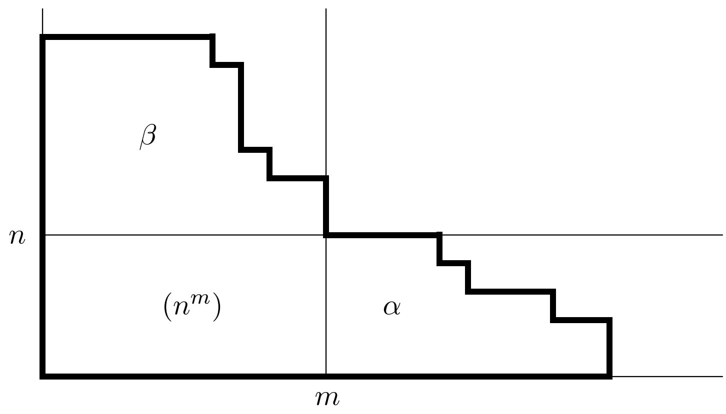


FIGURE 8. The shape of λ appearing in the factorization formula.

A proof can be found in eg. [**FP98**], Section 3.2. Note that $s_\alpha(-a) = (-1)^{|\alpha|}s_{\widetilde{\alpha}}(a)$, and $s_{(n^m)}(b-a)$ is just the (equivariant) Euler class $e(\mathsf{Hom}(A,B))$ (see Appendix A.2).

Compare this result to the observations above:

$$\varrho_{k,r+k}\mathsf{Tp}(r) = \begin{cases} 0 & k < i; \\ s_{(n^m)}(b-a)\cdot[\Sigma^{\bullet,j}(a,b)] & k = i. \end{cases}$$

These equations can be also interpreted as restriction equations (see Section 1.4), namely, we are restricting Σ_{ij} to Σ^k for $k \le i$.

Writing the universal polynomial $\mathsf{Tp}(r)$ and the class $[\Sigma^{\bullet,j}(a,b)]$ as a linear combination of Schur polynomials (resp. Schur classes, to be precise)

$$\mathsf{Tp}(r) = \sum_\lambda d_\lambda \cdot s_\lambda(c)$$

$$[\Sigma^{\bullet,j}(a,b)] = \sum_{\alpha,\beta} e_{\alpha\beta}\cdot s_\alpha(a)s_\beta(b), \tag{15}$$

the first case (actually it is enough to consider $k = i-1$) means that if d_λ is nonzero, then $i^{(r+i)} \subset \lambda$; and the second case ($k = i$) means that if $(i+1)^{(r+i+1)} \not\subset \lambda$, d_λ equals (up to sign) to the coefficient $e_{\widetilde{\alpha}\beta}$ of $s_{\widetilde{\alpha}}(a)s_\beta(b)$ in the class $[\Sigma^{\bullet,j}(i,r+i)]$. Note that we did not say anything about the coefficients d_λ where $(i+1)^{(r+i+1)} \subset \lambda$; fortunately, these are all zero, as we proved in Theorem 4.2.2.

To sum it up, we proved the following theorem.

THEOREM 4.4.2. *The Thom polynomial of the Σ^{ij} singularity in relative codimension r is*

$$\mathsf{Tp}(r) = \sum_{\alpha,\beta}(-1)^{|\alpha|}e_{\alpha\beta}\cdot s_{(i^{(r+i)}+\beta,\widetilde{\alpha})}(c),$$

where $e_{\alpha\beta} \in \mathbb{Z}$ are defined by (15) as the Schur coefficients of the class $[\Sigma^{\bullet,j}(a,b)]$.

Computing $e_{\alpha\beta}$ seems to be pretty difficult in general; however, the difficulties are mostly combinatorial. There are two special cases which are somewhat easier, namely, $r = -i+1$ or $i = 1$; in these cases, we will give closed formulae for these coefficients. Another interesting case is $j = i$, when we can give a nice combinatorial interpretation of the coefficients, but the problem of giving formulae or an enumerative recipe to compute them is unsolved. We can compute the case $i = j = 2$, however.

THEOREM 4.4.3. *Let $\pi : \mathsf{Gr}_j(A^i) \to \mathsf{pt}$ denote the projection map from the Grassmannian of j-planes in A to the one-point space, and J^j be the tautological (equivariant) vector bundle over $\mathsf{Gr}_j(A)$. Then*

$$\left[\Sigma^{\bullet,j}(A^i, B^{r+i})\right] = \pi_* e(\mathsf{Hom}((\pi^* A) \odot J, \pi^* B), \tag{16}$$

where e, as usual, is the equivariant Euler class, with the group $\mathsf{GL}(A) \times \mathsf{GL}(B)$ acting naturally.

Proof (compare with [**LP00**]*, Section 3).* Consider the diagram

$$\begin{array}{ccccccc}
\mathsf{Hom}(A\otimes A, B) & = & A^\vee \otimes A^\vee \otimes B & \xrightarrow{\sigma} & J^\vee \otimes A^\vee \otimes B & \longrightarrow & J^\vee \otimes J^\vee \otimes B \\
\downarrow & & \downarrow q_A & & \downarrow q_J & \swarrow & \\
\mathsf{Hom}(\wedge^2 A, B) & = & \wedge^2 A^\vee \otimes B & \longrightarrow & \wedge^2 J^\vee \otimes B & &
\end{array}$$

over a fixed $J \in \mathsf{Gr}_j(A)$. Note that $\mathsf{Hom}(\mathsf{Sym}^2 A, B) \cong \ker(q_A)$; thus any $\varphi \in \mathsf{Hom}(\mathsf{Sym}^2 A, B)$ gives us a section $\sigma(\varphi)$ of the vector bundle $\ker(q_J) \to \mathsf{Gr}_j(A)$. Combining these for different φ's, we get a section σ of the bundle

$$\ker(q_J) \to \mathsf{Gr}_j(A) \times \mathsf{Hom}(\mathsf{Sym}^2 A, B).$$

Observe that the image of the map pr_2 restricted to the zero locus Z of the section σ

$$\mathsf{pr}_2|_Z : Z = \sigma^{-1}\{0 \subset \ker(q_J)\} \to \mathsf{Hom}(\mathsf{Sym}^2 A, B)$$

is $\overline{\Sigma^{\bullet,j}}$, and it's one-to-one on $\Sigma^{\bullet,j}$; thus the locus Z is a resolution of $\overline{\Sigma^{\bullet,j}}$ (it is clear that σ is transversal to the zero section, thus Z is smooth). From that, it follows that

$$[\Sigma^{\bullet,j}] = \pi_* e(\ker(q_J));$$

but of course, $\ker(q_J) \cong \mathsf{Hom}(A \odot J, B)$. $\square$

Naturally, one would try to apply the pushforward formula (Theorem A.4.1) to evaluate (16). For this, we have to separate the "variable" J; the first step is

$$\pi_* e(\mathsf{Hom}(A \odot J,\, B) = \sum_{\lambda \subset (r+i)^{(i \odot j)}} (-1)^{|\lambda|} s_{\widetilde{\complement\lambda}}(B) \cdot \pi_* s_\lambda(A \odot J).$$

However, the problem of expanding $s_\lambda(A \odot J)$ into

$$s_\lambda(A \odot J) = \sum_{\varphi,\psi} f_{\varphi\psi} \cdot s_\varphi(A) s_\psi(J) \tag{17}$$

is unsolved in general. This question belongs to a larger family of similar expansion problems, most of which is unsolved. There are some tractable special cases, though:

(a) $j = 1$: in this case, $A \odot J = A \otimes J$, and the coefficients $f_{\varphi\psi}$ were computed by Lascoux in [**Las78**] (Lemma A.2.8);
(b) $r + i = 1$: in this case, $\lambda = (1^k)$ is special; $s_{(1^k)}$ is just the kth Chern class; for the *top* Chern class, a formula is proved in [**LP00**];

(c) $i = j$: in this case, $A \odot J = \mathsf{Sym}^2 A$; this is unsolved, but we can compute the smallest nontrivial case $i = j = 2$, see Section 4.5.2. Note that in this case π is trivial, so there is no pushforward; thus the coefficients in the Thom series are exactly the same as the coefficients $f_{\varphi\psi}$!

Nonetheless, we will not use Theorem 4.4.3 directly for cases (a) and (b), but handle them with different approaches. Case (c) serves as a motivation: It shows that the coefficients of the Thom polynomials have very rich combinatorics. We find this very important, so let us repeat it as a theorem:

Theorem 4.4.4. *The Thom series of the Σ^{ii} singularity is $\mathsf{Ts} = \sum f_{\nu_\pm} \mathsf{rs}_{\nu_\pm}$, where $f_{\nu_\pm}$ is the coefficient of $s_{\nu_+}(A^i)$ in the expansion of $s_{\nu_-}(\mathsf{Sym}^2 A^i)$. (Note that in this chapter we use the 'shifted base line' $(m-n+i, n-i)$, cf. Figure 3).*

Next, let us state two theorems about the cases (a) and (b).

Theorem 4.4.5. $[\Sigma^{\bullet,j}(A^i, L^1)] = 2^j \cdot s_{\lfloor j \rceil}(A^\vee \otimes \sqrt{L})$ *where* $\lfloor j \rceil = (j, j-1, \dots, 2, 1)$.

Remark. Note that the line bundle L has no square root, so the formula above should be understood formally: the only Chern root of $\sqrt{L}$ is $\beta/2$ where $\beta = \beta_1$ is the Chern root of L, and then the Chern roots of $A^\vee \otimes \sqrt{L}$ are $-\alpha_1 + \beta/2, \dots, -\alpha_n + \beta/2$.

Proof. Notice that the elements of $\mathsf{Hom}(\mathsf{Sym}^2\mathbb{C}^i, \mathbb{C})$ can be identified with symmetric $i \times i$ matrices and then the 'curried corank' becomes simply corank, so the class in question is given by the *twisted symmetric degeneracy locus* formula ([**HT84**], [**JLP82**], [**Pra90**], [**Ful96**]). A general explanation of twisting can be found in [**FNR05**]. □

Theorem 4.4.6. $[\Sigma^{\bullet,1}(A^i, B^{r+i})] = c_{i(r+i-1)+1}(A^\vee \otimes B - A)$.

Proof. The codimension of $\Sigma^{\bullet,1}(A^n, B^m) \subset \mathsf{Hom}(\mathsf{Sym}^2 A, B)$ is $mn - n + 1$, which equals to the codimension of $\Sigma^1(A, A^\vee \otimes B) \subset \mathsf{Hom}(A, A^\vee \otimes B) = \mathsf{Hom}(A \otimes A, B)$; so—exactly as noted in [**LP00**], where a similar degeneracy locus problem is considered—we are in the situation of the Giambelli-Thom-Porteous formula, since $\Sigma^{\bullet,1}(A^n, B^m)$ is just the transversal intersection $\Sigma^1(A, A^\vee \otimes W) \cap \mathsf{Hom}(\mathsf{Sym}^2 A, B)$:

$$\left[\Sigma^{\bullet,1}(A^n, B^m)\right] = [\Sigma^1(A, A^\vee \otimes W)] = c_{mn-n+1}(A^\vee \otimes B - A).$$

□

4.5. Combinatorics

In this section we will deal with the combinatorics of the special cases mentioned above.

4.5.1. The coefficients for $\Sigma^{i,1}$.

We will need the following statements of Lemma A.2.8:

Lemma ([**Las78**]). *Denote by $E_{\lambda/\mu}(n)$ the determinant*

$$E_{\lambda/\mu}(n) = \mathsf{det}\left[\binom{\lambda_i + n - i}{\mu_j + n - j}\right]_{i,j \in n \times n}.$$

(1) *Let A^n and B^m be an n-dimensional and a m-dimensional (equivariant) vector bundle, respectively. Then*

$$\sum_k c_k(A \otimes B) = \sum_{\mu \subset \lambda \subset m^n} E_{\lambda/\mu}(n) s_\mu(A) s_{\widetilde{\complement\lambda}}(B)$$

(2) *Furthermore, if L is a line bundle and λ is partition with $\ell(\lambda) \leq n$, then*

$$s_\lambda(A \otimes L) = \sum_{\mu \subset \lambda} E_{\lambda/\mu}(n) \cdot c_1(L)^{|\lambda|-|\mu|} \cdot s_\mu(A)$$

Remark. Our notation $E_{\lambda/\mu}(n)$ is motivated by the following formula. Suppose that $n \geq \ell(\lambda), \ell(\mu)$ (if this is not the case, one should take $(\lambda_1, \dots, \lambda_n)$ and $(\mu_1, \dots, \mu_n)$ instead of λ and μ in the RHS); then

$$E_{\lambda/\mu}(n) = s_{\lambda/\mu}\left(1, \frac{1}{2!}, \frac{1}{3!}, \frac{1}{4!}, \dots\right) \cdot \prod_{(i,j) \in \lambda/\mu} (n+i-j),$$

where we substitute $1/k!$ for the kth elementary symmetric polynomial in the (Jacobi-Trudi expansion of the) skew Schur polynomial $s_{\lambda/\mu}$ (this is called *exponential specialization* in the symmetric polynomial literature, see eg. [**Sta99**]). The proof of the formula is a straightforward computation (one observes that in the expansion of the determinant $E_{\lambda/\mu}(n)$ each term is the polynomial $\prod(n+i-j)$ up to a scalar). An important corollary is that $E_{\lambda/\mu}(n) = 0$ if $\mu \not\subset \lambda$.

The lemma, together with Theorem 4.4.5 immediately implies the following

Theorem 4.5.1. *The Thom polynomial of the second order Thom-Boardman singularity $\Sigma^{ij}(-i+1)$ in relative codimension $r = -i = 1$ is*

$$\left[\Sigma^{ij}(-i+1)\right] = \sum_{\mu \subset \lfloor j \rceil} 2^{|\mu| - j(j-1)/2} \cdot E_{\lfloor j \rceil/\mu}(i) \cdot s_{(d-|\mu|, \widetilde{\mu})}$$

where $\lfloor j \rceil$ is the 'stairway' partition $\lfloor j \rceil = (j, j-1, \dots, 2, 1)$ and

$$d = \mathsf{codim}\, \Sigma^{ij}(-i+1) = i + \tbinom{j+1}{2}.$$

Remark. This is the smallest relative codimension where the singularity appears at all. In the Thom series language, this theorem calculates the "lowest degree" part of the Thom series.

Similarly, Theorem 4.4.6 leads to

Theorem 4.5.2. *Using the shorthand notation $h = r + i$,*

$$\left[\Sigma^{i,1}(r)\right] = \sum_{(\lambda,\mu) \in K} s_{(i^h + \lambda, \mu)} \cdot \sum_{x \in \{0,1\}^{\ell(\mu)}} E_{\widetilde{\complement\lambda}/(\mu - x)^{\sim}}(i)$$

where x runs over the 0-1 sequences of length $\ell(\mu)$, and

$$K = \left\{ (\lambda, \mu) \;:\; \lambda \subset i^h,\ \mu_1 \leq i,\ |\lambda| + |\mu| = ih - i + 1,\ \textit{and } \mu - x \textit{ is a valid partition} \right\}.$$

Proof. According to Theorem 4.4.2, to express $[\Sigma^{i,1}]$ all we have to do is to expand the formula $c_{ih-i+1}(A^\vee \otimes B - A)$ into linear combination of products of Schur polynomials. For the sake of convenience, we calculate the total Chern class

$$\sum_{m \geq 0} c_m(A^\vee \otimes B - A) = \left(\sum_{k \geq 0} c_k(A^\vee \otimes B)\right) \cdot \left(\sum_{l \geq 0} c_l(-A)\right).$$

Using the Lemma, the Pieri formula, and

$$c(-A) = \sum_{l\geq 0} c_l(-A) = \sum_{k\geq 0}(-1)^k s_k(A),$$

we will get

$$c(A^\vee \otimes B - A) = \sum_{\mu\subset\lambda\subset i^h}\ \sum_{x\in\{0,1\}^{l(\mu)}} (-1)^{|\mu+x|} E_{\widetilde{\lambda}/\widetilde{\mu}}(i)\cdot s_{(\mu+x)^\sim}(A) s_{\complement\lambda}(B),$$

where the second sum runs over 0-1 sequences such that $\mu+x$ is a valid partition. From this the theorem follows directly, using the fact that $E_{\lambda/\mu}(k)=0$ if $\mu\not\subset\lambda$ and $k\geq\ell(\lambda),\ell(\mu)$. □

Note that in both cases, the Thom polynomial is a nonnegative linear combination of Schur polynomials. The same is true in general, for any Thom polynomial: This was conjectured by the author (based on numerical evidence), and independently by Pragacz; and then proved in [**PW07a, PW07b**].

With some work, we can get a more elegant formula. Introduce the notations

$$\begin{Bmatrix} n \\ k \end{Bmatrix} := \sum_{j=0}^{k}\binom{n}{j} \qquad \text{and} \qquad F_{\lambda/\mu}(n) := \mathsf{det}\left[\begin{Bmatrix}\lambda_k+n-k \\ \mu_l+n-l\end{Bmatrix}\right]_{k,l\in n\times n}.$$

Note that the numbers $\left\{{n \atop k}\right\}$ also form a Pascal-like triangle:

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 2 & & & & \\
 & & & 1 & & 3 & & 4 & & & \\
 & & 1 & & 4 & & 7 & & 8 & & \\
 & 1 & & 5 & & 11 & & 15 & & 16 & \\
1 & & 6 & & 16 & & 26 & & 31 & & 32
\end{array}$$

Theorem 4.5.3.

$$[\Sigma^{i,1}(r)] = \sum_{(\nu,\mu)\in K} F_{\nu/\mu}(i)\cdot s_{(i^h+\complement\widetilde{\nu},\widetilde{\mu})} \tag{18}$$

where $K=\{(\nu,\mu)\ :\ \nu\subset h^i,\ \ell(\mu)\leq i,\ \text{and}\ \ |\nu|-|\mu|=i-1\}$.

Proof. According to Theorem 4.5.2, the coefficient $a_{\nu,\mu}$ of $s_{(i^h+\complement\widetilde{\nu},\widetilde{\mu})}$ is a sum, which we can rewrite as follows:

$$a_{\nu,\mu} = \sum_{x\in\{0,1\}^{\mu_1}} E_{\nu/(\widetilde{\mu}-x)^\sim}(i) = \sum_{\alpha_1=\mu_2}^{\mu_1}\sum_{\alpha_2=\mu_3}^{\mu_2}\cdots\sum_{\alpha_i=0}^{\mu_i} E_{\nu/\alpha}(i)$$

Expanding the determinant $E_{\nu/\alpha}(i)$ and rearranging the sums yields

$$a_{\nu,\mu} = \mathsf{det}\left[\begin{Bmatrix}\nu_k+i-k \\ \mu_l+i-l\end{Bmatrix} - \begin{Bmatrix}\nu_k+i-k \\ \mu_{(l+1)}+i-(l+1)\end{Bmatrix}\right]_{k,l\in i\times i}$$

Observe that $a_{\nu,\mu}$ is of the form $\mathsf{det}(A-B)$ where

$$B_{k,l} = \begin{cases} A_{k,l+1} & \text{if } l<n \\ 0 & \text{if } l=n \end{cases}$$

It is an easy exercise then to prove that in such a situation $\mathsf{det}(A+\beta B)=\mathsf{det}(A)$ holds for any $\beta\in\mathbb{C}$. □

Note that in the formula (18), the coefficients are manifestly independent of $r = m - n$; thus what we got here is actually a closed formula for the *Thom series*:

COROLLARY 4.5.4. *The Thom series of the* $\Sigma^{i,1}$ *singularity is*

$$\mathsf{Ts}(\Sigma^{i,1}) = \sum_{\nu_\pm \in K} F_{\nu_-/\nu_+}(i) \cdot \mathsf{rs}_{\nu_\pm}$$

where

$$K = \big\{ (\nu_+, \nu_-) \ : \ \ell(\nu_+) = \ell(\nu_-) = i, \ \ |\nu_-| - |\nu_+| = i - 1 \big\}.$$

Note that with some abuse of notation, here we allow ν_+ and ν_- to be padded with zeros.

EXAMPLE 4.5.5. Specializing for Σ^{21},

$$\mathsf{Ts}(\Sigma^{21})] = \sum_K \left(\begin{Bmatrix} a+1 \\ d+1 \end{Bmatrix} \begin{Bmatrix} b \\ c \end{Bmatrix} - \begin{Bmatrix} a+1 \\ c \end{Bmatrix} \begin{Bmatrix} b \\ d+1 \end{Bmatrix} \right) \cdot \mathsf{rs}_{(d,c,-b,-a)}$$

where K is the set of quadruples

$$K = \{ (a,b,c,d) \in \mathbb{N}^4 \ : \ 0 \le b \le a, \ 0 \le c \le d, \ c+d = a+b-1 \}.$$

4.5.2. The coefficients for Σ^{22}. According to Theorem 4.4.4, the coefficients in the Thom series of Σ^{22} are the same as the coefficients in the expansion

$$s_{(a,b,c)}(2x, x+y, 2y) = \sum_r d^{(M-r,r)}_{(a,b,c)} \cdot s_{(M-r,r)}(x,y), \tag{19}$$

where $M = a + b + c$. In this section, we derive a formula for these numbers.

By definition,

$$s_{(a,b,c)}(2x, 2y, x+y) = \frac{-1}{2(x-y)^3} \begin{vmatrix} (2x)^{a+2} & (2x)^{b+1} & (2x)^c \\ (2y)^{a+2} & (2y)^{b+1} & (2y)^c \\ (x+y)^{a+2} & (x+y)^{b+1} & (x+y)^c \end{vmatrix}$$

Expanding the determinant by the last row, we get

$$\frac{-1}{2(x-y)^2} \left(2^{b+c+1} s_1^{a+2} s_{(b,c)} - 2^{a+c+2} s_1^{b+1} s_{(a+1,c)} + 2^{a+b+3} s_1^c s_{(a+1,b+1)} \right),$$

using the notational convention that $s_1 = s_1(x,y) = x + y$ and $s_{(a,b)} = s_{(a,b)}(x,y)$. One $(x - y)$ factor vanishes from the denominator, since the definition of $s_{(n,k)}$ contains that.

First, let us compute the subexpressions of the form $s_1^n s_{(b,c)}$. For this, recall the fact that

$$(x+y)^n = c_1^n = s_1^n = \sum_{i=0}^{\lfloor n/2 \rfloor} T_{(n-i,i)} s_{(n-i,i)},$$

where

$$\begin{aligned} T_{(n-i,i)} &= \binom{n}{i} - \binom{n}{i-1} = \text{the Catalan triangle} \\ &= \text{number of standard Young tableaux of shape } (n-i,i). \end{aligned}$$

We will need the Littlewood-Richardson rule for this very special case:

$$s_{(a,b)}s_{(p,q)} = \sum_{i=0}^{\min(a-b,p-q)} s_{(a+p-i,b+q+i)}.$$

Let us start.

$$s_1^n s_{(b,c)} = \sum_{i=0}^{\lfloor n/2\rfloor} T_{(n-i,i)} s_{(n-i,i)} s_{(b,c)} = \sum_{i=0}^{\lfloor n/2\rfloor} T_{(n-i,i)} \sum_{j=0}^{\min(n-2i,b-c)} s_{(n+b-i-j,c+i+j)}$$

Substituting $k = i + j$:

$$= \sum_{i=0}^{\lfloor n/2\rfloor} T_{(n-i,i)} \sum_{k=i}^{\min(n-i,b-c+i)} s_{(n+b-k,c+k)} = \sum_{k=0}^{b-c+\lfloor(n-b+c)/2\rfloor} s_{(n+b-k,c+k)} \sum_{i=\max(0,k-b+c)}^{\min(k,n-k)} T_{(n-i,i)} =$$

$$= \sum_{k=0}^{b-c+\lfloor(n-b+c)/2\rfloor} s_{(n+b-k,c+k)} \left[\binom{n}{\min(k,n-k)} - \binom{n}{k-b+c-1}\right] =$$

$$= \sum_{k=0}^{b-c+\lfloor(n-b+c)/2\rfloor} s_{(n+b-k,c+k)} \left[\binom{n}{k} - \binom{n}{k-b+c-1}\right]$$

since the inner sum is telescopic. Note that the last sum could start from $(-\infty)$, and if we declare $s_{(b<c)}$ to be zero, then it could stop at $(+\infty)$. We can rephrase this as follows: The coefficient of $s_{(n+p+q-l,l)}$ in $s_1^n \cdot s_{(p,q)}$ is

$$\text{coefficient of } s_{(n+p+q-l,l)} \text{ in } s_1^n \cdot s_{p,q} = \binom{n}{l-q} - \binom{n}{l-p-1}.$$

Thus we have

$$\begin{aligned}
&s_{(a,b,c)}(2x, x+y, 2y) = \\
&= \frac{-1}{(x-y)^2}\left(2^{b+c}c_1^{a+2}s_{b,c} - 2^{a+c+1}c_1^{b+1}s_{a+1,c} + 2^{a+b+2}c_1^{c}s_{a+1,b+1}\right) = \\
&= \frac{-1}{(x-y)^2}\sum_{k=0}^{\lfloor(M+2)/2\rfloor} s_{(M+2-k,k)}\left[2^{b+c}\binom{a+2}{k-c} - 2^{b+c}\binom{a+2}{k-b-1} - 2^{a+c+1}\binom{b+1}{k-c} + \right. \\
&\qquad \left. +2^{a+c+1}\binom{b+1}{k-a-2} + 2^{a+b+2}\binom{c}{k-b-1} - 2^{a+b+2}\binom{c}{k-a-2}\right].
\end{aligned} \tag{20}$$

What remains is to factor the term $(x-y)^2$ out of the expression above. To do this, let us introduce some notations: Let n be fixed, and $m = \lfloor n/2\rfloor$; furthermore,

$$\begin{aligned}
a^{(n)} &= (a_0, a_1, \ldots, a_m) \in \mathbb{Z}^{(m+1)} \\
b^{(n+2)} &= (b_0, b_1, \ldots, b_{m+1}) \in \mathbb{Z}^{(m+2)} \\
s^{(n)} &= (s_{(n)}, s_{(n-1,1)}, s_{(n-2,2)}, \ldots) \in \mathbb{Z}[x,y]^{(m+1)} \\
\langle a^{(n)}, s^{(n)}\rangle &= \sum_i a_i s_{(n-i,i)} \in \mathbb{Z}[x,y] \\
k^{(n)} &= (n+1, n-1, n-3, \ldots, n+1-2m) \in \mathbb{N}^{(m+1)} \\
k^{(n+2)} &= (n+3, n+1, n-1, n-3, \ldots, n+1-2m) \in \mathbb{N}^{(m+2)}
\end{aligned}$$

Then it is a straightforward computation to show that

$$(x-y)^2 \cdot \langle a^{(n)}, s^{(n)} \rangle = (s_{(2)} - 3s_{(1,1)}) \cdot \langle a^{(n)}, s^{(n)} \rangle = \langle A^{(n)} a^{(n)}, s^{(n+2)} \rangle$$

where the matrix $A^{(n)} \in \mathsf{Mat}_{(m+2)\times(m+1)}$ is defined as

$$A^{(n)} = \begin{bmatrix} 1 & & & & & \\ -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ & & & & 1 & x_n \end{bmatrix} \qquad \text{where } x_n = -3 + (n \bmod 2)$$

PROPOSITION 4.5.6. *$b^{(n+2)} \in \mathsf{im}(A^{(n)})$ if and only if $\langle k^{(n+2)}, b^{(n+2)} \rangle = 0$.*

Proof. $k^{(n+2)}$ generates $\mathsf{ker}([A^{(n)}]^\dagger)$. □

In this case, we have a left pseudo-inverse $J^{(n)} \in \mathsf{Mat}_{(m+1)\times(m+2)}$:

$$J^{(n)} = [A^{(n)}]^{-1} = \begin{bmatrix} 1 & & & & & & & 0 \\ 2 & 1 & & & & & & 0 \\ 3 & 2 & 1 & & & & & 0 \\ 4 & 3 & 2 & 1 & & & & 0 \\ \vdots & & \ddots & \ddots & \ddots & & & \vdots \\ m & \cdots & & 3 & 2 & 1 & 0 & 0 \\ m+1 & \cdots & & 4 & 3 & 2 & 1 & 0 \end{bmatrix}$$

With this, we can complete the computation: Just multiply the expansion vector of (20) by the matrix $J^{(M)}$. This results in

THEOREM 4.5.7. *The coefficient of $s_{(M-r,r)}(x+y)$ in (the expansion of) $s_{(a,b,c)}(2x, x+y, 2y)$ is*

$$d^{(M-r,r)}_{(a,b,c)} = -\sum_{k=0}^{r} (r+1-k) \cdot \Bigg[2^{b+c}\binom{a+2}{k-c} - 2^{b+c}\binom{a+2}{k-b-1} - 2^{a+c+1}\binom{b+1}{k-c} + \\ +2^{a+c+1}\binom{b+1}{k-a-2} + 2^{a+b+2}\binom{c}{k-b-1} - 2^{a+b+2}\binom{c}{k-a-2} \Bigg]. \tag{21}$$

Recall the following notation introduced in the previous section:

$$\begin{Bmatrix} n \\ k \end{Bmatrix} = \sum_{j=0}^{k} \binom{n}{j}.$$

With this notation, we can simplify a bit the type of sums appearing in (21):

LEMMA 4.5.8. *For $n, r, b \in \mathbb{N}$,*

$$\sum_{k=0}^{r} (r+1-k) \binom{n}{k-b} = \sum_{j=0}^{r-b} \begin{Bmatrix} n \\ j \end{Bmatrix}.$$

Proof. Straightforward. □

Corollary 4.5.9.

$$d^{(M-r,r)}_{(a,b,c)} = -\,2^{b+c}\sum_{j=c}^{b}\left\{{a+2 \atop r-j}\right\} + 2^{a+c+1}\sum_{j=c}^{a+1}\left\{{b+1 \atop r-j}\right\} - 2^{a+b+2}\sum_{j=b+1}^{a+1}\left\{{c \atop r-j}\right\}. \tag{22}$$

Putting together with Theorem 4.4.4:

Theorem 4.5.10. *The Thom series of the* Σ^{22} *singularity is*

$$\mathsf{Ts}(\Sigma^{22}) = \sum_{\nu_\pm} d^{\nu_+}_{\nu_-}\cdot \mathsf{rs}_{\nu_\pm},$$

where $\nu_\pm$ *runs over the pairs of partitions such that* $\ell(\nu_+)\le 2$, $\ell(\nu_-)\le 3$, *and* $|\nu_+| = |\nu_-|$, *and* $d^{\nu_+}_{\nu_-}$ *is defined by (22) above.*

Remark. Compare this formula with Example 4.5.5, which gives a formula for Σ^{21}. It is intriguing that both contain the numbers $\left\{{n \atop k}\right\}$; however, the connection between the two is not at all clear.

Chapter 5. Third order - A_3

In the last chapter, we sketch how to modify the "blow-up method" of Section 4.3.2.1 to work with the A_3 (or Σ^{111}) singularity. We believe it can be also adapted to other (small) singularities, eg. A_4, Σ^{211}, Σ^{221}; however, the computations did not work out yet. For this reason, some of the statements are presented in more generality than needed for the A_3 case.

We find it possible that this method could even work for all third order Thom-Boardman singularities (Σ^{ijk}), however this only of theoretical interest, since the vast number of fixed points (or possibly fixed components) make any computation inpractical already for small cases. A very rough estimation of the number of fixed points is $(\mu!)^2$, for the smallest case $n = \mu$; so for example Σ^{321}, with $\mu = 13$ seems to be completely out of reach with this method, while Σ^{221}, with $\mu = 8$, is around the limits of present-day personal computers.

5.1. The probe model for Σ^{ijk}

Recall Porteous' probe model (we concentrate on the $d = 3$ case here):

Proposition 5.1.1 ([**Por83**]). *$F \in \mathcal{J}_3(n, m)$ belongs to the Thom-Boardman class Σ^{ijk} if there exists a probe $(\alpha, \beta, \gamma) \in \mathcal{P}_{ijk} = \mathcal{J}_3^\circ(i, n) \times \mathcal{J}_2^\circ(j, i) \times \mathcal{J}_1^\circ(k, j)$ such that*

$$\begin{aligned} 0 &= \mathrm{d}(F \circ \alpha)|_0 \\ 0 &= \mathrm{d}(\mathrm{d}(F \circ \alpha) \circ \beta)|_0 \\ 0 &= \mathrm{d}(\mathrm{d}(\mathrm{d}(F \circ \alpha) \circ \beta) \circ \gamma)|_0, \end{aligned}$$

and no such probe exists for higher Boardman indices.

We can rewrite these equations to a linear form, using the tensor notation of Appendix A.1. The expansion of third, most complicated equation is shown in Figure 9.

Proposition 5.1.2 ([**Por83**]). *If (α, β, γ) is a good probe for F, then so is $(\alpha', \beta', \gamma')$ defined by the commutative diagram*

$$\begin{array}{ccccccc} \mathbb{C}^n & \xleftarrow{\alpha} & \mathbb{C}^i & \xleftarrow{\beta} & \mathbb{C}^j & \xleftarrow{\gamma} & \mathbb{C}^k \\ \| & & \downarrow\varphi & & \downarrow\psi & & \downarrow\chi \\ \mathbb{C}^n & \xleftarrow{\alpha'} & \mathbb{C}^i & \xleftarrow{\beta'} & \mathbb{C}^j & \xleftarrow{\gamma'} & \mathbb{C}^k \end{array} \tag{23}$$

where $(\varphi, \psi, \chi) \in G_{ijk} = \mathsf{Diff}_3(i) \times \mathsf{Diff}_2(j) \times \mathsf{Diff}_1(k)$ are jets of biholomorphisms.

Proposition 5.1.3. *The moduli space $\mathcal{M}_{ijk}$ of the probes is, <u>set-theoretically,</u>*

$$\begin{aligned} \mathcal{M}_{ijk} = \Big\{ (I, J, K, \widehat{\alpha}_2, \widehat{\alpha}_3, \widehat{\beta}_2) \ : \ & (I, J, K) \in \mathsf{Fl}_{ijk}(\mathbb{C}^n) \\ & , \ \widehat{\alpha}_2 \in \mathsf{Hom}(I \odot J, \ \mathbb{C}^n/I), \ \widehat{\alpha}_3 \in \mathsf{Hom}(I \odot J \odot K, \ \mathbb{C}^n/I) \\ & , \ \widehat{\beta}_2 \in \mathsf{Hom}(J \odot K, \ I/J) \Big\} \end{aligned}$$

Proof. Apply Gauss elimination to the equations written in a matrix form. The proof shows that $\widehat{\beta}_2$ should be probably called 'alpha' too, however, we think that would be equally confusing. ◻

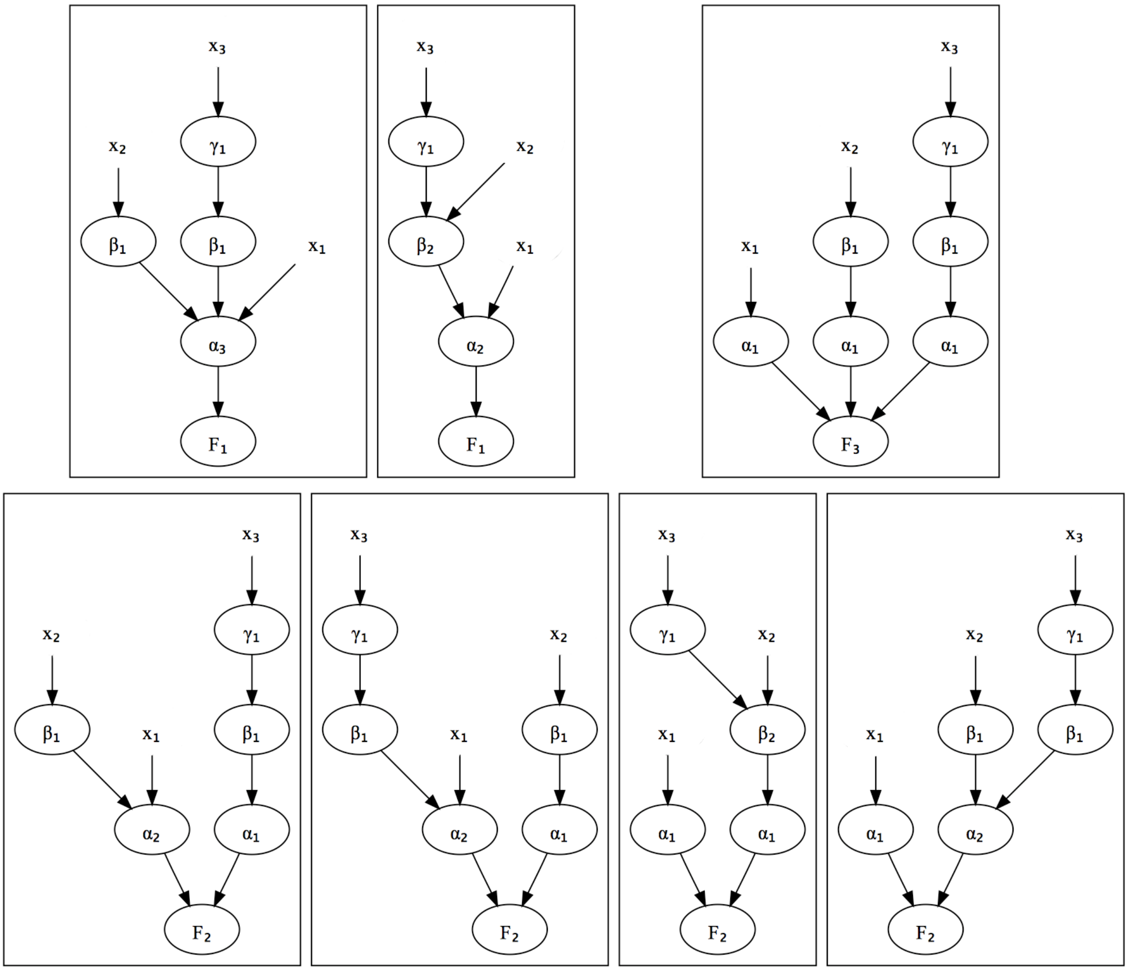


FIGURE 9. The 7 terms of the expression $\mathrm{d}(\mathrm{d}(\mathrm{d}(F \circ \alpha) \circ \beta) \circ \gamma)|_0$.

These spaces are remarkably subtle. At first sight, they look quite simple: They are fiber bundles (actually, towers of fiber bundles), and the fibers are vector spaces. But they are *not* holomorphic vector bundles, at least not with the complex structure defined by requiring the quotient map $\mathcal{P}_{ijk} \to \mathcal{M}_{ijk}$ to be analytic.

However, one aspect of the vector bundle structure (of $\mathcal{M}_{ij}$) survives the generalization, namely, a $\mathbb{C}^\times$-action, generalizing the multiplication by scalars. Suppose we multiply α_2 by a nonzero scalar $\omega \in \mathbb{C}^\times$; now, there is only way to extend this to an action of $\mathbb{C}^\times$ on the space of probes $\mathcal{P}$ which is compatible with the factor map $q : \mathcal{P} \to \mathcal{M}$, which is to multiply the degree d components by ω^{d-1}:

PROPOSITION 5.1.4. *Let $\mathbb{C}^\times$ act on the space of probes*

$$\mathcal{P}_I = \prod_{k=1}^{d} \mathcal{J}^\circ_{d+1-k}(i_k, i_{k-1})$$

by the formula

$$\begin{array}{rcccl} \omega & : & \mathcal{J}^\circ_{d+1-k}(i_k, i_{k-1}) & \to & \mathcal{J}^\circ_{d+1-k}(i_k, i_{k-1}) \\ & & \big(\beta_1,\, \beta_2,\, \beta_3,\, \ldots,\, \beta_{d+1-k}\big) & \mapsto & \big(\beta_1,\, \omega\beta_2,\, \omega^2\beta_3,\, \ldots,\, \omega^{d-k}\beta_{d+1-k}\big) \end{array}$$

This action induces a well-defined action on $\mathcal{M}_I = \mathcal{P}_I/_\sim$, via the formula

$$\omega \cdot x = q_I(\omega \cdot y), \qquad y \in q_I^{-1}(x).$$

Note that $\mathsf{im}(\alpha_1) \supset \mathsf{im}(\alpha_1 \circ \beta_1) \supset \mathsf{im}(\alpha_1 \circ \beta_1 \circ \gamma_1) \supset \dots$ define a fibration $\mathcal{M}_I \to \mathsf{Fl}_I(\mathbb{C}^n)$, and we act on the fibers (the action is trivial on Fl_I).

LEMMA 5.1.5 (Boardman). *The codimension of Σ^{ijk} is*

$$\mathsf{codim}(\Sigma^{ijk}) = (m - n + i)\mu_{ijk} - (i - j)\mu_{jk} - (j - k)k,$$

where

$$\mu_{ijk} = i + i \odot j + i \odot j \odot k$$
$$\mu_{jk} = j + j \odot k.$$

REMARK. This codimension formula has a straightforward generalization for higher order Thom-Boardman singularities, but we will not need the general case.

5.2. MORIN SINGULARITIES

The case $i = j = k = \cdots = 1$, called Morin singularities, deserves special attention. These were studied in [**BSz06, Bér08**], and it was that work where the idea of applying equivariant localization to the computation of Thom polynomials first appeared. This was also the case which motivated a large part of our work: The original goal was to find an alternate (and possibly more general) approach to the compactification of these moduli spaces, since even though they were able to derive iterated residue formulae for the Thom polynomials of A_d for $d \leq 6$, the spaces appearing in the process are not very well understood geometrically. Unfortunately, this program didn't really work, since the geometry is indeed intricate.

THEOREM 5.2.1 ([**BSz06**], [**Gaf83**]). *For $A_d = \Sigma^{11\dots1}$, the moduli of probes (which in this particular case is also called the moduli of* test curves*) is the quotient*

$$\mathcal{M}_d = \mathcal{J}_d^\circ(1, n)/\mathsf{Diff}_d(1),$$

that is, jets of curves in $\mathbb{C}^n$ up to reparameterization. The solutions in $\mathcal{J}_d(n, m)$ for a fixed test curve $\gamma \in \mathcal{J}_d^\circ(1, n)$ are

$$\mathsf{sol}(\gamma) = \left\{ F \in \mathcal{J}_d(n, m) \ : \ F \circ \gamma = 0 \right\}.$$

Proof. Recall Mather's theorem: Two germs are $\mathcal{K}$-equivalent if and only of their ideals are taken into each other by a diffeomorphism germ. The protype ideal for A_d is $I_d = (x_2, x_3, \dots, x_n) \lhd \mathcal{J}_d(n)$. Suppose we have a germ $F = (f_1, \dots, f_m)$ of type A_d: Then there exists a diffeomorhism germ $\varphi \in \mathsf{Diff}_d(n)$ such that the ideal $(f_1 \circ \varphi, \dots, f_m \circ \varphi)$ is I_d; thus $F \circ \gamma = 0$ for $\gamma(t) = \varphi(t \cdot \mathrm{d}x_1)$. Conversely, suppose that $F \circ \gamma = 0$; for any γ there is a $\psi \in \mathsf{Diff}_d(n)$ such that $\psi \circ \gamma = (t \mapsto t\mathrm{d}x_1)$, that is, $(f_1 \circ \psi^{-1}, \dots, f_m \circ \psi^{-1}) \subset I_d$. □

REMARK. It is not hard to see this from Porteous' model; in fact, we don't even need the Boardman indices $i = j = k = \dots$ to be exactly 1, we only need them to be equal. Look at diagram (23): For any fixed $\varphi \in \mathsf{Diff}_d(i)$, there is a unique $\psi \in \mathsf{Diff}_{d-1}(i)$, $\chi \in \mathsf{Diff}_{d-2}(i)$, etc. such that $\beta' = \gamma' = \cdots = \mathsf{id} : \mathbb{C}^i \to \mathbb{C}^i$; thus $\mathcal{P}_{iii\dots}/G_{iii\dots} = \mathcal{J}_d^\circ(i, n)/\mathsf{Diff}_d(i)$. Dimension calculation shows that there are no further ambiguities in the probes, so $\mathcal{M}_{iii\dots} = \mathcal{P}_{iii\dots}/G_{iii\dots}$.

We adopt the notation $x, u, v, w, \ldots \in \mathbb{C}^n$ for the components of our test curve:

$$\gamma \in \mathcal{J}_d^\circ(1,n)$$
$$\gamma(t) = xt + ut^2 + vt^3 + wt^4 + \cdots$$

(these corresponds to $\alpha_1, \alpha_2, \alpha_3, \ldots$ in the old notation), and $A, B, C, D, \ldots$ for the components of the singularity (which corresponds to $F_1, F_2, F_3, \ldots$ in the old notation). Note that $x \neq 0$. Then the equations can be written as

$$\begin{aligned}
0 &= Ax \\
0 &= Au + Bxx \\
0 &= Av + 2Bux + Cxxx \\
0 &= Aw + Buu + 2Bvx + 3Cuxx + Dxxxx \\
&\vdots
\end{aligned}$$

In general, the terms in the dth equations correspond to partitions of d, and the coefficients (which are mostly irrelevant) are the number of automorphisms of the partition (see [**BSz06**]). Note that there are at least two different conventions for the coordinatization of the symmetric tensors $B, C, D, \ldots$; the other convention results in different coefficients.

We can represent the group $\mathsf{Diff}_d(1)$ and its action on the space of test curves with matrices: Denoting the components of a diffeomorphism jet by $(\alpha, \beta, \gamma, \ldots)$, using the $d = 4$ case as an example, the action reduces to the matrix multiplication

$$\begin{bmatrix} \alpha & & & \\ \beta & \alpha^2 & & \\ \gamma & 2\alpha\beta & \alpha^3 & \\ \delta & 2\alpha\gamma + \beta^2 & 3\alpha^2\beta & \alpha^4 \end{bmatrix} \cdot \begin{bmatrix} x_1 & x_2 & \ldots & x_n \\ u_1 & u_2 & \ldots & u_n \\ v_1 & v_2 & \ldots & v_n \\ w_1 & w_2 & \ldots & w_n \end{bmatrix} = \begin{bmatrix} x'_1 & x'_2 & \ldots & x'_n \\ u'_1 & u'_2 & \ldots & u'_n \\ v'_1 & v'_2 & \ldots & v'_n \\ w'_1 & w'_2 & \ldots & w'_n \end{bmatrix}$$

Note that $\mathsf{Diff}_d(1)$ acts on $\mathcal{J}_d^\circ(1,n)$ on the left, while GL_n acts on the right; thus, these two actions commute, and GL_n also acts on the quotient space $\mathcal{M}_d = \mathcal{J}_d^\circ(1,n)/\mathsf{Diff}_d(1)$.

THEOREM 5.2.2 (cf. [**BSz06**], Prop. 4.4). *For any test curve $(x, u, v, w, \ldots)$ with $x_i \neq 0$, there is a unique diffeomorphism jet $(\alpha, \beta, \gamma, \ldots) \in \mathsf{Diff}_d(1)$ such that for the new curve $(x', u', v', w', \ldots)$ we have $x'_i = 1$, and $u'_i = v'_i = w'_i = \cdots = 0$.*

Proof. This is basically the Lagrange inversion theorem. The unique $(\alpha, \beta, \ldots)$ can be written down explicitly:

$$\begin{aligned}
\alpha &= \frac{1}{x_i} \\
\beta &= \frac{-u_i}{x_i^3} \\
\gamma &= \frac{2u_i^2 - x_i v_i}{x_i^5} \\
\delta &= \frac{5x_i u_i v_i - 5u_i^3 - x_i^2 w_i}{x_i^7} \\
&\vdots
\end{aligned}$$

The coefficients in these formulae are Sloane's A111785 [**OEIS**]. ☐

This theorem gives us an atlas on the moduli space of test curves: The charts are $\{x_i = 1, u_i = v_i = w_i = \dots = 0\}$ for $1 \le i \le n$, and the transition functions are compositions of two diffemorphisms. The atlas gives a down-the-earth definition of the complex structure on the moduli space of test curves $\mathcal{M}_d = \mathcal{J}_d^\circ(1,n)/\mathsf{Diff}_d(1)$.

Example. Consider the transition function from ith chart to the jth chart. The notation will be such that $\{x_i = 1, u_i = v_i = w_i = 0\}$ and $\{x'_j = 1, u'_j = v'_j = w'_j = 0\}$, and $k \ne i, j$. These transition functions can be also written down explicitly:

$$\begin{aligned} x'_i &= \tfrac{1}{x_j} & x'_k &= \tfrac{x_k}{x_j} \\ u'_i &= \tfrac{-u_j}{x_j^3} & u'_k &= \tfrac{u_k x_j - x_k u_j}{x_j^3} \\ v'_i &= \tfrac{-v_j x_j + 2u_j^2}{x_j^5} & v'_k &= \tfrac{v_k x_j^2 - 2u_k u_j x_j - x_k v_j x_j + 2x_k u_j^2}{x_j^5} \\ w'_i &= \tfrac{-w_j x_j^2 + 5v_j u_j x_j - 5u_j^2}{x_j^7} & w'_k &= \tfrac{w_k x_j^3 - 3v_k u_j x_j^2 - 2u_k v_j x_j^2 + 5u_k u_j^2 x_j - x_k w_j x_j^2 + 5x_k v_j u_j x_j - 5x_k u_j^3}{x_j^7} \end{aligned}$$

Remark. Note that these expressions for $\alpha, \beta, \dots$ are homogeneous with respect to *two different grading*: The first one is the standard grading $\mathsf{deg}(x) = \mathsf{deg}(u) = \mathsf{deg}(v) = \dots = 1$; and the second one is $\mathsf{deg}(x) = 0$, $\mathsf{deg}(u) = 1$, $\mathsf{deg}(v) = 2$, $\mathsf{deg}(w) = 3$, etc. The transition functions are also homogeneous wrt. the second grading; this observation motivated our $\mathbb{C}^\times$ action.

Theorem 5.2.3. *The $\mathbb{C}^\times$ action on $\mathcal{J}_d^\circ(1,n)$ defined by*

$$\omega \cdot (x, u, v, w, \dots) = (x, \omega u, \omega^2 v, \omega^3 w, \dots)$$

induces a well-defined action on $\mathcal{M}_d = \mathcal{J}_d^\circ(1,n)/\mathsf{Diff}_d(1)$.

Proof. We have to prove that for any two $y_1, y_2 \in \mathcal{J}_d^\circ(1,n)$ such that $y_1 \sim y_2$, and any $\omega \in \mathbb{C}^\times$, we have $\omega \cdot y_1 \sim \omega \cdot y_2$. But the equivalence relation $\sim$ is defined by a group action, thus there is an $H \in \mathsf{Diff}_d(1)$ such that $y_2 = Hy_1$; and we want to find an H' such that $\omega \cdot (Hy_1) = \omega \cdot y_2 = H'(\omega \cdot y_1)$. In fact, an H' exists (independently of y_1) such that $H' \circ \omega = \omega \circ H$, and it is very easy to write down: If $H = (\alpha, \beta, \gamma, \delta, \dots)$, then $H' = (\alpha, \omega\beta, \omega^2\gamma, \omega^3\delta, \dots)$. □

This $\mathbb{C}^\times$ action allows us to imitate the process of Section 4.3.2.1: The initial, wrong compactification will be

$$B^{(1)} = \Big((\mathbb{C} \times \mathcal{M}_d) - \{\text{zero section}\} \Big) / _{\mathbb{C}^\times},$$

which is a bundle over $\mathbb{P}^{n-1}$ (the projection map is given by $[x] \in \mathbb{P}^{n-1}$), whose fibers are *weighted projective spaces* with weights

$$\{1; 1^{n-1}, 2^{n-1}, \dots, (d-1)^{n-1}\}$$

($\mathbb{C}^\times$ acts on the new direction $\mathbb{C}\langle\xi\rangle$ with weight 1, that is, $\omega \cdot \xi = \omega\xi$). Note that we have a well defined zero section of $\mathcal{M}_d$, which can be defined for example as the set of limits $\lim_{\omega=0}(\omega \cdot [\gamma])$; alternatively, the transition functions leave the set $\{u = v = w = \dots = 0\}$ invariant.

Weighted projective spaces are singular; however, their singularities are very mild, namely, cyclic quotient singularities: Locally, they look like $\mathbb{C}^N/\mathbb{Z}_k$ for some cyclic group $\mathbb{Z}_k$ acting diagonally. In other words, they are complex orbifolds. This allows us to pretend that they are smooth, since we can work over a smooth finite cover instead (but we have to count multiplicities): For a weighted projective space $\mathbb{P}^{[d]}$ with weights $d = (d_0, \dots, d_N)$ we have a natural degree $\prod_i d_i$ branched covering $\pi : \mathbb{P}^N \to \mathbb{P}^{[d]}$ given by the formula

$$\pi\,[x_0 : x_1 : \cdots : x_N] = [x_0^{d_0} : x_1^{d_1} : \cdots : x_N^{d_N}],$$

assuming that $\gcd(d_0, \dots, d_N) = 1$ (otherwise we shall factor out by the common divisor).

Of course, we have to homogenize the equations. Written in convenient matrix form, for the $d = 4$ case they are $M \cdot [D|C|B|A]^t = 0$, where M is the matrix

$$M = \begin{array}{cccc} D & C & B & A \\ \hline & & & x \\ & & \xi x^2 & u \\ & \xi^2 x^3 & 2\xi u x & v \\ \xi^3 x^4 & 3\xi^2 u x^2 & \xi(u^2 + 2vx) & w \end{array}$$

(note that while ξ is a scalar, x, u, v, w are vectors; multiplication of vectors in this picture means symmetric tensor product). The rank of this matrix is clearly d if $\xi \neq 0$, and $\mathsf{rk}\,[x|u|v|w]$ if $\xi = 0$ (recall that $x \neq 0$, thus the rank is always at least 1; actually, it is at least 2, since we removed the zero section). Note that the rank is invariant for all three group actions: The $\mathsf{Diff}_d(1)$ action, the GL_n action, and the $\mathbb{C}^\times$ action.

The image of (the linear map represented by) this matrix (or equivalently, the kernel of the adjoint) determines the map

$$\begin{aligned} \mathsf{sol} : \mathcal{M}_d \subset B^{(1)} &\dashrightarrow \mathsf{Gr}^d(\mathcal{J}_d(n)) \\ \mathsf{sol}\big([x, u, v, \dots]\big) &= [M_1 \wedge M_2 \wedge \cdots \wedge M_d] \end{aligned} \tag{24}$$

where M_i are the rows of the matrix; the map is not defined when the rank of the matrix is less than d. Again, the natural thing to try is to blow up the rank varieties, and indeed that is what we will do.

The weights of the tangent space of $\mathcal{M}_d \to \mathbb{P}^{n-1}$ at a torus-fixed point of $\mathbb{P}^{n-1}$, eg. $[x] = [1 : 0 : 0 : \cdots : 0]$ can be read off from Theorem 5.2.2 (the first row represents the tangent space $T_{[x]}\mathbb{P}^{n-1}$, and the rest is the fiber):

$$\begin{bmatrix} \text{n/a} & (\alpha_2 - \alpha_1) & (\alpha_3 - \alpha_1) & \cdots & (\alpha_n - \alpha_1) \\ \text{n/a} & (\alpha_2 - 2\alpha_1) & (\alpha_3 - 2\alpha_1) & \cdots & (\alpha_n - 2\alpha_1) \\ \text{n/a} & (\alpha_2 - 3\alpha_1) & (\alpha_3 - 3\alpha_1) & \cdots & (\alpha_n - 3\alpha_1) \\ \vdots & \vdots & \vdots & & \vdots \\ \text{n/a} & (\alpha_2 - d\alpha_1) & (\alpha_3 - d\alpha_1) & \cdots & (\alpha_n - d\alpha_1) \end{bmatrix} \tag{25}$$

The torus acts on the 'extra direction' ξ trivially (otherwise the embedding into the compactification wouldn't be equivariant; that is, the compactification would not respect the torus action).

From that, we can compute the torus weights of the corresponding weighted projective space at the fixed points. Since these fixed points are typically singular, the weights should be understood in the sense of Lemma A.3.8: They are the weights of $\mathbb{C}^N$ where our space looks locally like $\mathbb{C}^N/\mathbb{Z}_k$. Given a weighted projective space[1] with projective weights $d_0, d_1, \dots, d_N$ and torus weights $\beta_0, \beta_1, \dots, \beta_N$, these are easy to compute: Since the equivalence relation is

$$(x_0,\, x_1,\, \dots,\, x_N) \sim (\omega^{d_0}x_0,\, \omega^{d_1}x_1,\, \dots,\, \omega^{d_N}x_N),$$

around the kth fixed point $[0 : \dots : 0 : 1 : 0 : \dots : 0]$ we have a local chart given by $x_k \neq 0$, and then choosing $\omega = x_k^{-1/d_k}$ (note that there are d_k such roots, forming the cyclic group $\mathbb{Z}_{d_k}$),

$$(x_0, \dots, x_{k-1}, x_k, x_{k+1}, \dots, x_N) \sim \Big(\frac{x_0}{x_k^{d_0/d_k}}, \dots, \frac{x_{k-1}}{x_k^{d_{k-1}/d_k}}, 1, \frac{x_{k+1}}{x_k^{d_{k+1}/d_k}}, \dots, \frac{x_N}{x_k^{d_N/d_k}} \Big).$$

Thus the jth weight at the kth fixed point will be

$$w_j = \beta_j - \frac{d_j}{d_k}\beta_k, \tag{26}$$

and the tangent Euler class at the kth fixed point is, according to Lemma A.3.8,

$$e^{(k)} = d_k \cdot \prod_{j \neq k} \Big(\beta_j - \frac{d_j}{d_k}\beta_k \Big). \tag{27}$$

5.2.1. The A_3 singularity. Let us now concentrate on the $d = 3$ case (that is, the A_3 singularity). In this case we only have to do a single blow-up $\pi : B^{(2)} = \mathsf{Bl}_{\Sigma_1} B^{(1)} \to B^{(1)}$. For brevity, let's work in the fiber over a given fixed point $x_p = [0 : \dots : 0 : 1 : 0 : \dots : 0]$ of the projective space $\mathbb{P}^{n-1}$; the situtation is of course symmetric. There are three types of fixed points in $B^{(1)}$ (see the middle row of Figure 10 and also Figure 11, left):

- type 0: $\xi = 1$, $u = v = 0$; this is a single smooth fixed point;
- type 1: $\xi = 0$, $u_k = 1$, $u_{\neq k} = 0$, $v = 0$, which is again smooth;
- type 2: $\xi = 0$, $u = 0$, $v_k = 1$, $v_{\neq k} = 0$, which has a $\mathbb{Z}_2$ cyclic quotient singularity.

Note that the singular locus of $B^{(1)}$ is defined by the equations $u = \xi = 0$, and is therefore fully contained in the rank variety Σ_1 we want to blow up. There is two ways we can proceed from here: we can either blow-up the singular locus first, so that everything becomes smooth; or we can just accept and live with the (mild) singularities. Both works equally well in this situation, but only the second version has any chance to scale to more complicated examples, hence we will concentrate on that.

Proposition 5.2.4. *The map* sol *extends to a regular (dominant, birational) map* $\mathsf{sol} : B^{(2)} \to \mathsf{Gr}^3(\mathcal{J}_d(n))$.

Proof. We imitate the the proof used for Σ^{ij}. It is actually much simpler, since we have only a single blow-up here; however, there is a bit less symmetry present. We want to show that approaching any point $z \in E$ in the exceptional divisor $E = \pi^{-1}(\Sigma_1)$ on a curve $\gamma(t)$, the limit of $\mathsf{sol}(\gamma(t))$ is independent of the curve, and depends continuously on z; hence the extension is continuous. Then by the Riemann extension theorem (see eg. [**GH78**]) the extended map is also holomorphic.

A generic point in $\pi(z) \in \Sigma_1$ (over a fixed x) is $u = \lambda a$, $v = \mu a$, where $y \in \mathbb{C}^n$, $[\lambda : \mu] \in \mathbb{P}^{[1,2]} \cong \mathbb{P}^1$, and y is not a multiple of x. Because of the GL_n symmetry, we can

[1] We will always assume that 1 appears among the weights, and that leaving out any weight, the gcd. of the rest is still 1. These assumptions hold for our weighted projective spaces.

assume that $x = (1, 0, 0, \dots, 0)$ and $y = (0, 1, 0, \dots, 0)$. Then a curve approaching $\pi(z)$ looks like

$$\begin{aligned} x &= (1, 0, \dots, 0) \\ u &= (0, \lambda + a_2 t, a_3 t, \dots, a_n t) \qquad\qquad \xi = et \\ v &= (0, \mu + b_2 t, b_3 t, \dots, b_n t) \end{aligned}$$

where $a, b \in \mathbb{C}^{n-1} \subset \mathbb{C}^n$ and $e \in \mathbb{C}$ are parameters. We don't have to worry about higher order terms (eg. t^2), since sol is already analytic outside E; but it will be also clear that adding them wouldn't change the proof. The matrix M thus is

$$M = \begin{array}{|c|c|c|c|c|} \hline & & & & 1 \\ \hline & & \xi & u & \\ \hline \xi^2 & 2\xi u x & & v & \\ \hline \end{array} = \begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & 1 \\ \hline & & & et & \lambda + a_2 t & a' t & \\ \hline e^2 t^2 & 2\lambda e t + 2\lambda e a_2 t^2 & 2\lambda a' t^2 & & \mu + b_2 t & b' t & \\ \hline \end{array}$$

where we used the shorthand $a' = (a_3, \dots, a_n)$ and $b' = (b_3, \dots, b_n)$. The map sol is defined by taking the row span of M. We will distinguish two cases: $\lambda \neq 0$ and $\lambda = 0$, $\mu \neq 0$. In the first case, from the second row, the term λ will dominate in the limit $t \to 0$; thus we have to apply one step of Gaussian elimination, subtracting c times the second row from the third one where

$$c = \frac{\mu + b_2 t}{\lambda + a_2 t} = \frac{\mu}{\lambda} + \frac{b_2 \lambda - \mu a_2}{\lambda^2} \cdot t + O(t^2).$$

The new third row will be then, modulo t^2,

$$M_3' \mod t^2 = \begin{array}{|c|c|c|c|c|c|c|} \hline 0 & 2\lambda e t & 0 & -\frac{\mu}{\lambda} e t & 0 & b' t - \frac{\mu}{\lambda} a' t & 0 \\ \hline \end{array}$$

which means that the limit of $M_1 \wedge M_2 \wedge M_3'$ will depend only on $\chi = [e : b' - \frac{\mu}{\nu} a'] \in \mathbb{P}^{n-2}$ (we assume that they are not both 0, which would mean the we are approaching from a direction inside $T_z E$), which is determined by $z \in E = \mathbb{P} N_{\Sigma_1} B^{(1)}$, since $N_{\Sigma_1} B^{(1)}|_z = T_z B^{(1)} / T_z \Sigma_1$, and the (translation) action of $T_z \Sigma_1$ leaves χ invariant. Indeed, $T_z(\Sigma_1|_x) \subset T_z(\mathsf{B}^{(1)}|_x)$ is spanned by $(u = \lambda y, v = \mu y)$, $y \in \mathbb{C}^{n-1}$, and those (u, v) pairs where only u_2 and v_2 is nonzero; but a_2 and b_2 is not present in χ, and

$$(b' + \mu y) - \frac{\mu}{\lambda}(a' + \lambda y) = b' - \frac{\mu}{\lambda} a'.$$

The other case ($\lambda = 0$) is similar, but even simpler; we omit it.

Finally, consider the dependence of $\mathsf{sol}(z) = \lim_{t=0} [M_1 \wedge M_2 \wedge M_3]$ on z. Again, because of the GL_n symmetry, the only interesting part is dependence on $[\lambda : \mu] \in P^{[1,2]}$; setting $a = b = 0$, it is easy to see, separately on the two charts $\lambda \neq 0$ and $\mu \neq 0$, that $\mathsf{sol}(z)$ depends on z continuously (in fact, we can write down the solution explicitely). □

name	indices	description	mult.	solution weights		
type 0	$p \in \{1, 2, \dots, n\}$	$x_p = 1,\ \xi = 1$	1	α_p	$2\alpha_p$	$3\alpha_p$
type 1a	$p,\ q \neq p,\ r \neq p, q$	$x_p = 1,\ u_q = 1,\ v_r = \varepsilon$	1	α_p	α_q	α_r
type 1b	$p,\ q \neq p$	$x_p = 1,\ u_q = 1,\ \xi = \varepsilon$	1	α_p	α_q	$\alpha_p + \alpha_q$
type 2a	$p,\ q \neq p,\ r \neq p, q$	$x_p = 1,\ v_q = 1,\ u_r = \varepsilon$	2	α_p	α_q	α_r
type 2b	$p,\ q \neq p$	$x_p = 1,\ v_q = 1,\ \xi = \varepsilon$	2	α_p	$2\alpha_p$	α_q

TABLE 4. Table of types of fixed points in $B^{(2)}$

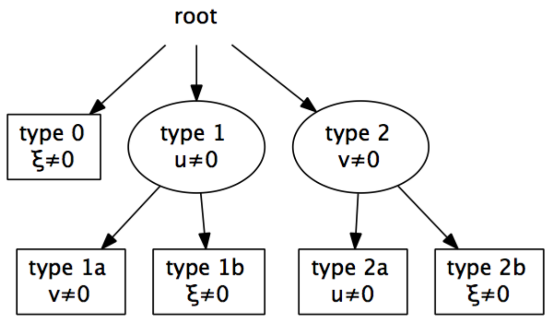


FIGURE 10. The different fixed point types of the $B^{(2)}$ compactification for A_3.

REMARK. The naive generalization of this proposition fails for $d \geq 4$. For the first problematic case A_4, it seems that the trouble is caused by the locus $v^2 = 2uw$; we conjecture that by blowing up this locus first, we can make the method work in that case too.

Blowing up the rank variety $\Sigma_1(u, v)$, the fixed point types 1 and 2 branch into 2-2 new types; the 5 types of fixed points of $B^{(2)}$ are summarized in Table 4, and illustrated in Figure 11. The solution space over a fixed point can be easily read off from the matrix M, using (24); the corresponding weights are indicated in the table. The number of fixed points is alltogether

$$\#\text{fixp} = n + 2n(n-1) + 2n(n-1)(n-2).$$

The most involved (though straightforward) part is to compute the tangent Euler classes at the different fixed points. To do that, we have to combine (25), (26), (27) and the blow-up. To make life easier, let us introduce the shorthand notations

$$U_i = \alpha_i - 2\alpha_p$$
$$V_i = \alpha_i - 3\alpha_p$$
$$\tau_p = \prod_{j \neq p} (\alpha_j - \alpha_p).$$

Here τ_p is just the tangent Euler class of the projective space $\mathbb{P}^{n-1}$ at the pth fixed point $x_p = 1$; since the everything is fibered over this projective space, this will be a common factor in all the Euler classes.

The simplest case is type 0, where the tangent Euler class is just

$$E_0(p) = \tau_p \cdot \prod_{j \neq p} (U_j - 0)(V_j - 0) = \tau_p \cdot \prod_{j \neq p} (U_j V_j).$$

Next, consider type 1. Before blowing up, the weights of the fiber over $x_p = 1$ at the fixed point given by $u_q = 1$ are, according to (26),

$$\{U_j - U_q \; : \; j \neq p, q\} \cup \{V_j - 2U_q \; : \; j \neq p\} \cup \{0 - U_q\}.$$

This can be partitioned to the tangent space of Σ_1 and the corresponding normal space:

$$T_1 = \{V_q - 2U_q\} \cup \{U_j - U_q \; : \; j \neq p, q\}$$
$$N_1 = \{V_j - 2U_q \; : \; j \neq p, q\} \cup \{0 - U_q\}.$$

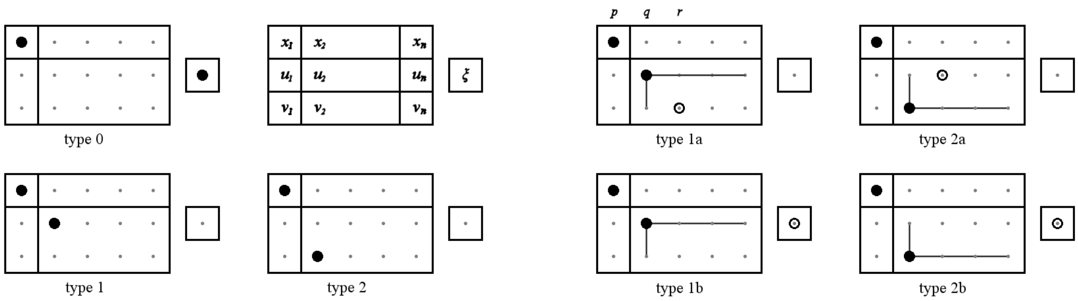


FIGURE 11. The fixed points of $B^{(1)}$ (left) and $B^{(2)}$ (right, plus the top-left one).

Thus after the blow-up, the tangent Euler classes of the fixed points of type 1a and 1b are, respectively

$$E_{1a}(p,q,r) = \tau_p \cdot (V_q - 2U_q) \cdot \Big[\prod_{j \neq p,q} (U_j - U_q) \Big] \cdot (V_r - 2U_q) \cdot$$
$$\cdot \Big[\prod_{i \neq p,q,r} \underbrace{\big((V_i - 2U_q) - (V_r - 2U_q)\big)}_{V_i - V_r = \alpha_i - \alpha_r} \Big] \cdot \underbrace{\big((0 - U_q) - (V_r - 2U_q)\big)}_{U_q - V_r = \alpha_p + \alpha_q - \alpha_r}$$

$$E_{1b}(p,q) = \tau_p \cdot (V_q - 2U_q) \cdot \Big[\prod_{j \neq p,q} (U_j - U_q) \Big] \cdot (0 - U_q) \cdot \Big[\prod_{i \neq p,q} \underbrace{\big((V_i - 2U_q) - (0 - U_q)\big)}_{V_i - U_q = \alpha_i - \alpha_p - \alpha_q} \Big]$$

Similarly, for type 2, the weights are $T_2 \cup N_2$ where

$$T_2 = \big\{ U_q - \tfrac{1}{2} V_q \big\} \cup \big\{ V_j - V_q \ : \ j \neq p,q \big\}$$
$$N_2 = \big\{ U_j - \tfrac{1}{2} V_q \ : \ j \neq p,q \big\} \cup \big\{ 0 - \tfrac{1}{2} V_q \big\},$$

and the Euler classes for type 2a and 2b are

$$E_{2a}(p,q,r) = 2 \cdot \tau_p \cdot (U_q - \tfrac{1}{2} V_q) \cdot \Big[\prod_{j \neq p,q} (V_j - V_q) \Big] \cdot (U_r - \tfrac{1}{2} V_q) \cdot$$
$$\cdot \Big[\prod_{i \neq p,q,r} \underbrace{\big((U_i - \tfrac{1}{2} V_q) - (U_r - \tfrac{1}{2} V_q)\big)}_{U_i - U_r = \alpha_i - \alpha_r} \Big] \cdot \underbrace{\big((0 - \tfrac{1}{2} V_q) - (U_r - \tfrac{1}{2} V_q)\big)}_{-U_r = 2\alpha_p - \alpha_r}$$

$$E_{2b}(p,q) = 2 \cdot \tau_p \cdot (U_q - \tfrac{1}{2} V_q) \cdot \Big[\prod_{j \neq p,q} (V_j - V_q) \Big] \cdot (0 - \tfrac{1}{2} V_q) \cdot \Big[\prod_{i \neq p,q} \underbrace{\big((U_i - \tfrac{1}{2} V_q) - (0 - \tfrac{1}{2} V_q)\big)}_{U_i} \Big].$$

Note the factors of two, which account to the multiplicities, according to (27).

Putting everything together, we get a localization formula for the Thom polynomials of the A_3 singularity. Using the notation

$$\Theta(w_1, w_2, w_3) = \prod_{j=1}^{m} \Big[(\theta_j - w_1)(\theta_j - w_2)(\theta_j - w_3) \Big],$$

we have, for $n \geq 3$,

$$\mathsf{Tp}_{A_3}(n,m) = \sum_p \frac{\Theta(\alpha_p, 2\alpha_p, 3\alpha_p)}{E_0(p)} + \sum_{p,q}\left[\frac{\Theta(\alpha_p,\alpha_q,\alpha_p+\alpha_q)}{E_{1b}(p,q)} + \frac{\Theta(\alpha_p,2\alpha_p,\alpha_q)}{E_{2b}(p,q)}\right] + \\ + \sum_{p,q,r}\left[\frac{\Theta(\alpha_p,\alpha_q,\alpha_r)}{E_{1a}(p,q,r)} + \frac{\Theta(\alpha_p,\alpha_q,\alpha_r)}{E_{2a}(p,q,r)}\right]. \tag{28}$$

Indeed, implementing (28) as a computer program, and converting the results to Schur polynomials in the difference alphabet $\theta - \alpha$ (otherwise they would be way too large to fit in a page), we get

$$\begin{aligned}
\mathsf{Tp}_{A_3}(3,3) &= 6s_{1,1,1} + 5s_{2,1} + s_3\\
\mathsf{Tp}_{A_3}(3,4) &= 36s_{1,1,1,1,1,1} + 30s_{2,1,1,1,1} + 19s_{2,2,1,1} + 5s_{2,2,2} + 6s_{3,1,1,1} + 5s_{3,2,1} + s_{3,3}\\
\mathsf{Tp}_{A_3}(3,5) &= 36s_{3,1,1,1,1,1,1} + 6s_{3,3,1,1,1} + 216s_{1,1,1,1,1,1,1,1,1} + 65s_{2,2,2,1,1,1}+\\
&\quad + 24s_{2,2,2,2,1} + 5s_{3,2,2,2} + s_{3,3,3} + 114s_{2,2,1,1,1,1,1} + 5s_{3,3,2,1}+\\
&\quad + 19s_{3,2,2,1,1} + 180s_{2,1,1,1,1,1,1,1} + 30s_{3,2,1,1,1,1}
\end{aligned}$$

The phenomenon that these polynomials fit into a Thom series can be observed on these examples already: The terms of $\mathsf{Tp}(n,m)$ appear in $\mathsf{Tp}(n,m+1)$, but with a 3 preprended to the partition. Unfortunately, the RHS of (28) is in a form which makes it rather hard to evaluate for larger n, m-s.

Remark. Compare the formula (28) above with Section 3.1, in particular with equations (4) and (6) there. The observation is that by computing the Euler classes at the fixpoints of a $B^{(2)}$, which has a birational dominant map $\mathsf{sol} : B^{(2)} \to \bar{\mathcal{M}}_3 \subset \mathsf{Hilb}^3(\mathcal{J}_3(n))$, we can derive the same data for $\bar{\mathcal{M}}_3$ itself, which was not clear how to do directly! We only have to rearrange our Euler classes to the form (6), so that a term of the localization formula looks like

$$\frac{\Theta(w_1,w_2,w_3)}{P(\alpha_K)\cdot\prod_{i\notin K}\left[(\alpha_i-w_1)(\alpha_i-w_2)(\alpha_i-w_3)\right]}$$

where $K = \{i_1,\dots,i_k\} \in \binom{n}{k}$, and w_1, w_2, w_3 resp. P are linear resp. rational in $\alpha_K = \{\alpha_{i_1},\dots,\alpha_{i_k}\}$. In our case, K will be either $\{p\}$, $\{p,q\}$ or $\{p,q,r\}$, and the P-s can be readily read off from the Euler classes; we summarized them in Table 5.

name	k_Q	ideal	the rational function P
type 0	1	(x^4)	$P_0(p) = 1$
type 1a	3	(x^2,y^2,z^2,xy,xz,yz)	$P_{1a}(p,q,r) = (\alpha_p-\alpha_q)^2(\alpha_r+\alpha_p-2\alpha_q)\cdot$ $\cdot(\alpha_q+\alpha_p-\alpha_r)(\alpha_r-\alpha_p)(\alpha_q-\alpha_r)$
type 1b	2	(x^2,y^2)	$P_{1b}(p,q) = (\alpha_p-\alpha_q)^2(\alpha_q-2\alpha_p)$
type 2a	3	(x^2,y^2,z^2,xy,xz,yz)	$P_{2a}(p,q,r) = \frac{1}{2}(\alpha_p-\alpha_q)^2(\alpha_q+\alpha_p-2\alpha_r)\cdot$ $\cdot(2\alpha_p-\alpha_r)(\alpha_r-\alpha_p)(\alpha_q-\alpha_r)$
type 2b	2	(x^3,xy,y^2)	$P_{2b}(p,q) = \frac{1}{2}(\alpha_p-\alpha_q)^2(3\alpha_p-\alpha_q)$

Table 5. Table of fixed points for A_3 from the viewpoint of Section 3.1.

To fully convert to the form used in Chapter 3 and [**FR08**], we have to combine the different fixed points with the same ideal; note that this also includes permutations of the parameters p, q, r when the ideal has some symmetry! Therefore the final data (compare with the table in [**FR08**], Section 8) can obtained as follows:

$$\begin{aligned}
P_{(x^4)} &= P_0 = 1 \\
P_{(x^2,y^2)} &= \Big[P_{1\mathrm{b}}(1,2)^{-1} + P_{1\mathrm{b}}(2,1)^{-1}\Big]^{-1} = \frac{(\alpha_1-\alpha_2)^2(2\alpha_1-\alpha_2)(\alpha_1-2\alpha_2)}{\alpha_1+\alpha_2} \\
P_{(x^3,xy,y^2)} &= P_{2\mathrm{b}}(1,2) = \tfrac{1}{2}(\alpha_1-\alpha_2)^2(3\alpha_1-\alpha_2) \\
P_{(x^2,y^2,z^2,xy,xz,yz)} &= \left[\sum_{(i,j,k)\in\mathfrak{S}_3} \Big(P_{1\mathrm{a}}(i,j,k)^{-1} + P_{2\mathrm{a}}(i,j,k)^{-1}\Big)\right]^{-1} = \dots
\end{aligned}$$

For reference (it's omitted from [**FR08**]), the last expression equals to

$$\dots = 4\,\frac{6(\alpha_1^3+\alpha_2^3+\alpha_3^3) - 7(\alpha_1^2\alpha_2+\alpha_1^2\alpha_3+\alpha_2^2\alpha_1+\alpha_2^2\alpha_3+\alpha_3^2\alpha_1+\alpha_3^2\alpha_2) + 10\alpha_1\alpha_2\alpha_3}{(\alpha_1-\alpha_2-\alpha_3)(\alpha_2-\alpha_1-\alpha_3)(\alpha_3-\alpha_1-\alpha_2)\prod_{i\neq j\in\{1,2,3\}}(\alpha_i-2\alpha_j)}.$$

We could now in principle apply the ideas of Sections 3.3 and 3.4 to compute the Thom *series* of the A_3 singularity (which is known by the way, see [**BFR02**] and [**LP09**]; however, the methods used in those work are not very elegant and have no chance to scale); unfortunately the actual calculations present rather profound challenges, which are yet to overcome.

Appendix

The aim of the Appendix is to collect together results, sometimes with proofs, which are used in the main body of the thesis, but would break the flow if presented there.

A.1. Multivariate differentials

Porteous' probe model for the Thom-Boardman singularities (see Chapter 4, Section 4.3.1) deals with higher order differentials of composite functions in many variables; in particular, differentials of the form

$$\mathrm{d}(\mathrm{d}(\mathrm{d}(F\circ\alpha)\circ\beta)\circ\gamma)|_0.$$

Since differential calculus is usually not covered in this generality in university classes and textbooks, we present a simple graphical calculus to deal with such expressions.

For $\alpha\in\mathcal{J}(n,m)$, the first differential at $x\in\mathbb{C}^n$ is $(\mathrm{d}\alpha)(x)\in\mathsf{Hom}(T_x\mathbb{C}^n,T_{\alpha(x)}\mathbb{C}^m)=\mathsf{Hom}(\mathbb{C}^n,\mathbb{C}^m)$. Thus

$$\begin{aligned}\mathrm{d}\alpha &\in \mathcal{J}\big(\mathbb{C}^n,\ \mathsf{Hom}(\mathbb{C}^n,\mathbb{C}^m)\big)\\ \mathrm{d}^2\alpha &\in \mathcal{J}\big(\mathbb{C}^n,\ \mathsf{Hom}(\mathbb{C}^n,\mathsf{Hom}(\mathbb{C}^n,\mathbb{C}^m))\big)\\ \mathrm{d}^3\alpha &\in \mathcal{J}\big(\mathbb{C}^n,\ \mathsf{Hom}(\mathbb{C}^n,\mathsf{Hom}(\mathbb{C}^n,\mathsf{Hom}(\mathbb{C}^n,\mathbb{C}^m)))\big)\end{aligned}$$

and so on; but of course $\mathsf{Hom}(\mathbb{C}^n,\mathsf{Hom}(\mathbb{C}^n,\mathbb{C}^m))=\mathsf{Hom}(\mathbb{C}^n\otimes\mathbb{C}^n,\mathbb{C}^m)$, and we also know from Young's Theorem that $\mathrm{d}^2\alpha$ is actually symmetric: $\mathrm{d}^2\alpha:\mathbb{C}^n\to\mathsf{Hom}(\mathsf{Sym}^2\mathbb{C}^n,\mathbb{C}^m)$. In general

$$\mathrm{d}^k\alpha\ \in\ \mathcal{J}\big(\mathbb{C}^n,\ \mathsf{Hom}(\mathsf{Sym}^k\mathbb{C}^n,\mathbb{C}^m)\big).$$

Since we work with both (multi)linear maps and smooth maps between vector spaces, in order to not mix them up, we adopt the (temporary) convention that $\circ$ denotes the composition of arbitrary maps, while $*$ denotes the composition of linear maps.

Proposition A.1.1 (Chain rule).

$$\big(\mathrm{d}(\alpha\circ\beta\circ\gamma\circ\cdots\circ\zeta)\big)[x]=(\mathrm{d}\alpha)\big[(\beta\circ\gamma\circ\cdots\circ\zeta)(x)\big]*(\mathrm{d}\beta)\big[(\gamma\circ\cdots\circ\zeta)(x)\big]*\cdots*(\mathrm{d}\zeta)[x].$$

The next ingredient is the product rule; however, we will need to apply it to various tensor contractions, so it's time to introduce a graphical notation. Tensor contractions will be represented by trees drawn vertically: The nodes correspond to the tensors, the edges to the contractions, and composition "flows downwards". For example, the pictures on Figure 12 represent the two tensors (written in redundant Einstein notation)

$$S_k^{ij}=\sum_{a,b}(F_1)_k^a(\alpha_2)_a^{ib}(\beta_1)_b^j$$

$$\text{and}\qquad T_k^{ij}=\sum_{a,b,c}(F_2)_k^{ab}(\alpha_1)_a^i(\alpha_1)_b^c(\beta_1)_c^j,$$

respectively. These are actually the two terms of the expression $\mathrm{d}(\mathrm{d}(F\circ\alpha)\circ\beta)|_0$.

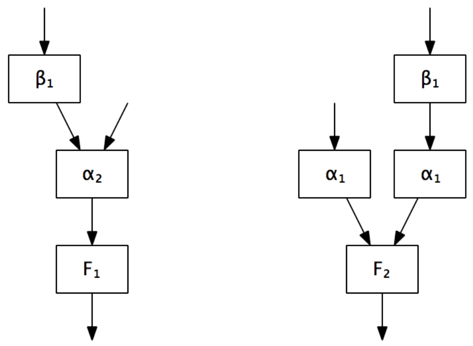


FIGURE 12. Examples of our graphical tensor notation.

PROPOSITION A.1.2 (The product rule). *Suppose we have such a tensor expression, represented by a tree, as a smooth function of a parameter $x \in V$. Then its differential wrt. x is a sum over the nodes of the tree, and the term corresponding to a fixed node can be drawn by replacing the node with its differential, and attaching a new 'input leg', labelled with V, to symbolize the dependence of this differential on $T_xV \cong V$.*

We can incorporate more complex dependencies on the parameter space by drawing horizontal arrows, representing composition of functions. For an example, consider the picture on the left in Figure 13; this represents the expression

$$T[x]^{ij}_k = \sum_{a,b} (\mathrm{d}F)[\alpha(\beta(x))]^a_k \cdot (\mathrm{d}^2\alpha)[\beta(x)]^{ib}_a \cdot (\mathrm{d}\beta)[x]^j_b.$$

Putting together the chain rule and the product rule, we get

PROPOSITION A.1.3 (Pictorial rule of tensor differentials). *The differential of an expression of the form shown on Figure 13 is a sum over the (boxed) nodes of the tree, where the term corresponding to a fixed node can be obtained by replacing that node with its differential, attaching a new leg to it, and attaching to that leg the string of differentials of the incoming horizontal thread (representing the dependence on the parameter space), in accordance with the chain rule A.1.1.*

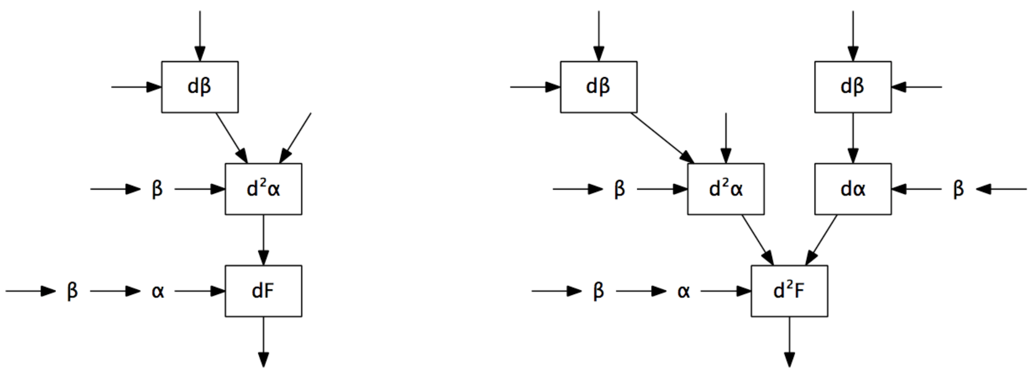


FIGURE 13. A more complex example (left), and one of the 3 terms appearing in its derivative (right).

As an example, the picture on the right in Figure 13 is the term corresponding to the node $\mathrm{d}F$ in the differential of the picture on the left.

COROLLARY A.1.4. *The terms of the kth equation in Porteous' model are in bijection with the rooted trees with k leaves such that the set of the depths of the leaves is $\{2, 3, \ldots, k+1\}$, where the depth is measured as the number of edges from to root the leaf.*

Proof. What we need to show is that the terms in the derivatives of the trees with k leaves are exactly the trees with $k+1$ leaves. But the derivative process, as described above, simply attaches a "long" thread (with the leaf having depth $k+1$) to each existing node in turn. Conversely, starting with a tree with $k+1$ leaves, removing the thread of the "deepest" node (with depth $k+1$) gives back the k-tree and the node (the attach point of the removed thread) whose derivative this tree is. □

For an illustration, see Figure 9 on page 69, which shows the $k = 3$ case (also Figure 12 above shows the $k = 2$ case). The number of such terms (or trees) is growing fast; for $k \leq 8$, the counts are

$$1,\ 2,\ 7,\ 39,\ 321,\ 3686,\ 56516,\ 1118159,\ \ldots$$

A.2. Formulae for symmetric polynomials

In this section we collect together various useful formulae for symmetric polynomials, mostly involving Schur polynomials. The canonical reference is [**Mac98**].

DEFINITION A.2.1. The *Schur polynomial* indexed by the partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a symmetric polynomial in the variables $x_1, \ldots, x_n$ defined as the following quotient of two alternating polynomials:

$$s_\lambda(x_1, \ldots, x_n) = \frac{\mathsf{det}[x_i^{\lambda_j+n-j}]_{n\times n}}{\mathsf{det}[x_i^{n-j}]_{n\times n}} \tag{29}$$

Note that the denominator is a Vandermonde determinant (up to sign).

REMARK A.2.2. The Schur polynomials can be defined for arbitrary sequences of integers, not just partitions, with the same formulae; and it is true that such a "generalised" Schur polynomial is either zero or equals to a "usual" Schur polynomial up to sign. Specifically, applying the following transformation finitely many times, we can always obtain a partition (or zero):

$$s_{(\ldots,a,b,\ldots)} = \begin{cases} -s_{(\ldots,b-1,a+1\ldots)} & a < b-1 \\ 0 & a = b-1 \end{cases}$$

In formula (29), this corresponds to exchanging columns of the matrix in the numerator, or having two identical columns, respectively.

The *Jacobi-Trudi formulae* express the Schur polynomials in terms of elementary (resp. complete) symmetric polynomials:

$$s_\lambda(x) = \mathsf{det}[c_{\mu_i+j-i}(x)] = \mathsf{det}[s_{\lambda_i+j-i}(x)]$$

where $\mu = \widetilde{\lambda}$ is the dual partition.

Let us introduce two variations of Schur polynomials, which also depend on a second alphabet $y = (y_1, \dots, y_m)$. Define $c_k(x-y)$ via the equation

$$\frac{\prod_i (1+x_i t)}{\prod_j (1+y_j t)} = \sum_{k=0}^{\infty} c_k(x-y) t^k,$$

where t is a formal variable. Then the *supersymmetric Schur polynomials* (or Schur polynomials in the 'difference alphabet') can be defined via the Jacobi-Trudi formula as

$$s_\lambda(x-y) = \mathsf{det}\big[c_{\mu_i+j-i}(x-y)\big]_{\ell(\mu)\times\ell(\mu)}, \tag{30}$$

where $\mu = \widetilde{\lambda}$ is the dual partition. $s_\lambda(x-y)$ is a symmetric polynomial in both set of variables.

The *double Schur polynomials* (or factorial Schur polynomials) $s_\lambda(x|y)$, defined as

$$(x_i|y)^k = (x_i+y_1)(x_i+y_2)\cdots(x_i+y_k) = \prod_{j=1}^{k}(x_i+y_j)$$

$$s_\lambda(x|y) = \frac{\mathsf{det}\big[(x_i|y)^{\lambda_j+n-j}\big]_{n\times n}}{\mathsf{det}\big[(x_i|y)^{n-j}\big]_{n\times n}} = \frac{\mathsf{det}\big[(x_i|y)^{\lambda_j+n-j}\big]_{n\times n}}{\mathsf{det}\big[x_i^{n-j}\big]_{n\times n}}, \tag{31}$$

are, in general, symmetric only in the x_i variables. For this definition to make sense, we have either to assume that $m > n + \lambda_1$; or define $y_{>m}$ to be zero.

The two constructions are related by the surprising fact that $s_\lambda(x-y) = s_\lambda(x|y^\vee)$ in the limit $n \to \infty$ (see [**Mac92**]); also both specialize to the usual Schur polynomials when substituting $y = 0$.

In the following, instead of explicit variables $x_1, \dots, x_n$, we will work with the Grothendieck ring of representations (or even more generally, equivariant vector bundles); for non-virtual representations, the variables are the roots of the representation, but in general they do not exist.

A family of natural questions ask for expressing the Schur polynomials of various derived representations, eg. symmetric and antisymmetric tensor powers, in terms of the Schur polynomials of the original representation(s). A compact way to ask these questions is to find some kind of formula for the coefficients $g^{\nu_1,\dots,\nu_k}_{\lambda;\,\mu_1,\dots,\mu_k} \in \mathbb{Z}$ in the equation

$$s_\lambda\big[\mathbb{S}^{\mu_1}(X_1)\otimes\mathbb{S}^{\mu_2}(X_2)\otimes\cdots\otimes\mathbb{S}^{\mu_k}(X_k)\big] = \sum_{\nu_1}\sum_{\nu_2}\cdots\sum_{\nu_k} g^{\nu_1,\dots,\nu_k}_{\lambda;\,\mu_1,\dots,\mu_k} s_{\nu_1}(X_1)s_{\nu_2}(X_2)\cdots s_{\nu_k}(X_k)$$

where λ, μ_i, ν_i are partitions, and $\mathbb{S}^\mu$ is the Schur functor corresponding to the partition μ. This is very far from being solved[2], however, a few special and very useful cases are known.

Lemma A.2.3 (Pieri's rule).

$$s_\mu(X)\cdot s_k(X) = \sum_{\lambda\in K} s_\lambda(X)$$

where λ runs over the partitions K which are obtained from μ by adding k boxes to the Young diagram of μ, but no two boxes in the same column.

[2]For example, as we prove in this thesis, these coefficients for $k = 1$, $\mu_1 = (2)$ coincide with the coefficients of the Thom polynomials of Σ^{ii} singularities, expressed in Schur polynomials.

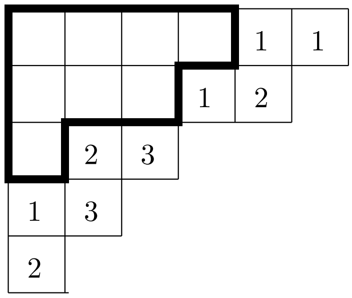


FIGURE 14. A strict $(4,3,2)$-expansion of $(4,3,1)$. The boxes read from the right to the left and from the top to the bottom are $1,1,2,1,3,2,3,1,2$.

LEMMA A.2.4 (multiplication of Schur polynomials).

$$s_\mu(X)\cdot s_\nu(X)=\sum_\lambda c^\lambda_{\mu\nu}s_\lambda(X)$$

where $c^\lambda_{\mu\nu}$ are the Littlewood-Richardson coefficients.

The numbers $c^\lambda_{\mu\nu}$ are determined by the *Littlewood-Richardson rule*:

PROPOSITION A.2.5. *$c^\lambda_{\mu\nu}$ equals the number of ways the Young diagram of ν can be expanded to the Young diagram of λ by a strict μ-expansions. A μ-expansion of a Young diagram is obtained by adding μ_1 boxes, according to Pieri's rule, and filling them with the number 1; then adding μ_2 boxes and filling them with the number 2, etc. The expansion is* strict *if, when these integer numbers are listed from the right to the left and from the top to the bottom (in the English notation), in any initial segment of this list, any number k appears at least as many times as the next number $k+1$.*

REMARK. Note that while $c^\lambda_{\mu\nu}$ is symmetric for the exchange of μ and ν, the Littlewood-Richardson rule is *not*. As far as we know, there is no known enumerative interpretation for $c^\lambda_{\mu\nu}$ in which this symmetry is manifest.

LEMMA A.2.6 (branching rule).

$$\begin{aligned} s_\lambda(X^{-1})=s_{\widetilde{\lambda}}(X^\vee) &= (-1)^{|\lambda|}s_{\widetilde{\lambda}}(X)\\ s_\lambda(X\oplus Y) &= \sum_{\mu,\nu} c^\lambda_{\mu\nu}s_\mu(X)s_\nu(Y)\\ s_\lambda(X\ominus Y) &= \sum_{\mu,\nu} c^\lambda_{\mu\nu}s_\mu(X)s_{\widetilde{\nu}}(Y^\vee). \end{aligned}$$

An important special case is when $\lambda=(n^k)$ is a rectangle, because in this case

$$c^\lambda_{\mu\nu}=\begin{cases}1 & \text{if } \mu\subset\lambda \text{ and } \nu=\mathsf{C}\mu;\\ 0 & \text{otherwise.}\end{cases}$$

COROLLARY A.2.7.

$$\begin{aligned} c_{\mathrm{top}}(X^n\otimes Y^k) &= \sum_{\mu\subset n^k} s_{\widetilde{\mu}}(X)s_{\mathsf{C}\mu}(Y)\\ c_{\mathrm{top}}(\mathsf{Hom}(X^n,Y^k)) &= \sum_{\mu\subset n^k} s_{\widetilde{\mu}}(X^\vee)s_{\mathsf{C}\mu}(Y)=s_{(n^k)}(Y\ominus X) \end{aligned}$$

Let us introduce the determinant (see Chapter 4, Section 4.5)

$$E_{\lambda/\mu}(n) = \mathsf{det}\left[\binom{\lambda_i + n - i}{\mu_j + n - j}\right]_{n\times n}.$$

Lemma A.2.8 ([**Las78**]). *Let L be a 1 dimensional (virtual) representation. Then*

$$\begin{aligned} s_\lambda(X^n \otimes L) &= \sum_{\mu\subset\lambda} E_{\lambda/\mu}(n) s_\mu(X) s_{|\lambda/\mu|}(L) \\ c(X^n \otimes Y^k) &= \sum_{\mu\subset\lambda\subset n^k} E_{\lambda/\mu}(k) s_\mu(Y) s_{\widetilde{\complement\lambda}}(X) \\ c(\wedge^2 X^n) &= \sum_{\mu\subset\lfloor \mathrm{n-1}\rceil} 2^{|\mu|-n(n-1)/2}\cdot E_{\lfloor \mathrm{n-1}\rceil/\mu}(n)\cdot s_\mu(X) \\ c(\mathsf{Sym}^2 X^n) &= \sum_{\mu\subset\lfloor \mathrm{n}\rceil} 2^{|\mu|-n(n-1)/2}\cdot E_{\lfloor \mathrm{n}\rceil/\mu}(n)\cdot s_\mu(X) \end{aligned}$$

where c denotes the total Chern class: $c(X) = \sum c_i(X)$, and $\lfloor \mathrm{n}\rceil = (n, n-1, n-2, \dots, 1)$ is the 'stairway' partition.

A.3. Localization of equivariant cohomology classes

Localization in equivariant cohomology has a rich history, dating back to Duistermaat and Heckman [**DH82**], Atiyah and Bott [**AB84**] and Berline and Vergne [**BV82, BV83**].

We build our treatment on the algebraic theory developed by Edidin and Graham in [**EG98a, EG98b**], so that the singular case fits into the theory naturally. We can then pass to the cohomology version via the so called *cycle map*. Regarding the intersection theory background, we refer to the standard source [**Ful98**]. We will constrain ourselves to torus actions, which causes no problems in our context, since the GL_n-equivariant cohomology is a subring of the $\mathbb{T}^n$-equivariant cohomology for a maximal torus $\mathbb{T}^n \subset \mathsf{GL}_n$.

Lemma A.3.1 ([**Ive72**]). *Let Y be smooth variety with a torus action. Then the fixed point set $Y^{\mathbb{T}}$ is also smooth.*

Theorem A.3.2 ([**EG98b**], Proposition 6). *Let $f : X \to Y$ be a $\mathbb{T}$-equivariant embedding of X into a nonsingular variety Y. Assume that every component of $Y^{\mathbb{T}}$ which intersects X is contained in X. For a component $F \subset X^{\mathbb{T}}$ write $i_F : F \to X$ and $j_F = f \circ i_F : F \to Y$ for the corresponding embeddings. Let $R = A^*_{\mathbb{T}}(\mathsf{pt}) \cong \mathbb{Q}[t_1, \dots, t_n]$ and $\mathcal{Q} = (R_+)^{-1}R$, where $R_+ \subset R$ is the multiplicative system of homogeneous elements of positive degree. Then*

- *$f_* : A^{\mathbb{T}}_*(X) \otimes_R \mathcal{Q} \to A^{\mathbb{T}}_*(Y) \otimes_R \mathcal{Q}$ is injective;*
- *Let $\alpha \in A^{\mathbb{T}}_*(X) \otimes_R \mathcal{Q}$. Then*

$$\alpha = \sum_{F\subset X^{\mathbb{T}}} (i_F)_* \frac{j_F^* f_* \alpha}{c_{\mathrm{top}}(N_F Y)},$$

where F runs over the components of $X^{\mathbb{T}}$, and $c_{\mathrm{top}}(N_F Y)$ is the $\mathbb{T}$-equivariant top Chern class (in the Chow ring of F) of the normal bundle of F in Y.

Remark. Implicit in the theorem is the fact that $c_{\mathrm{top}}(N_F Y)$ is invertible in $A^{\mathbb{T}}_*(F) \otimes \mathcal{Q}$, see [**EG98b**], Proposition 4.

For smooth varieties, we can use the cycle map to pass to the cohomology: For a smooth n-dimensional variety X, we have

$$A^{\mathbb{T}}_{n-k}(X) = A^k_{\mathbb{T}}(X) \xrightarrow{cl} H^{2k}_{\mathbb{T}}(X).$$

See [**Ful98**], Chapter 19 and [**EG98a**], sections 2.6, 2.8.

Equivariant localization can be used to compute the pushforward:

Corollary A.3.3. *Let $\pi : M \to \mathsf{pt}$ be the map collapsing a smooth compact variety M to a point, and $\alpha \in H^*_{\mathbb{T}}(M)$ a cohomology class (it could be a Chow class, too). Assume that M has isolated fixed points. Then*

$$\pi_*\alpha = \sum_{p\in M^{\mathbb{T}}} \frac{i_p^*\alpha}{e(T_pM)}$$

It can also be used to compute classes of subvarieties.

Corollary A.3.4. *In the situation of Theorem A.3.2, we have*

$$[X \subset Y] = \sum_{F\subset X^{\mathbb{T}}} (j_F)_* \frac{j_F^*[X \subset Y]}{e(N_FY)}$$

Proof. Use the theorem with $\alpha = [X]\otimes 1 \in A^{\mathbb{T}}_n(X)\otimes \mathcal{Q}$ and apply f_* to the resulting formula. Note that $f_*[X] = [X \subset Y] \in \mathbb{A}^{\mathbb{T}}_n(Y)$. Finally use the cycle map to pass to the cohomological version. □

Lemma A.3.5. *$j_F^*[X \subset Y] = [N_FX \subset N_FY]$ where N_FX is the bundle of normal cones of X along F, embedded into the normal bundle of F in Y.*

A direct consequence of the Lemma and the Corollary is the following

Theorem A.3.6 (Localization of classes of subvarieties).

$$[X \subset Y] = \sum_{F\subset X^{\mathbb{T}}} (j_F)_* \frac{[N_FX \subset N_FY]}{e(N_FY)}$$

Topological sketch of proof of Lemma A.3.5. Consider a sequence of "smaller and smaller" tubular neighbourhoods of F in Y:

$$Y \supset N_1 \supset N_2 \supset \cdots \supset N_k \supset \cdots \supset F$$

with inclusion maps $i_k : N_k \to Y$ and $j_k : F \to N_k$. By the excision property of cohomology, we have

$$j_F^*[X \subset Y] = j_k^* i_k^*[X \subset Y] = j_k^*[(X \cap N_k) \subset N_k],$$

but the latter is "closer and closer" to $[N_FX \subset N_FY]$. □

Algebraic sketch of proof of Lemma A.3.5. Apply the 'deformation to the normal cone' construction ([**Ful98**], Chapter 5) to $F \subset X$: Let

$$\mathcal{X} = \mathsf{Bl}_{F\times\{\infty\}}(X \times \mathbb{P}^1) \ \subset\ \mathcal{Y} = \mathsf{Bl}_{F\times\{\infty\}}(Y \times \mathbb{P}^1);$$

the pair $(\mathcal{X}, \mathcal{Y}) \to \mathbb{P}^1$ is now a (flat) family of embeddings: a deformation from $F \subset X \subset Y$ at $0 \in \mathbb{P}^1$ to $F \subset N_FX \subset N_FY$ at $\infty \in \mathbb{P}^1$. Apply the 'principle of continuity' for flat families. □

Our main tool will be the following, less well-known but very useful variation of the above.

THEOREM A.3.7 ([**BSz06**], [**Ros89**]). *Let V be a representation of a torus $\mathbb{T}$, M be a smooth compact variety equipped with an action of the torus $\mathbb{T}$, with isolated fixed points, and $X \subset M$ be a (possibly singular) $\mathbb{T}$-invariant closed subvariety. Consider the classifying map $\varphi : M \to \mathsf{Gr}_r(V)$ of a rank r equivariant vector bundle $\mathsf{pr}_1 : E \subset M \times V \to M$, and let $Y \to X$ the pullback (restriction) of E to X Suppose that $\varphi|_X : X \to \varphi(X)$ is birational; then $Z = \mathsf{pr}_2(Y)$ is a closed invariant subvariety of V of dimension $\mathsf{dim}(Z) = r + \mathsf{dim}(X)$, and*

$$[Z \subset V]_{\mathbb{T}} = \sum_{p \in X^{\mathbb{T}}} [Y_p \subset V]_{\mathbb{T}} \cdot \frac{[N_p X \subset T_p M]_{\mathbb{T}}}{e_{\mathbb{T}}(T_p M)} \in H^*_{\mathbb{T}}(\mathsf{pt}) = H^*_{\mathbb{T}}(V),$$

where $Y_p = \mathsf{pr}_1^{-1}(p) \subset V \times \{p\} \cong V$ is the fiber over the point $p \in X$, and $N_p X$ is the tangent cone of X at p (the normal cone of X "along" p).

REMARK. The quotient $\frac{e(T_pM)}{[N_pX \subset T_pM]}$ is simply $e(T_pX)$ if X is smooth at p, and can be thought as the generalization of the tangent Euler class for singular points. Its inverse is sometimes called *equivariant multiplicity* ([**Ros89**], [**Bri97**]). Note that it is actually independent of the embedding, since the normal cone N_pX embeds into the Zariski tangent space T_pX which further embeds into T_pM; thus

$$\frac{e(T_pM)}{[N_pX \subset T_pM]} = \frac{e(T_pX) \cdot e(T_pM/T_pX)}{[N_pX \subset T_pX] \cdot [T_pX \subset T_pM]} = \frac{e(T_pX)}{[N_pX \subset T_pX]}.$$

LEMMA A.3.8. *Let a cyclic group $\mathbb{Z}_k$ act (diagonally and faithfully) on a torus representation $\mathbb{C}^n$. Then for the cyclic quotient singularity $X = \mathbb{C}^n/\mathbb{Z}_k$, the virtual Euler class is just k times the Euler class of $\mathbb{C}^n$ (that is, k times the product of weights).*

Proof. Direct application of [**Bri97**], 4.3. □

Proof of Theorem A.3.7. First we apply Theorem A.3.6 to the embedding $(\Delta \circ K) : Y \subset M \times V$. Note that since M, and thus X has isolated fixed points, we can classify the fixed components of Y by recording which fixed point p of X they lie over. The following diagram summarises the situation and the notations:

$$\begin{array}{ccccccccccc}
F & \xrightarrow{j_F} & Y_p & \xrightarrow{I_p} & Y & \xrightarrow{K} & E & \xrightarrow{\Delta} & M \times V & \xrightarrow{\mathsf{pr}_2} & V \\
& {\scriptstyle \pi_F} \searrow & \downarrow {\scriptstyle \pi_p} & & \downarrow {\scriptstyle \pi} & & \downarrow {\scriptstyle \pi} & \swarrow {\scriptstyle \mathsf{pr}_1} & & & \\
& & p & \xrightarrow{i_p} & X & \xrightarrow{k} & M & & & &
\end{array}$$

Now

$$[Z] = (\mathsf{pr}_2)_*[Y \subset M \times V] = (\mathsf{pr}_2)_* \sum_{F \subset Y^{\mathbb{T}}} (\Delta \circ K \circ I_p \circ j_F)_* \frac{[N_F Y \subset N_F(M \times V)]}{e(N_F(M \times V))} =$$

$$= (\mathsf{pr}_2)_* \sum_{p \in X^{\mathbb{T}}} (\Delta \circ K \circ I_p)_* \sum_{F \subset Y_p^{\mathbb{T}}} (j_F)_* \left(\frac{[N_F Y_p \subset N_F V_p]}{e(N_F V_p)} \cdot (\pi_F)^* \frac{[N_p X \subset T_p M]}{e(T_p M)} \right)$$

since $N_F Y$ is just the product $N_p X \times N_F Y_p$ (note that $F \subset Y_p$ is smooth by Lemma A.3.1), and the class of a product is the product of the classes. Now, observe that all our maps respect the local product structure around p:

$$\begin{array}{ll}
I_p = i_p \otimes \mathsf{id} & \Delta = \mathsf{id} \otimes \Delta_p \\
K = k \otimes \mathsf{id} & \mathsf{pr}_2 = \mathsf{pt} \otimes \mathsf{id}
\end{array}$$

where pt is the collapsing map $\mathsf{pt}: M \to \mathsf{pt}$ and Δ_p is the embedding $\Delta_p : Y_p \subset V_p$. Thus rearranging and applying Theorem A.3.6 again, now in the reverse direction:

$$[Z] = \sum_{p \in X^{\mathbb{T}}} \left((\mathsf{pt} \circ k \circ i_p)_* \frac{[N_p X \subset T_p M]}{e(T_p M)} \right) \sum_{F \subset Y_p^{\mathbb{T}}} (\Delta_p \circ j_F)_* \frac{[N_F Y_p \subset N_F V_p]}{e(N_F V_p)}$$
$$= \sum_{p \in X^{\mathbb{T}}} \frac{[N_p X \subset T_p M]}{e(T_p M)} \cdot [Y_p \subset V],$$

which is what we wanted to prove. □

A.3.1. *Application.* Equivariant localization can be used to prove algebraic identities. Consider a $\mathbb{T}$-representation V with different nonzero weights $w_1, \dots, w_n$, and the blow-up $\pi : U \to V$ of the origin $\{0\} \subset V$. We can use Theorem A.3.2 to give two different formula for any $\alpha \in A_k^{\mathbb{T}}(V) \otimes \mathcal{Q}$. First, apply the theorem to the α and $X = Y = V$:

$$\alpha = i_* \frac{i^* \alpha}{c_{\text{top}}(V)} = i_* \frac{i^* \alpha}{\prod_{j=1}^n w_j},$$

where $i : \{0\} \to V$. But we can also apply it to $\pi^* \alpha$ and U:

$$\pi^* \alpha = \sum_{k=1}^n (i_k)_* \frac{(i_k)^* \pi^* \alpha}{c_{\text{top}}(T_{p_k} U)} = \sum_{k=1}^n (i_k)_* \frac{(i_k)^* \pi^* \alpha}{w_k \cdot \prod_{j \neq k} w_i}, \tag{32}$$

where $i_k : p_k \to U$ and $p_k \in \mathbb{P}V = \pi^{-1}(0) \subset U$ are the fixed points of the blow-up. Apply π_* to (32), note that $\pi_* \pi^* \alpha = \alpha$ for blow-ups, and that $\pi \circ i_k = i$ (after identifying the points p_k and $\{0\}$):

$$\alpha = i_* \frac{i^* \alpha}{\prod_{j=1}^n w_j} = i_* \sum_{k=1}^n \frac{i^* \alpha}{w_k \cdot \prod_{j \neq k} w_i}.$$

Since $i_* : A_*^{\mathbb{T}}(\mathsf{pt}) \otimes \mathcal{Q} \to A_*^{\mathbb{T}}(V) \otimes \mathcal{Q}$ is an isomorphism ([**EG98b**] Theorem 1), and thus so is i^* (by the self-intersection formula $i^* i_* \beta = c_{\text{top}}(V) \cdot \beta$) and this is true for all α, we have

$$\frac{1}{\prod_{j=1}^n w_j} = \sum_{k=1}^n \frac{1}{w_k \cdot \prod_{j \neq k} w_i} \in A_{\mathbb{T}}^*(\mathsf{pt}) \otimes \mathcal{Q} \cong \mathcal{Q}. \tag{33}$$

In general, we can use a sequence of blow-ups, or even flip-flops to get nontrivial identities.

For a very simple example, let V be the standard $\mathbb{T}^n$-representation V. Then the above argument gives the formal identity

$$\frac{1}{\prod_{j=1}^n t_i} = \sum_{k=1}^n \frac{1}{t_k \cdot \prod_{j \neq k} (t_j - t_k)}.$$

A more complex example is used in Section 4.3.2.1.

A.4. Pushforward formulae

The situation we are considering here is the following. Let M be a compact manifold, and $E^n \to M$ a complex vector bundle. The Grassmann bundle $\pi : \mathsf{Gr}_r E = \mathsf{Gr}^q E \to M$ has the the tautological exact sequence of vector bundles

$$0 \to R^r \to \pi^* E \to Q^q \to 0$$

over it. We are interested in formulae expressing the pushforward map

$$\pi_* : H^*(\mathsf{Gr}^q E) \to H^{*-2rq}(M).$$

There are variations of this theme for partial flag bundles, sequences of vector bundles, etc.

THEOREM A.4.1. *Assuming the situation described above, and that $\ell(\lambda) \le q$, $\ell(\mu) \le r$, we have*

$$\pi_* s_\lambda(Q) = s_{(\lambda - r^q)}(E); \tag{34}$$

$$\pi_* s_\mu(R) = (-1)^{rq} s_{(\mu - q^r)}(E); \tag{35}$$

$$\pi_*[s_\mu(R) s_\lambda(Q)] = s_{(\lambda - r^q, \mu)}(E). \tag{36}$$

Furthermore, if $F \to M$ is another vector bundle,

$$\pi_*[s_\mu(R|F) s_\lambda(Q|F)] = s_{(\lambda - r^q, \mu)}(E|F); \tag{37}$$

$$\pi_*[s_\mu(R - F) s_\lambda(Q - F)] = s_{(\lambda - r^q, \mu)}(E - F). \tag{38}$$

The same formulae are true for the universal bundle, and equivariant vector bundles too; and also for Chow groups instead of cohomology.

REMARKS. Both (34) and (35) are special cases of (36), which itself is special case of both (37) and (38). The RHS of these formulae should be understood according to Remark A.2.2, which explains the sign in (35). The most useful variation (38) is proved in [**JLP82, Pra88**]. The case (35) was also proved in [**Ron72**]. As far as we now, (37) is new.

In the remaining part of the section, we give a simple geometric proof of (37), using equivariant localization. We believe this proof can be adapted to the last case (38) too, for example using the so-called Sergeev-Pragacz formula for the supersymmetric Schur polynomials.

A geometric representation. For the proof, we will need a geometric representation for the Schur classes $s_\lambda(E)$; consider the following construction. Let E^n be the standard GL_n representation; fix a large integer $N \gg 0$, the standard representation F^N of GL_N, and a complete flag $K_\bullet$ in $F^\vee$:

$$0 = K_0 < K_1 < K_2 < \cdots < K_N = F^\vee, \quad \mathsf{dim}(K_j) = j.$$

Denote by B_N the Borel subgroup of GL_N fixing $K_\bullet$. Let τ be a partition

$$N \ge \tau_1 \ge \tau_2 \ge \cdots \ge \tau_n \ge 0.$$

Consider $\mathsf{Fl}_\tau(E^\vee)$, the variety of partial flags in the dual space $E^\vee$ with dimensions corresponding to $\{\widetilde{\tau}_j : j\}$: points of $\mathsf{Fl}_\tau(E^\vee)$ correspond to sequences $A_\bullet$ of linear subspaces

$$0 < \cdots < A_i < \cdots \le E^\vee, \quad \mathsf{dim}(A_i) = i, \quad (i, *) \in \mathsf{corner}(\tau).$$

Here corner denotes the set of outer corners of the Young diagram of a partition:

$$\mathsf{corner}(\mu) = \left\{\, (i,j) \in \mathbb{N} \times \mathbb{N} \;:\; \mu_i = j,\ \widetilde{\mu}_j = i \,\right\}.$$

If τ is a *strict partition*, Fl_τ is simply the complete flag variety. Let $\{e_k\}$ denote the (ordered) set of the dimensions of the subspaces in $A_\bullet$:

$$\left\{\, e_1, e_2, \ldots, e_l \,\right\} = \left\{\, i \;:\; (i,*) \in \mathsf{corner}(\tau) \,\right\}, \quad l = |\mathsf{corner}(\tau)|,$$

(note that $0 < e_k \le n$; for convenience, set $e_0 = 0$), and let d_τ denote the dimension of Fl_τ:

$$d_\tau = \mathsf{dim}(\mathsf{Fl}_\tau(E^\vee)) = \sum_{k=1}^{l}(e_k - e_{k-1})(n - e_k) \le \binom{n}{2}.$$

Now consider the vector bundle $\mathsf{pr}_1 : \widetilde{S}_\tau \to \mathsf{Fl}_\tau(E^\vee)$ of subspaces of $E \otimes F$, where elements of $E \otimes F$ are thought as bilinear functions on $E^\vee \times F^\vee$:

$$\widetilde{S}_\tau = \left\{ \ (A_\bullet, f) \in \mathsf{Fl}_\tau(E^\vee) \times E \otimes F \ : \ f|_{A_i \otimes K_j} = 0, \ \ (i,j) \in \mathsf{corner}(\tau) \ \right\}$$

and let $S_\tau = \mathsf{pr}_2(\widetilde{S}_\tau)$ be the image of $\widetilde{S}_\tau$ in $E \otimes F$.

Lemma A.4.2. *S_τ is a $\mathsf{GL}_n \times \mathsf{B}_N$-invariant closed subvariety of $E \otimes F$ of codimension*

$$\mathsf{codim}(S_\tau) \ = \ |\tau| - d_\tau \ \ge \ |\tau| - \binom{n}{2},$$

with equality if and only if τ is a strict partition.

Proof. The only thing not clear here is the (co)dimension. However, $\widetilde{S}_\tau \to \mathsf{Fl}_\tau$ is GL_n-equivariant by construction, and it is easy to see that for any flag $A_\bullet$ the stabiliser of the fibrum $X_{A_\bullet} = \mathsf{pr}_1^{-1}(A_\bullet)$ is the same as the stabiliser of $\mathsf{pr}_2(X_{A_\bullet})$, from which the codimension formula follows using the simple fact that $\mathsf{codim}(X_{A_\bullet}) = |\tau|$. □

Remark. The unique flag $A_\bullet \in \mathsf{Fl}_\tau$ corresponding to a generic map $f \in S_\tau$ is

$$\left\{ \ A_i = \mathsf{coker}(f^\dagger|_{K_j}) \ : \ (i,j) \in \mathsf{corner}(\tau) \ \right\}$$

where $f^\dagger$ is the image of f at the canonical isomorphism $E \otimes F \to \mathsf{Hom}(F^\vee, E)$. Alternatively, it can be also found algorithmically, by applying the Gaussian elimination process to the matrix of f, where on F we choose a basis compatible with $K_\bullet$.

If τ is a *strict* partition, let us denote by λ the partition $\lambda = \tau - \lfloor \mathrm{n} - 1 \rceil$ (that is, $\lambda_i = \tau_i - n + i$), and use the alternative name Z_λ for S_τ. In this case Fl_τ is the complete flag variety, $\mathsf{codim}(Z_\lambda) = |\lambda|$, and

Lemma A.4.3. *Z_λ represents the classes*

$$[Z_\lambda]_{\mathsf{GL}_n \times \mathsf{B}_N} = s_\lambda(E|F) \in H^*_{\mathsf{GL}_n \times \mathsf{B}_N}(E \otimes F).$$

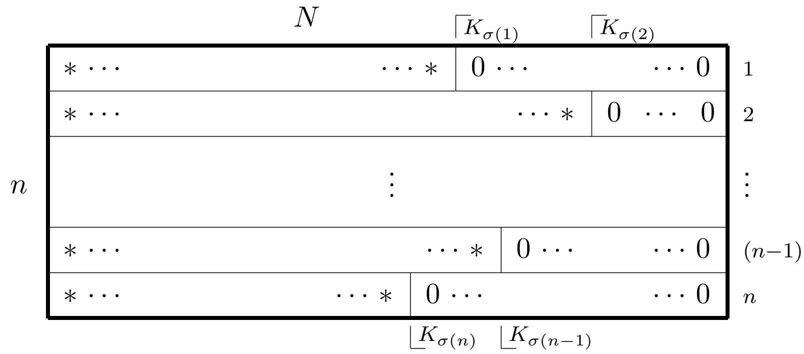
Proof. The proof is a direct application of Theorem A.3.7. As always, we can reduce $\mathsf{GL}_n \times \mathsf{B}_N$ to its maximal torus $\mathbb{T}^n \times \mathbb{T}^N$. The fixed points of the complete flag variety $\mathsf{Fl}(E^\vee)$ are the coordinate flags, indexed by permutations of n. Let

$$x_1, \ldots, x_n \quad \text{and} \quad y_1, \ldots, y_N$$

denote the weights of E and F, respectively; then the weights of the tangent space representation $T_\sigma\mathsf{Fl}(E^\vee)$ at the fixed flag corresponding to the permutation $\sigma \in \mathfrak{S}_n$ are $\{ -x_{\sigma(j)} + x_{\sigma(i)} : j > i \}$, thus the equivariant Euler class $e_\mathbb{T}(T_\sigma\mathsf{Fl}(E^\vee))$ is

$$e_\mathbb{T}\big(T_\sigma\mathsf{Fl}(E^\vee)\big) = \mathsf{sgn}(\sigma)\prod_{j>i}(x_i - x_j) = \mathsf{det}\big[x_i^{n-j}\big].$$

Choosing a basis of $F^\vee$ such that the K_j are coordinate subspaces, the fiber $Z_\sigma \subset \mathsf{Hom}(E^\vee, F)$ over $\sigma \in \mathfrak{S}_n$ consists of the matrices of the form



thus its class is

$$\big[Z_\sigma \subset E \otimes F\big]_{\mathsf{GL}_n \times \mathsf{B}_N} = \prod_{i=1}^{n} \prod_{k=1}^{\lambda_i+n-i} (x_{\sigma(i)} + y_k).$$

Summing over $\sigma \in \mathfrak{S}_n$, we get the right hand side of Equation (31). ☐

The proof. Now we are prepared to prove the pushforward formula (37). We present two variations of the proof. The first one is more intuitive, but leaves the algebraic category. The second one fixes this problem.

THEOREM A.4.4. *Let λ and μ be partitions with $\ell(\lambda) \leq q$ and $\ell(\mu) \leq r$. In the situation described above, we have*

$$\pi_*[s_\mu(R|F)s_\lambda(Q|F)] = s_{(\lambda - r^q, \mu)}(E|F) \in H^*_{\mathsf{GL}_n}(\mathsf{pt}),$$

where the right hand side should be understood according to Remark A.2.2.

Proof variation A. Since the s_λ are characteristic classes, they are *universal*; thus $s_\lambda(Q|F)$ is represented by subvariety $Z_Q \subset Q \otimes F$ which we get by applying the construction of the previous section *fiberwise* to the bundle $Q \to \mathsf{Gr}^q(E)$:

$$\widetilde{Z}_Q = \big\{\ (A_\bullet, f, Q) \in \mathsf{Fl}(Q^\vee) \times (Q \otimes F) \to \mathsf{Gr}^q(E)\ :\ f|_{A_i \otimes K_{\lambda_i+n-i}} = 0\ \big\},$$

and $Z_Q = \mathsf{pr}_2(\widetilde{Z}_Q)$; similarly for μ and $Z_R \subset R \otimes F$. At this point we step out of the algebraic category, since we want to identify Q with a complement of R; but a holomorphic complement of R does not exist. However, if we don't want holomorphicity, we can simply identify Q with $R^\perp$. Again by universality, the fibre product

$$X = Z_R \times_{\mathsf{Gr}} Z_Q \subset (R \oplus Q) \otimes F \cong \pi^* E \otimes F \to \mathsf{Gr}_r(E)$$

represents $[X] = s_\mu(R|F)s_\lambda(Q|F) \in H^*(\mathsf{Gr}_r E)$. Consider the projection

$$Y = \bar{\pi}(X) \subset E \otimes F.$$

There are two cases here: $s_{(\lambda - r^q, \mu)}$ is either $\pm s_\nu$ for a honest partition ν, or 0 otherwise. It is easy to see (see Remark A.2.2) that this is equivalent to ask whether

$$\tau = \big(\mu + \lfloor \mathrm{r} - 1 \rceil\big) \cup \big(\lambda + \lfloor \mathrm{q} - 1 \rceil\big)$$

is a *strict* partition or not. On the other hand, *set-theoretically* $Y = S_\tau$; thus, according to Lemma A.4.2 it has the "right" codimensions if and only if τ is strict.

In the latter case, consider the following resolution of Y:

$$\begin{aligned}\widetilde{Y} = \{ \ (U_\bullet, f_R, f_Q) &\in \mathsf{Fl}_\tau(E^\vee) \times (U_r \otimes F) \times (U_r^\perp \otimes F) \\ &: f|_{U_i \otimes K_{\mu_i+r-i}} = 0 \ \ \forall i \le r \\ &, g|_{(U_j \cap U_r^\perp)\otimes K_{\lambda_{j-r}+q-(j-r)}} = 0 \ \ \forall j > r \ \}.\end{aligned}$$

The following diagram summarises the situation:

$$\begin{array}{ccccc} & & \mathsf{Hom}(\pi^*E, F) & \xrightarrow{\bar\pi} & \mathsf{Hom}(E, F) \\ & & \uparrow & & \uparrow \\ \widetilde{Y} & \xrightarrow{\bar\alpha} & Z_R \times_{\mathsf{Gr}} Z_Q & \xrightarrow{\bar\pi} & Y \\ \downarrow & & \downarrow & & \downarrow \\ \mathsf{Fl}(E) & \xrightarrow{\alpha} & \mathsf{Gr}_r(E) & \xrightarrow{\pi} & \mathsf{pt} \end{array}$$

where $\alpha(U_\bullet) = U_r$ and $\bar\alpha(U_\bullet, f, g) = (U_r, f, g)$. Applying Theorem A.3.7 to $\widetilde{Y}$ gives the desired result, similarly as in the proof of Lemma A.4.3. The sign comes from the different order of $(\mu + \lfloor \mathrm{r}-1\rceil) \cup (\lambda + \lfloor \mathrm{q}-1\rceil)$ and $(\mu + \lfloor \mathrm{r}-1\rceil, \lambda + \lfloor \mathrm{q}-1\rceil)$ (it will be the sign of the permutation between the two; cf. Remark A.2.2). □

Remark. It is in fact not surprising that the above proof does not work in the algebraic category: equivariant cohomology classes (also called *multidegrees*) represented by complex algebraic varieties are always "positive", similarly as degrees of projective varieties are positive. By "positive", we mean that it is in the cone spanned by the weights of the ambient representation, which is $E \otimes F$ in our case. However, we have a sign in our formula, depending on the relation of λ and μ, that is, depending on the geometry and not just, say, the conventions. Nonetheless, the proof works in the algebraic category if either $\mu = 0$ or $\lambda = 0$, which gives the idea for the second variation below.

Proof variation B. We will calculate the pushforward of

$$s_\mu(R^\vee|F^\vee)s_\lambda(Q|F) = (-1)^{|\mu|}s_\mu(R|F)s_\lambda(Q|F).$$

For this, consider the varieties

$$\begin{aligned} X_Q &= (q \otimes \mathsf{id})^{-1}(Z_Q) \subset E \otimes F \to \mathsf{Gr}^q(E) \quad \text{and} \\ X_{R^\vee} &= (i^\vee \otimes \mathsf{id})^{-1}(Z_{R^\vee}) \subset E^\vee \otimes F^\vee \to \mathsf{Gr}_r(E^\vee) = \mathsf{Gr}^q(E), \end{aligned}$$

where i and q are the tautological inclusion and factor maps:

$$\begin{array}{ccccc} R & \xrightarrow{i} & E & \xrightarrow{q} & Q \quad ; \\ R^\vee & \xleftarrow{i^\vee} & E^\vee & \xleftarrow{q^\vee} & Q^\vee \quad . \end{array}$$

We still have $[X_Q] = s_\lambda(Q|F)$ and $[X_{R^\vee}] = s_\mu(R^\vee|F^\vee)$, and thus

$$\left[X_Q \times X_{R^\vee} \subset (E \otimes F) \oplus (E^\vee \otimes F^\vee) \to \mathsf{Gr}^q(E)\right] = s_\mu(R^\vee|F^\vee)s_\lambda(Q|F).$$

Note that there is a canonical isomorphism

$$\begin{array}{ccc} \mathsf{Fl}(Q^\vee)\times_{\mathsf{Gr}}\mathsf{Fl}(R) & \longrightarrow & \mathsf{Fl}(E^\vee) \\ \{A_1, \ldots, A_q\} \ \ , \ \ \{B_1, \ldots, B_r\} & \longmapsto & \left\{A_1, \ldots, A_q = (E/B_r)^\vee, \ldots, (E/B_1)^\vee, E^\vee\right\} \end{array}$$

and we can compute the class $\pi_*[X] = [\pi(X)]$ using Theorem A.3.7, as before. □

A.5. Basic hypergeometric series

Here we collect the definitions and theorems we use from the theory of basic hypergeometric[3], or q-hypergeometric series. We refer to [**GR90**] for the details.

Definition A.5.1 (The q-shifted factorial).

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k)$$

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty} = \begin{cases} (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}) & n>0 \\ 1 & n=0 \\ 1/\big[(1-aq^{-1})(1-aq^{-2})\cdots(1-aq^{-n})\big] & n<0 \end{cases}$$

We will sometimes use the shorthand notation

$$(a_1,a_2,\ldots a_k;q)_n = \prod_{i=1}^{k}(a_i;q)_n$$

Definition A.5.2 (The q-binomial coefficient).

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}$$

Definition A.5.3 (The basic hypergeometric (or q-hypergeometric, or Heine's) series).

$${}_2\Phi_1\left[\begin{matrix} a,b \\ c \end{matrix}\,\middle|\, q,z\right] = \sum_{n=0}^{\infty}\frac{(a;q)_n(b;q)_n}{(q;q)_n(c;q)_n}z^n$$

Definition A.5.4 (The generalized q-hypergeometric series).

$${}_r\Phi_s\left[\begin{matrix} a_1,a_2,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix}\,\middle|\, q,z\right] = \sum_{n=0}^{\infty}\frac{(a_1,\ldots,a_r;q)_n}{(q,b_1,\ldots,b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{(1+s-r)} z^n$$

One of the fundemental results in the subject is the *q-binomial theorem*:

Theorem A.5.5. *For $|z|<1$ and $|q|<1$*

$$\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}z^n = \frac{(az;q)_\infty}{(z;q)_\infty}$$

A general trick is letting a parameter tend to infinity. For example:

Corollary A.5.6. *Setting $z=z/a$ and letting $a\to\infty$ in the q-binomial theorem, we get*

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}}{(q;q)_n}z^n = (z;q)_\infty$$

Theorem A.5.7 (Finite q-binomial theorem).

$$(ab;q)_n = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q b^k(a;q)_k(b;q)_{n-k}$$

The Theorem A.5.5 follows from this one by setting $b=z$ and letting n tend to infinity.

[3]The word 'basic' refers to 'base q'; for example there are 'bibasic' series too, which contain two parameters p and q.

Corollary A.5.8 (Finite version of Corollary A.5.6).

$$(z;q)_n = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} z^k$$

Theorem A.5.9 (Heine's transformation formulae). *For* $|z| < 1$ *and* $|b| < 1$

$$ {}_2\Phi_1\left[\begin{matrix} a, b \\ c \end{matrix} \,\middle|\, q, z\right] = \frac{(b, az; q)_\infty}{(c, z; q)_\infty} \; {}_2\Phi_1\left[\begin{matrix} c/b, z \\ az \end{matrix} \,\middle|\, q, b\right] \tag{39}$$

and, iterating it:

$$= \frac{(c/b, bz; q)_\infty}{(c, z; q)_\infty} \, {}_2\Phi_1\left[\begin{matrix} abz/c, b \\ bz \end{matrix} \,\middle|\, q, c/b\right] \tag{40}$$

$$= \frac{(abz/c; q)_\infty}{(z; q)_\infty} \; {}_2\Phi_1\left[\begin{matrix} c/a, c/b \\ c \end{matrix} \,\middle|\, q, abz/c\right] \tag{41}$$

Theorem A.5.10 (Jacobi's triple product identity).

$$(q, zq, z^{-1}; q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{\binom{n+1}{2}} z^n$$

Theorem A.5.11 (Finite version of Jacobi's triple product identity).

$$(zq; q)_n (z^{-1}; q)_m = \sum_{k=-m}^{n} (-1)^k \begin{bmatrix} m+n \\ m+k \end{bmatrix}_q q^{\binom{k+1}{2}} z^k$$

Proof (Cauchy). Applying Corollary A.5.8:

$$\sum_{j=0}^{m+n} (-1)^j \begin{bmatrix} m+n \\ j \end{bmatrix}_q q^{\binom{j}{2}} \left(zq^{1-m}\right)^j = (zq^{1-m}; q)_{m+n} = (zq^{1-m}; q)_m (zq; q)_n$$

$$= (-1)^m q^{-\binom{m}{2}} z^m (z^{-1}; q)_m (zq; q)_n$$

Rearranging and substituting $j \mapsto m + k$ gives the desired result. □

Letting m and n tend to infinity gives Theorem A.5.10.

Definition A.5.12 (The bilateral basic hypergeometric series).

$$ {}_r\Psi_s\left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix} \,\middle|\, q, z\right] = \sum_{n=-\infty}^{\infty} \frac{(a_1, \ldots, a_r; q)_n}{(b_1, \ldots, b_s; q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{(s-r)} z^n$$

Theorem A.5.13 (Ramanujan's summation formula for ${}_1\Psi_1$). *For* $|b/a| < |z| < 1$

$$ {}_1\Psi_1\left[\begin{matrix} a \\ b \end{matrix} \,\middle|\, q, z\right] = \frac{(q,\ b/a,\ az,\ q/az\,; q)_\infty}{(b,\ q/a,\ z,\ b/az\,; q)_\infty}$$

Note how Jacobi's triple product identity follows from this by setting $b = 1/a$, $z = qz/a$ and letting a tend to infinity.

Theorem A.5.14 (Bailey's transformation formula for ${}_2\Psi_2$).

$$ {}_2\Psi_2\left[\begin{matrix} a, b \\ c, d \end{matrix} \,\middle|\, q, z\right] = \frac{(az,\ d/a,\ c/b,\ dq/abz\,; q)_\infty}{(z,\ d,\ q/b,\ cd/abz\,; q)_\infty} \cdot {}_2\Psi_2\left[\begin{matrix} a, abz/d \\ c, az \end{matrix} \,\middle|\, q, \frac{d}{a}\right]$$

Bibliography

[AB84] Michael F. Atiyah and Raoul Bott, *The moment map and equivariant cohomology*, Topology **23** (1984), no. 1, 1–28.

[AVGL98] V. I. Arnold, V. A. Vasil'ev, V. V. Goryunov, and O. V. Lyashko, *Singularity theory I.*, Springer, 1993, 1998, Originally published as Vol. 6 in the series: Encyclopaedia of Mathematical Sciences.

[BSz06] Gergely Bérczi and András Szenes, *Thom polynomials of Morin singularities*, arXiv:math/0608285, 2006.

[Bér08] Gergely Bérczi, *Multidegrees of singularities and nonreductive quotients*, Ph.D. thesis, Budapest University of Technology, 2008.

[BFR02] Gergely Bérczi, László M. Fehér, and Richárd Rimányi, *Expressions for resultants coming from the global theory of singularities*, Topics in Algebraic and Noncommutative Geometry (L. McEwan, J. P. Brasselet, C. Melles, G. Kennedy, and K. Lauter, eds.), Contemporary Mathematics, vol. 324, AMS, 2002.

[Boa67] J. Michael Boardman, *Singularities of differentiable maps*, Inst. Hautes Études Sci. Publ. Math **33** (1967), 21–57.

[Bri97] Michel Brion, *Equivariant Chow groups for torus actions*, Transformation Groups **2** (1997), no. 3, 225–267.

[BV82] Nicole Berline and Michéle Vergne, *Classes caractéristiques équivariantes. Formule de localisation en cohomologie équivariante*, C. R. Acad. Sci. Paris Sér. I Math **295** (1982), 539–541.

[BV83] ______, *Zéros d'un champ de vecteurs et classes charactéristiques équivariantes*, Duke Math. J. **50** (1983), no. 2, 539–549.

[Dam72] James Damon, *Thom polynomials for contact class singularities*, Ph.D. thesis, Harvard, 1972.

[DH82] J. J. Duistermaat and G. J. Heckman, *On the variation in the cohomology of the symplectic form of the reduced phase space*, Invent. Math. **69** (1982), no. 2, 259–268.

[EG98a] Dan Edidin and William Graham, *Equivariant intersection theory*, Invent. Math. **131** (1998), no. 3, 595–634.

[EG98b] ______, *Localization in equivariant intersection theory and the Bott residue formula*, Amer. J. Math. **120** (1998), no. 3, 619–636.

[Eis95] David Eisenbud, *Commutative algebra with a view toward algebraic geometry*, GTM, vol. 150, Springer, 1995.

[FK06] László M. Fehér and Balázs Kőműves, *On second order Thom-Boardman singularities*, Fundamenta Mathematicae **191** (2006), no. 3, 249–264.

[FNR05] László M. Fehér, András Némethi, and Richárd Rimányi, *Degeneracy of 2-forms and 3-forms*, Canad. Math. Bull. **48** (2005), no. 4, 547–560.

[FP98] William Fulton and Piotr Pragacz, *Schubert varieties and degeneracy loci*, LNM, vol. 1689, Springer, 1998.

[FR04] László M. Fehér and Richárd Rimányi, *Calculation of Thom polynomials and other cohomological obstructions for group actions*, Real and complex singularities, Contemp. Math, vol. 354, AMS, Providence, R.I., 2004, pp. 69–93.

[FR06] ______, *Thom polynomial computing strategies. A survey*, Singularity Theory and Its Applications, Adv. Studies in Pure Math., vol. 43, Math. Soc. Japan, 2006, pp. 45–53.

[FR07] ______, *On the structure of Thom polynomials of singularities*, Bull. London Math. Soc. **39** (2007), no. 4, 541–549.

[FR08] ______, *Thom series of contact singularities*, arXiv:0809.2925, 2008.

[Fra09] Matthias Franz, *Convex - a Maple package for convex geometry*, published electronically at http://www.math.uwo.ca/∼mfranz/convex/, 2009.

[Ful96] William Fulton, *Schubert varieties in flag bundles for the classical groups*, Procs. of the Hirzebruch 65 Conference on Algebraic Geometry, Israel Math. Conf. Proc., vol. 9, 1996, pp. 241–262.

[Ful98] ______, *Intersection theory*, Springer-Verlag, 1994, 1998.

[Gaf83] Terence Gaffney, *The Thom polynomial of* $\overline{\sum^{1111}}$, Singularities, Part 1 (Arcata, Calif., 1981), Proc. Sympos. Pure Math, vol. 40, AMS, 1983, pp. 399–408.

[GH78] Phillip Griffiths and Joseph Harris, *Principles of algebraic geometry*, Wiley-Interscience, 1978.

[GHC] *The Glasgow Haskell Compilation System*, http://haskell.org/ghc/.

[GR90] George Gasper and Mizan Rahman, *Basic hypergeometric series*, Cambridge Univ. Press, 1990.

[Har95] Joe Harris, *Algebraic geometry. A first course*, GTM, vol. 133, Springer-Verlag, 1995.

[HK57] A. Haefliger and A. Kosinski, *Un théorème de Thom sur les singularités des applications différentiables*, Séminaire Henri Cartan, vol. 9, 1956–1957, Exposé 8.

[HT84] Joe Harris and Loring W. Tu, *On symmetric and skew-symmetric determinantal varieties*, Topology **23** (1984), no. 1, 71–84.

[Ive72] Birger Iversen, *A fixed point formula for action of tori on algebraic varieties*, Invent. Math. **16** (1972), 229–236.

[JLP82] Tadeusz Józefiak, Alain Lascoux, and Piotr Pragacz, *Classes of determinantal varieties associated with symmetric and antisymmetric matrices*, Math. USSR Izvestija **18** (1982), 575–586.

[Kaz06] Maxim E. Kazarian, *Remark on Thom polynomials for $I_{2,2}$ and $\Sigma^{a,b}$*, private communication, 2006.

[Las78] Alain Lascoux, *Classes de Chern d'un produit tensoriel*, C. R. Acad. Sci. Paris, Sér. A **286** (1978), 385–387.

[LP00] Alain Lascoux and Piotr Pragacz, *Schur Q-functions and degeneracy locus formulas for morphisms with symmetries*, Recent Progress in Intersection Theory (G. Ellingsrud, W. Fulton, and A. Vistoli, eds.), Trends in Mathematics, 2000, pp. 239–263.

[LP09] ———, *Thom polynomials and Schur functions: the singularities $A_3(-)$*, to appear in Publications of RIMS; http://www.impan.pl/∼pragacz, 2009.

[Mac92] Ian G. Macdonald, *Schur functions: Theme and variations*, Séminaire Lotharingien de Combinatoire **28** (1992), 5–39.

[Mac98] ———, *Symmetric functions and Hall polynomials*, Oxford University Press, 1979, 1998.

[Mat73] John N. Mather, *On Thom-Boardman singularities*, Dynamical Systems (Proc. Sympos. Univ. Bahia, Salvador, 1971), 1973, pp. 233–248.

[OEIS] Neil J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, published electronically at http://http://www.research.att.com/∼njas/sequences/.

[PJ03] Simon Peyton-Jones, *The Haskell 98 language and libraries: The revised report*, Journal of Functional Programming **13** (2003), no. 1, http://www.haskell.org.

[Por71] Ian R. Porteous, *Simple singularities of maps*, Proceedings of Liverpool Singularities Symposium, I (1969/70), LNM, vol. 192, Springer, 1971, pp. 286–307.

[Por83] ———, *Probing singularities*, Singularities, Part 2 (Arcata, Calif., 1981), Proc. Sympos. Pure Math, vol. 40, AMS, 1983, pp. 395–406.

[Pra88] Piotr Pragacz, *Enumerative geometry of degeneracy loci*, Ann. Sci. École Norm. Sup. **21** (1988), no. 3, 413–454.

[Pra90] ———, *Cycles of isotropic subspaces and formulas for symmetric degeneracy loci*, Topics in Algebra (S. Balcerzyk et al., ed.), vol. 26, Banach Center Publications, no. 2, 1990, pp. 189–199.

[Pra07] ———, *Thom polynomials and Schur functions: the singularities $I_{2,2}(-)$*, Ann. Inst. Fourier **57** (2007), no. 5, 1487–1508.

[PW07a] Piotr Pragacz and Andrzej Weber, *Positivity of Schur function expansions of Thom polynomials*, Fundamenta Mathematicae **195** (2007), 85–95.

[PW07b] ———, *Thom polynomials of invariant cones, Schur functions, and positivity*, Algebraic cycles, sheaves, shtukas, and moduli, Trends in Math., 2007, pp. 117–129.

[Rim01] Richárd Rimányi, *Thom polynomials, symmetries and incidences of singularities*, Inv. Math. **143** (2001), 499–521.

[Ron72] Felice Ronga, *Le calcul des classes duales aux singularités de Boardman d'ordre deux*, Comment. Math. Helv. **47** (1972), 15–35.

[Ros89] Wulf Rossmann, *Equivariant multiplicities on complex varieties*, Orbites unipotentes et représentations, III. Orbites et faisceaux pervers, Astérisque, vol. 173–174, 1989, pp. 313–330.

[Sta99] Richard P. Stanley, *Enumerative combinatorics*, Cambridge University Press, 1997, 1999.

[Ste05] John R. Stembridge, *A Maple package for symmetric functions*, published electronically at http://www.math.lsa.umich.edu/∼jrs/software/SF.ps, 2005.

[Tho56] René Thom, *Les singularités des applications différentiables*, Ann. Inst. Fourier, Grenoble **6** (1956), 43–87.