Thom Polynomials via Restriction Equations

Theory and Practice

by

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THESIS

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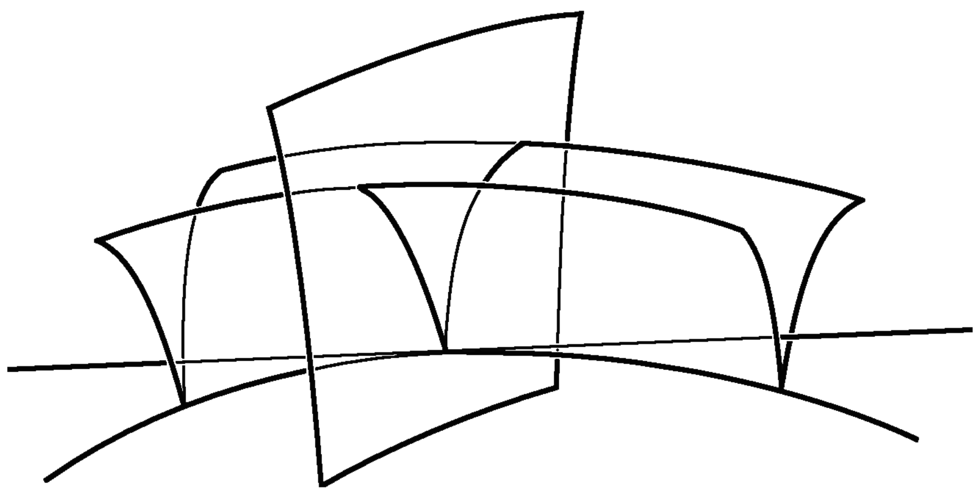
Table of Contents



Introduction

"I believe that proving is not a natural activity for mathematicians."
(René Thom)

The problem we would like to address is a basic question in homotopy theory: Whether a given fiber bundle admits a section? In many cases this can be reformulated to the question whether a given *vector bundle* ξ with fibers V admits a section *avoiding* a (closed) subset $\eta \subset V$. The latter is of course meaningful in general only if η is invariant to the action of the structure group G of the bundle (so we can "spread" η over the bundle, that is, define it in every fiber in a compatible way, which set we also denote by η). In this work we will study the cohomology class represented by the subset $\sigma^{-1}(\eta)$ of the base space M over which a generic section σ falls into η. This class is an obstruction to the existence of a section avoiding η: If this class is nonzero, then such a section can't exist. In fact there is always an universal class in $\mathrm{Tp}(\eta) \in H^*(BG)$ such that $[\sigma^{-1}(\eta)] = f^*\mathrm{Tp}(\eta)$, if $f : M \to BG$ is the classifying map of the bundle. This universal class, called the *Thom polynomial*, is our object of study.

Calculating Thom polynomials has its own interest far beyond our original question, connecting different areas of mathematics. Although from the modern viewpoint the original definition of characteristic classes by Stiefel can be regarded as the first appearance of Thom polynomials, it was René Thom who initiated their study in the case of singularities of smooth maps. The so called *degeneracy loci formulas* in algebraic geometry are also Thom polynomials for group action; to stay with algebraic geometry, as a by-product of the theory we can apply it to calculate degrees of invariant varieties. Study of Thom polynomials seems relevant to the theory of moduli spaces. There is also a strong connection to Schur and Schubert calculus, and algebraic combinatorics in general. It would be nice to list all threads connecting the subject to other areas, but it seems impossible, for at least two reasons; first, there are so many of them, second, we don't understand them at all.

In this work we will study Thom polynomials via a particular method, the so called 'restriction equations', based on an idea of R. Rimányi. There are (at least) two other known methods, the method of resolutions — this is the 'classical' way of calculating Thom polynomials (and also other invariants of singular varieties) —; and the Duistermaat-Heckman formula, which works only for smooth varieties. We describe the latter briefly in Section §7.

Organization of the text. The text is divided into three parts. In the first part, we recall some well-known material from the theory of bundles and equivariant cohomology (section §1), from the representation theory of Lie groups and Lie algebras (section §2) and from the topology of algebraic varieties (section §3). The second part contains the theoretical core of the work: In §4 we introduce the notion of Thom polynomial and discuss its basic properties; in §5 we present the methods used here to calculate Thom polynomials, and we also give detailed examples of deriving known formulas using these methods; in §6 we show the relation between projective and affine Thom polynomials, and that how one can calculate degrees of invariant projective varieties from their Thom polynomials; finally in §7 we outline a different approach, the so called Duistermaat-Heckman formula, which can be used for calculating Thom polynomials of smooth varieties. The aim of the last part is to present concrete calculations: In §8 a modest example (concerning the parameter space of plane cubics) is worked out in full details; and in the last section, §9 we derive closed formulas for the Thom polynomials of a very natural stratification of the representation $\mathrm{Sym}^n\mathbb{C}^2$. At the end of text we list some unsolved problems fitting well in the context of this work; we plan to work on them in the near future.

New results. The main result of the work is presented in §9 — although we should note that parallel to our work, L. Fehér, A. Némethi and R. Rimányi calculated the same Thom polynomials with quite different methods, see [**FNRb**]. Actually the whole material of Part 3 is new, although §8 is not a really "big thing". Apart from these, modest contributions (maybe not new, but at least ours) can be found scattered in the text, the most important being Lemma 4.8; and there is also Theorem 5.1.8, Theorem 5.2.3.1 and the proof of Theorem 5.1.1.

Prerequisites. We assume that the reader is familiar with the notions of linear algebra, differentiable manifolds[1], the very basic notions of algebraic geometry[2], Lie groups and Lie algebras[3] and finally singular homology and cohomology theory[4].

Some comments and conventions. Through the whole text we are working over the algebraic closed field of complex numbers $\mathbb{C}$; that is, vector space are complex vector spaces, algebraic varieties are complex algebraic varieties, etc... One exception is (co)homology theory, where we choose the field of real numbers as a coefficient ring. We have two reasons for doing that: First, as we wanted to avoid messing with torsion elements, we had to choose a field[5]; and from the fields $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ we chose $\mathbb{R}$ since then there is a natural isomorphism $H^2_T = H^2(BT) \cong \mathfrak{t}^*$ and more generally $H^{2k}_T \cong \mathrm{Sym}^k\mathfrak{t}^*$ for a torus (compact abelian Lie group) T.

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The thesis was typesetted using LaTeX 2_ε and a customly modified version of the AMS macro package (the modifications concern aesthetic aspects, the numbering of propositions, and an annoying "bug" in the bibliography).

[1] A short introduction can be found in eg. [**GHL93**] or in [**MS74**]

[2] See eg. [**Hrr92**] or [**Sha94**]; a more advanced book is [**Hrt77**]

[3] See eg. [**Ada69**], [**DK00**], [**FH91**] or [**Csi02**] (in hungarian)

[4] See eg. [**Hat01**] or [**MS74**] Appendix A.

[5] Almost everything described in this work should work with integer coefficients too; however, we would then have to explain the validity of the statements (mainly the lack of torsions) each time.

Part 1. Background

"In mathematics you don't understand things. You just get used to them."
(John von Neumann)

1. Bundles and Equivariant Cohomology

Fiber bundles in topology are somewhat analogous to short exact sequences in algebra: they describe a space assembled by two smaller spaces, a direct product locally, but somehow "twisted" globally.

Definition 1.1. For a given (left) action $\rho : G \to \mathrm{Aut}(F)$ of a Lie group G on a space F (where Aut is the group of homeomorphisms if F is just a topological space; the group of diffeomorphisms if F is a differentiable manifold, etc...), a *fiber bundle* with base space M, total space E, fiber F and structural group G is a map $\pi : E \to M$ (called the *projection map*) such that there is an open covering $\mathcal{U}$ of M and $\varphi_U : \pi^{-1}(U) \to U \times F$ isomorphisms (homeomorphisms, diffeomorphisms, etc...) that we have the following commutative diagrams for every $U, V \in \mathcal{U}$:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\varphi_U"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "\mathrm{pr}_1"] \\ & U & \end{tikzcd} \qquad \begin{tikzcd} (U\cap V) \times F \arrow[rr, "\rho\psi_{UV}\mathrm{pr}_1\times \mathrm{id}"] & & (U\cap V) \times F \\ & \pi^{-1}(U\cap V) \arrow[ul, "\varphi_U"] \arrow[ur, "\varphi_V"'] & \end{tikzcd}$$

where $\psi_{UV} : U\cap V \to G$ are (continuous, smooth, etc...) maps and the compositions $\psi_{VU}\psi_{UV}$ and $\psi_{WU}\psi_{VW}\psi_{UV}$ are the constant identity map on $U\cap V$ and $U\cap V\cap W$, respectively, for every $U, V, W \in \mathcal{U}$ (the latter is called the *cocycle condition*). We will call such a gadget a *ρ-bundle* and use the notation $F \to E \to M$ (the first map being an injection of the fiber into the total space and the second map being the projection).

Example 1.2. *Vector bundles* are a special case of the previous general definition, when $F = V$ is a vector space, and ρ is the standard representation of $\mathrm{GL}(V)$.

Example 1.3. Another very important special case is when $F = G$ is a Lie group and $\rho : G \to \mathrm{Aut}(G)$ is given by the left multiplication; the resulting object is called a *principal G-bundle.* Since G also acts on itself by the *right* multiplication, and this commutes with the left multiplication, the total space of a principal G-bundle admits a free right G-action, and the factor by this action is just the projection map.

We will use the following short notations: if ξ is a bundle, then denote by B_ξ, E_ξ, F_ξ and π_ξ its base space, total space, fiber and projection map, respectively.

Definition 1.4. A *section* of a bundle $F \to E \to M$ is map $\sigma : M \to E$ such that $\pi \circ \sigma = \mathrm{id}_M$.

Sections of a trivial bundle $F \times M$ are just (graphs of) maps $M \to F$; in this sense sections are generalizations of maps. As we can speak of continuous, smooth, etc... maps we can also speak of continuous, smooth, etc... sections (of course smooth sections are meaningful only if the bundle we are talking about is also smooth, and same holds for other structures). Note that not every bundle admits a section.

DEFINITION 1.5. A *bundle map* between G-bundles ζ and ξ (that is, we have two G-actions $\rho_\zeta : G \to \mathrm{Aut}(F_\zeta)$ and $\rho_\xi : G \to \mathrm{Aut}(F_\xi)$) is a map $\Phi : E_\zeta \to E_\xi$ which is a lifting of another map $\varphi : M_\zeta \to M_\xi$, that is, we have the commutative square

$$\begin{array}{ccc} E_\zeta & \xrightarrow{\Phi} & E_\xi \\ {\scriptstyle \pi_\zeta}\downarrow & & \downarrow{\scriptstyle \pi_\xi} \\ M_\zeta & \xrightarrow{\varphi} & M_\xi \end{array}$$

and the restrictions to each fiber $\Phi : \pi_\zeta^{-1}(x) \to \pi_\xi^{-1}(fx)$ are G-equivariant.

Given a bundle $\xi : F \to E \to M$ and a map $p : M' \to M$ we can define $p^*\xi$, the *pullback* of ξ, which is a bundle over M' with fiber F:

$$E' = \{(x', e) \in M' \times E \ : \ px' = \pi e\} \xrightarrow{\mathrm{pr}_1} M'$$

There is another construction (being somewhat dual to the previous one) called the *associated bundle*: if $\xi : G \to P \to M$ is a principal G-bundle and $\rho : G \to \mathrm{Aut}(F)$ is a (left) action, then define $\rho_*\xi$ as the quotient by the (left) diagonal action

$$E = P \times_\rho F := (P \times F) \,/\, \{(e, f) \sim (eg^{-1}, gf) \ \ \forall\, g \in G\} \xrightarrow{\pi \mathrm{pr}_1} M$$

It is easy to check that $\rho_*\xi$ is a ρ-bundle over the base space M. The importance of this latter construction is the following

FACT. Every ρ-bundle is associated to a principal G-bundle.

Sections of the bundle $P \times_\rho V$ are in bijection with equivariant maps $P \to V$.

There exist *universal bundles*, which embody all the complexity a ρ-bundle can have; that is, if γ is a universal ρ-bundle, then for every ρ-bundle ξ there is map $f : B_\xi \to B_\gamma$ (unique up to homotopy) such that $\xi = f^*\gamma$. In the light of the above fact, the most important ones are the universal principal G-bundles.

THEOREM 1.6 (consequence of theorems of Milnor [**Mil56**] and Dold [**Dld63**]). *For every Lie group G there exists a contractible CW-complex EG with a free (right) G-action (unique up to homotopy equivalence), and the quotient map $EG \to BG = EG/G$ is a universal principal G-bundle. Furthermore, every universal principal G-bundle is obtained such a way.*

Universal ρ-bundles can obtained by association: $V \to B_\rho V = EG \times_\rho V \to BG$ is a universal ρ-bundle. The total spaces $B_\rho V$ are sometimes denoted $B_G V$ or even BV, if there is no danger of misunderstanding. Note that B_G behaves like a functor (although it is defined only up to homotopy): If $f : X \to Y$ is an equivariant map between G-spaces than we also have a map $B_G f : B_G X \to B_G Y$, whose homotopy class — thus also the induced map on cohomology — is well-defined. If we fix the space BG, then already the map $B_G f$ is well-defined: It is just the bundle map lifting the identity of BG which gives f restricted to each fiber.

DEFINITION 1.7. The space BG (unique up to homotopy equivalence, if we require it to be a CW-complex) is called the *classifying space* of G.

The classifying space "mapping" $G \mapsto BG$ is also a functor: Let $f : H \to G$ be morphism of Lie groups, then define $Bf : BH \to BG$ to be the classifying map of the G-bundle associated to the universal H-bundle $H \to EH \to BH$ by the (left) action of H on G given by $hg = f(h)g$ (that is of course unique up to homotopy).

Armand Borel introduced the idea that topological properties of an action of a group G on a space X should be investigated not by studying the quotient space — which can have quite pathological topological structure — but by studying the space $B_G X = EG \times_G X$ (in some sense $B_G X$ is "the proper homotopy-theoretic quotient"). It is then natural to introduce the *equivariant cohomology ring* of X by the definition

$$H_G^*(X) := H^*(EG \times_G X)$$

This definition gives an *extraordinary cohomology theory* on the category of G-spaces: All axioms of cohomology is satisfied, expect that one on the triviality of the cohomology of the one-point space.[1] Indeed, the ring $H_G^* = H_G^*\mathrm{pt}$ can be quite complicated; for example, if $G = \mathrm{GL}_n$ then this ring is the (graded) polynomial ring

$$H_{\mathrm{GL}_n}^* = \mathbb{R}[c_1, c_2, \ldots, c_n]$$

where the variable c_i has degree $2i$ (c_i are the *universal Chern classes*).

Remark. A fundamental observation by Borel, motivating the above idea, is that if G acts on X *freely*, then $B_G X$ is homotopy equivalent to X/G, particularly $H_G^*(X) = H^*(X/G)$.

Almost all concepts of topology works also in the equivariant realm; for example an equivariant map $f : X \to Y$ between the G-spaces X and Y induces ring homomorphism $f_G^* : H_G^* Y \to H_G^* X$; we have long exact sequences and excision; for an equivariant vector bundle ξ (that is, E_ξ and B_ξ are G-spaces and the projection map π_ξ is G-equivariant) we have the Thom isomorphism[2]$H_G^*(B_\xi) \cong H_G^{*+n}(E_\xi|B_\xi)$ (where n is the rank of ξ), etc... However, because of the extra structure, there are also new phenomenons appearing; probably the most noticeable of them is that the cohomology group $\oplus H_G^i X$ is also a module over the cohomology ring of the point $H_G^* = H_G^*\mathrm{pt}$, the module structure being induced by the collapse map $X \to \mathrm{pt}$.

If we would like to calculate with these equivariant cohomologies, we have to understand the ring structure of the cohomologies of at least some spaces. A fundamental theorem of Borel gives us this information for the one-point space:

Theorem 1.8 (A. Borel). *Let's fix a maximal torus (compact abelian subgroup) T^r of G, and denote by $\mathfrak{W}$ the Weyl group $\mathfrak{W} = N_G T/T$, where $N_G T$ is the normalizer of T in G (the biggest subgroup of G in which T is still a normal subgroup). Then*

$$H^*(BG) = H_G^* = [H_T^*]^{\mathfrak{W}} = \mathbb{R}[\alpha_1, \ldots, \alpha_r]^{\mathfrak{W}}$$

where the upper index $\mathfrak{W}$ denotes the $\mathfrak{W}$-invariant part.

The action of the Weyl group is the following. The normalizer $N_G T$ acts on T by conjugations, the action of the subgroup $T \subset N_G T$ on itself being trivial, as T is commutative. Thus there is a well-defined action of the factor group $\mathfrak{W} = N_G T/T$

[1] See eg. [**Hat01**] or [**MS74**] for the definition and properties of singular cohomology.

[2] We use the shorthand notation $H^*(X|A)$ for the relative cohomology groups $H^*(X, X - A; \mathbb{R})$.

on T (note that the Weyl group is always a finite group). It acts then on $\mathfrak{t}$ and $\mathfrak{t}^*$ (since the identity element of T is fix under conjugations), and we will see later that H_T^* is naturally isomorphic to $\oplus \mathrm{Sym}^i \mathfrak{t}^*$, thus it also acts on H_T^*. Or, more directly, we have seen that a map $T \to T$ gives a map (up to homotopy) $BT \to BT$, thus an action of $\mathfrak{W}$ on T gives an action on $H^*(BT)$.

For example if $G = \mathrm{GL}_n$ then we can choose the diagonal matrices as T, and $\mathfrak{W}$ is the n-th symmetric group $\mathfrak{S}_n$; this proves the above remark about $H^*_{\mathrm{GL}_n}$ as c_i will be the elementary symmetric polynomials $c_i = \sigma_i(\alpha_1, \dots, \alpha_n)$ (α_i are the generators of H_T^*, called the *universal Chern roots*). Thus this theorem generalize the famous *splitting principle* of vector bundles.

A simple observation gives us the equivariant cohomology of homogeneous spaces:

LEMMA 1.9. *Let the Lie group G act (from the left) transitively on the space X, then $H_G^* X \cong H_{G_x}^* \mathrm{pt}$, where G_x is the stabilizer subgroup of a point $x \in X$.*

Proof. Observe that EG will do for EG_x too, since if G acts freely then $G_x \subset G$ acts "even more freely". Since $X \cong_G G/G_x = \{gG_x : g \in G\}$, the space of *right* cosets, we have

$$\begin{aligned} H_G^* X = H_G^*(G/G_x) = H^*(EG \times_G (G/G_x)) = H^*(G \backslash (EG \times G)/G_x) = \\ = H^*(EG/G_x) = H^*(BG_x) = H_{G_x}^* \mathrm{pt} \end{aligned}$$

as the left action of G on $E \times G$, given by $g(e,h) = (eg^{-1}, gh)$, and the right action of G_x, given by $(e,h)g = (e, hg)$, commutes. □

Moreover, if X is an orbit of a G-representation V (thus it is a homogeneous G-space) we can calculate the map on the cohomology ring induced by the embedding $i : X \hookrightarrow V$. In fact, we have the following commutative diagram

$$\begin{array}{ccc} H_G^*(X) & \cong & H^*(BG_x) \\ {\scriptstyle i^*}\uparrow & & \uparrow{\scriptstyle (Bj)^*} \\ H_G^*(V) & \cong & H^*(BG) \end{array}$$

where G_x is the stabilizer subgroup of a point $x \in X$ and j is the embedding (homomorphism) of Lie groups $j : G_x \hookrightarrow G$. Note that from the viewpoint of homotopy the embedding $X \hookrightarrow V$ is just the same as the projection map $X \to \mathrm{pt}$. The proof the commutativity of the above diagram is almost trivial; it is encoded in the following diagram of universal bundles:

$$\begin{array}{ccccccc} G_x & \longrightarrow & EG & \longrightarrow & EG/G_x & = & BG_x \\ {\scriptstyle j}\downarrow & & \| & & \downarrow & & \downarrow{\scriptstyle Bj} \\ G & \longrightarrow & EG & \longrightarrow & EG/G & = & BG \end{array}$$

More generally, if $j : H \to G$ is a morphism of Lie groups, we can explicitly calculate the induced map $j^* : H^*(BG) \to H^*(BH)$ (with some abuse of notation, we will write simply j^* instead of $(Bj)^*$ in the rest of this work). To do this, let's fix maximal tori $S^s \subset G$ and $T^t \subset H$ such that $j(T) \subset S$, and introduce coordinates $a_1, \dots, a_s$ and $b_1, \dots, b_t$ on these tori ($a_m : S \to \mathrm{U}_1$ and $b_k : T \to U_1$) such that

$$S = a_1(S) \times \cdots a_s(S) \qquad \text{and} \qquad T = b_1(T) \times \cdots b_t(T)$$

In this situation we have $j(y_k) = \prod x_i^{\lambda_{ik}}$ for some $\lambda_{ik} \in \mathbb{Z}$ integer numbers, where by $x_m \in S$ and $y_k \in T$ we mean elements of the tori which satisfy $a_l(x_m) = 1$ and $b_i(y_k) = 1$ if $l \neq m$ and $i \neq k$, respectively; and they are connected by the relation $a_m(x_m) = b_k(y_k)$. The numbers λ_{ik} thus determine the map $j : T \to S$ completely.

LEMMA 1.10. *Let's denote by $\alpha_1, \ldots, \alpha_s \in H^*(BS)$ and $\beta_1, \ldots, \beta_t \in H^*(BT)$ the corresponding generators of the cohomology rings. Then we have $j^*(\alpha_m) = \sum \lambda_{mi}\beta_i$ (thus in some sense j^* is the "adjoint" of j).*

Proof. It is enough to check the $T = U_1$ case. By definition we have

$$\alpha_i = c_1(ES \times_{a_i} \mathbb{C}) \qquad \text{and} \qquad \beta = c_1(E\mathrm{U}_1 \times_b \mathbb{C})$$

Also almost by definition[3] Bj is the classifying map of the vector bundle $E\mathrm{U}_1 \times_j \mathbb{C}^s$, that is, we have the following commutative diagram:

$$\begin{array}{ccc} \xi = E\mathrm{U}_1 \times_j \mathbb{C}^s & \longrightarrow & ES \times_{\mathrm{std}} \mathbb{C}^s = \gamma \\ \downarrow & & \downarrow \\ B\mathrm{U}_1 & \xrightarrow{Bj} & BS \end{array}$$

Both bundles split to the direct sum of line bundles, actually, they do that in a compatible way:

$$\xi = \oplus_i \xi_i = \oplus_i (E\mathrm{U}_1 \times_{(b^{\lambda_i})} \mathbb{C})$$
$$\gamma = \oplus_i \gamma_i = \oplus_i (ES \times_{a_i} \mathbb{C})$$

thus we have $\lambda_i \beta = c_1(\xi_i) = c_1(j^*\gamma_i) = j^*c_1(\gamma_i) = j^*\alpha_i$, which completes the proof. $\square$

Of course, the same is true for $j^* : H^*(BG) \to H^*(BH)$, by Borel's theorem.

We will need the following generalization of the notion of *characteristic classes* for equivariant vector bundles.[4]

DEFINITION 1.11. A *G-equivariant (vector) bundle* is just (vector) bundle whose total and base space are G-spaces and the projection map is G-equivariant.

For a G-equivariant complex vector bundle ξ of rank n define the *equivariant Chern classes* $c_1^G(\xi), \ldots, c_n^G(\xi)$ as the "ordinary" Chern classes of the vector bundle

$$B_G\xi \,:\, \mathbb{C}^n \longrightarrow B_G(E_\xi) = EG \times_G E_\xi \xrightarrow{B_G\pi_\xi} B_G(B_\xi) = EG \times_G B_\xi$$

thus $c_G(\xi) = 1 + c_1^G(\xi) + \cdots + c_n^G(\xi) \in H^*(EG \times_G B_\xi)$.

DEFINITION 1.12. The *equivariant Euler class* $e_G(\xi)$ of an equivariant vector bundle ξ is just the top (equivariant) Chern class $c_{\mathrm{top}}^G(\xi)$.

A new phenomenon in the equivariant world is that every G-space is a nontrivial bundle over the one-point space; particularly, a representation $\rho : G \to \mathrm{GL}(V)$ is an equivariant vector bundle over the point, and it has nontrivial Chern classes.

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[3] That is because the standard representation of S on $\mathbb{C}^s$ is faithful.

[4] We refer the reader to the excellent book [**MS74**] regarding the theory of characteristic classes.

PROPOSITION 1.13. *The Chern classes of a G-representation V^n are the elementary symmetric polynomials $c_i(V) = \sigma_i(w_1, \dots, w_n)$ where $w_i \in \mathfrak{t}^* \cong H^2_T$ are the weights of V.*[5]

Proof. It is enough to check the $n = 1$ case (splitting principle), where it is true by definition. Note that the (multi)set of the weights is invariant to the action of the Weyl group, thus the c_i is indeed an element of H^*_G. □

Thus the Euler class $e_G(V)$ is just to product of the weights.

2. Representation Theory

In this section we recall some notions and results from the representation theory of Lie groups and Lie algebras. A good reference (although from the algebraic viewpoint, instead of the more geometric one fitting better here) is [**FH91**].

DEFINITION 2.1. A *representation* of the Lie group G on the (complex) vector space V is just a Lie group morphism $G \to \mathrm{GL}(V)$. A representation of the Lie algebra $\mathfrak{g}$ is a Lie algebra morphism $\mathfrak{g} \to \mathfrak{gl}(V)$.

For every Lie group G there is a canonical representation, the *adjoint representation*, on its Lie algebra $\mathfrak{g}$. To define this, consider the (left) action of G on itself given by the conjugation:

$$\begin{aligned} \Phi_g : G &\longrightarrow G \\ h &\longmapsto ghg^-1 \end{aligned}$$

The identity element $e \in G$ is fix under the action Φ_g, thus by differentiating we get an action on the tangent space $T_eG \cong \mathfrak{g}$:

$$\mathrm{Ad}\, g := d_e\Phi_g : \mathfrak{g} \longrightarrow \mathfrak{g}$$

It is easy to check that Ad is a representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$. We can even go one step further, and take the differential of *this* map; this way we get the *adjoint representation of the Lie algebra* $\mathfrak{g}$:

$$\mathrm{ad} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$$

FACT. $\mathrm{ad}(x)(y)$ is just the Lie bracket $[x, y]$.

Introduce the notations $\mathcal{D}_1\mathfrak{g} = \mathcal{D}^1\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, and inductively $\mathcal{D}_k\mathfrak{g} = [\mathfrak{g}, \mathcal{D}_{k-1}\mathfrak{g}]$ and $\mathcal{D}^k\mathfrak{g} = [\mathcal{D}^{k-1}\mathfrak{g}, \mathcal{D}^{k-1}\mathfrak{g}]$. The first series $\mathcal{D}_k\mathfrak{g}$ is called the *lower central series*, while $\mathcal{D}^k\mathfrak{g}$ is called the *derived series.*

DEFINITION 2.2. A Lie algebra $\mathfrak{g}$ is called

(i) *nilpotent* if $\mathcal{D}_k\mathfrak{g} = 0$ for some k;
(ii) *solvable* if $\mathcal{D}^k\mathfrak{g} = 0$ for some k;
(iii) *semisimple* if $\mathfrak{g}$ has no nonzero solvable ideals;
(iv) *reductive* if $[\mathfrak{g}, \mathfrak{g}]$ is semisimple.

Note that a reductive Lie algebra $\mathfrak{g}$ is the sum of its center and a semisimple Lie algebra: $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$.

DEFINITION 2.3. A Lie group G is called

[5] See the next section for details.

(i) *semisimple* if its Lie algebra $\mathfrak{g}$ is semisimple;
(ii) *reductive* if its Lie algebra $\mathfrak{g}$ is reductive.

For example GL_n is reductive but not semisimple; SL_n is semisimple.

DEFINITION 2.4. A representation is called *irreducible* if it has no nontrivial subrepresentations.

Unfortunately, *complete reducibility* — that is, that for every invariant subspace there is an invariant complementary subspace, or equivalently, every representation decomposes (uniquely) as a direct sum of irreducible representations — fails for general Lie groups; however, it holds for *compact*, and more generally for *semisimple*[6] groups.

EXAMPLE 2.5. Consider the following two-dimensional representation of the reductive Lie algebra $\mathfrak{gl}_n$:

$$X \longmapsto \begin{bmatrix} 0 & \mathrm{tr}(X) \\ 0 & 0 \end{bmatrix}$$

The subspace $A = \{(x, 0) \in \mathbb{C}^2 \;:\; x \in \mathbb{C}\}$ is an invariant subspace, but no invariant complementary subspace exists. However, there is no way to extend this counterexample to a representation of GL_n.[7]

Nevertheless, at least for 'nice' representation of GL_n complete reducibility holds:

PROPOSITION 2.6. *Let $\rho : \mathrm{GL}_n \to \mathrm{GL}(V)$ a representation of GL_n such that the image of the scalars $\rho(\lambda \cdot \mathrm{id})$ is contained in the scalars $\{\mu \cdot \mathrm{id}_V\} \subset \mathrm{GL}(V)$. Then complete reducibility holds for ρ.*

Proof. We can restrict the representation to $\mathrm{SL}_n \subset \mathrm{GL}_n$; from the extra condition it follows clearly that a subspace of V is SL_n-invariant if and only if it is GL_n-invariant. On the other hand, complete reducibility holds for SL_n since it is semisimple. $\square$

THEOREM 2.7. *Any representation of a compact abelian group decomposes to the direct sum of one-dimensional representations. This decomposition is unique in the sense that the set of these representations and their multiplicity in the decomposition is uniquely determined.*

Now fix a maximal torus $T \subset G$. The Lie algebra $\mathfrak{t}$ of T is a maximal abelian subalgebra $\mathfrak{t} < \mathfrak{g}$ (called a *Cartan subalgebra*). The kernel of the mapping $\exp : \mathfrak{t} \to T$ defines a lattice $\Lambda = \exp^{-1}(1) \subset \mathfrak{t}$; the dual lattice

$$\Gamma = \Lambda^* = \{w \in \mathfrak{t}^* \;:\; w(x) \in \mathbb{Z} \;\; \forall x \in \Lambda\} \subset \mathfrak{t}^*$$

is called the *weight lattice*. Elements of the weight lattice (called *weights*) are in natural bijection with the set of characters of T, that is, Lie group morphisms $T \to \mathrm{U}_1$; the bijection is given by $w \mapsto \exp(w)$. Given a representation $\rho : G \to \mathrm{GL}(V)$, the restriction $\rho \restriction_T$ decomposes to the direct sum of one-dimensional representations (that is, characters) of T; thus we can associate a set[8] of weights $\mathcal{W} =$

[6] Weyl's theorem. In fact, for a Lie group (with finitely many connected components) or for a Lie algebra semisimplicity is equivalent with the complete reducibility.

[7] It can be done locally, in a neighbourhood of $1 \in \mathrm{GL}_n$, but not globally.

[8] To be precise, a *multiset*: a set where every element has a multiplicity (a positive integer).

$\{w_1, \dots, w_n\} \subset \Gamma$ to the any representation ρ of G. Moreover, this set is invariant to the natural action of the Weyl group $\mathfrak{W} = N_G T$ on $\mathfrak{t}^*$.

DEFINITION 2.8. The nonzero weights of the adjoint representation of G are called the *roots*; the set of roots is denoted by Δ. The lattice generated by the roots is called the *root lattice*.

For a given weight the corresponding (T-invariant) subspace of V in the decomposition is called the corresponding *weight space* (in the case of the adjoint representation the *root space*). Its dimension is the multiplicity of that weight (since we can't canonically "subdivide" that space to the sum of one-dimensional invariant subspaces, as we could if the weights were different). A nonzero vector in the weight space corresponding to a weight w is called a *weight vector*; we can always fix an arbitrary set of weight vectors, one for each weight, which we will denote by $\mathcal{X} \subset V$. If all the weights are different (that is, all multiplicities are 1), then there is a canonical bijection between the set of weights $\mathcal{W}$ and weight vectors $\mathcal{X}$; the weight corresponding to a weight vector x will be denoted by w_x. In the case of the adjoint representation of a semisimple Lie group, all roots are different, thus the root spaces (denoted by $\mathfrak{g}_\alpha$) are one dimensional; this gives a decomposition (called a *Cartan decomposition*) of the Lie algebra $\mathfrak{g}$ to

$$\mathfrak{g} = \mathfrak{t} \oplus \Big(\bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \Big)$$

Now fix a (nondegenerate, positive definite) inner product $\langle \bullet, \bullet \rangle$ on $\mathfrak{t}^*$ which is invariant to the action of the Weyl group $\mathfrak{W}$. Such a inner product always exists if G is reductive; particularly if G is semisimple, the only choice (up to nonzero scalars) is that coming from the Killing form: $K(X,Y)$ is defined for $X, Y \in \mathfrak{g}$ as the trace of the endomorphism of $\mathfrak{g}$ given by $\mathrm{ad}(X) \circ \mathrm{ad}(Y)$. Fixing a vector $v \in \mathfrak{t}^*$ — such that the intersection of the hyperplane $v^\perp$ with the weight lattice Γ contains only the zero vector — gives a decomposition $\Delta = \Delta^+ \cup \Delta^-$ where

$$\Delta^+ = \{\alpha \in \Delta \ : \ \langle \alpha, v \rangle > 0\}$$

Or equivalently we can choose a set $\Delta_S \subset \Delta$ of *simple roots*, where Δ_S should satisfy the following properties:

i) $\Delta_S = \{\beta_1, \dots, \beta_r\}$ is a basis in $\mathfrak{t}^*$;
ii) all the coefficients q_i in $\alpha = \sum q_i \beta_i$ have the same sign for every $\alpha \in \Delta$.

Then let $\Delta^\pm$ be the set of roots where the coefficients has positive (resp. negative) sign. The inner product and the fixed vector v gives an ordering on the set of the weights; a weight w maximal to this ordering (that is, $\langle w, v \rangle > \langle w_i, v \rangle$ for every weight $w_i \neq w$) is called a *highest weight*; a corresponding weight vector is called a *highest weight vector*. The set of highest weights is denoted by $\mathcal{H} \subset \mathcal{W}$.

THEOREM 2.9. *The set of highest weights $\mathcal{H}$ is invariant to the action of the Weyl group $\mathfrak{W}$. Furthermore if V is an irreducible representation of the semisimple Lie group G, then the projective orbit $\{[x] \in \mathbb{P}V \ : \ x \in Gv\}$ — where v is an arbitrary highest weight vector — is a smooth closed subvariety of $\mathbb{P}V$.*

The (open) *Weyl chamber* $\mathcal{C} \subset \mathfrak{t}^*$ corresponding to the decomposition $\Delta = \Delta^+ \cup \Delta^-$ is defined as

$$\mathcal{C} = \{\gamma \in \mathfrak{t}^* \ : \ \langle \gamma, \alpha \rangle > 0 \text{ for each } \alpha \in \Delta^+\} \subset \mathfrak{t}^*$$

The closed Weyl chamber is just the closure of $\mathcal{C}$. The Weyl group acts on the set of the Weyl chambers transitively (simple transitively, at least if G is semisimple). There is a highest weight in each Weyl chamber.

Pursuing this approach further would lead to the classification of the irreducible representations of semisimple groups; but we won't need that.

Very often we will need the tangent space of an orbit of a representation. Here we present an easy trick to deal with this problem. Let $\mathcal{A}$ be a finite dimensional associative algebra (with an identity element) equipped with an involution $a \mapsto a^\dagger$, and $j \in \mathcal{A}$ be an arbitrary element. Denote by $G(\mathcal{A})$ the group of invertible elements of $\mathcal{A}$ (that is of course a Lie group), and define the subgroup $G_j = \{a \in G(\mathcal{A}) : aja^\dagger = j\}$; this is again a Lie group, and its Lie algebra is the linear subspace $\mathfrak{g}_j = \{a \in \mathcal{A} : aj + ja^\dagger = 0\} < \mathcal{A}$ with the Lie bracket defined by $[a,b] = ab - ba$ (see eg. [**Csi02**]). For example all classical groups can be obtained from this construction. Suppose that G is such a group; then we have the following

LEMMA 2.10. *Let ρ be a representation of G on the vector space V which extends to a representation of $G(\mathcal{A})$; $X \subset V$ an orbit, and $x \in X$ a point. Choose a basis $A_1, \dots, A_m$ in $\mathfrak{g}$; then $T_xX < T_xV \cong V$ is spanned by the vectors*

$$B_i = \left[\frac{\partial}{\partial t}\rho(\mathrm{id} + A_i t)x\right]_{t=0}$$

We need the assumptions to give meaning to the expression $\rho(\mathrm{id} + A_i t)$.

Proof. It is clear from the definitions that

$$T_xX = \left\{ \left[\frac{\partial}{\partial t}\rho\big(\exp(At)\big)x\right]_{t=0} \;:\; A \in \mathfrak{g} \right\}$$

But we have

$$\left[\frac{\partial}{\partial t}\rho\big(\exp(At)\big)x\right]_{t=0} = \left[\frac{\partial}{\partial t}\rho\big(\mathrm{id} + At)\big)x\right]_{t=0}$$

since for a (local) one-parameter subgroup $\gamma(t)$ of $G(\mathcal{A})$

$$\left[\frac{\partial}{\partial t}\rho\big(\gamma(t)\big)\right]_{t=0} = (d\rho)\big(\dot\gamma(0)\big)$$

by definition, and $t \mapsto \exp(At)$ and $t \mapsto (\mathrm{id} + At)$ are both (local) one-parameter subgroups whose tangent vector at 0 is $A \in \mathfrak{g} \subset \mathcal{A}$. On the other hand it is clear again that it is enough to consider a basis of $\mathfrak{g}$ instead of the whole $\mathfrak{g}$. $\square$

Another thing we will use very often — already mentioned in the introduction — is that for a torus T there exists a canonical isomorphism $\mathfrak{t}^* \xrightarrow{\sim} H^2_T$, and more generally $\mathrm{Sym}^k\mathfrak{t}^* \xrightarrow{\sim} H^{2k}_T$. This can be constructed as follows. First we construct an isomorphism $\Gamma \xrightarrow{\sim} H^2(BT;\mathbb{Z})$ between the weight lattice $\Gamma \subset \mathfrak{t}^*$ and $H^2(BT;\mathbb{Z})$: For a weight $w \in \Gamma$ we can take the exponential $\exp(w) : T \to \mathbb{C}$ and associate a line bundle $\lambda_w = ET \times_{exp(w)} \mathbb{C}$ to the universal T-bundle with it; then we have a natural mapping $w \mapsto c_1(\lambda_w)$, which is easy to check that is an isomorphism. Than the simple observation that $\mathfrak{t}^* = \Gamma \otimes_{\mathbb{Z}} \mathbb{R}$ and $H^*_T = H^*(BT;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$ extends this isomorphism to that mentioned above. The more general statements $\mathrm{Sym}^k\mathfrak{t}^* \cong H^{2k}_T$ is again a trivial corollary.

3. Topology of Algebraic Varieties

In this section we summarize the topological properties of complex algebraic varieties we will use later. See [**Ful97**] Appendix B for a good overview, or [**Ful98**] (mainly chapter 19) for details.

Definition 3.1. By the words *algebraic variety* we mean a not necessary irreducible quasiaffine or quasiprojective variety; that is, the intersection of a Zariski-open subset of $\mathbb{A}^n$ or $\mathbb{P}^n$ with the common zero locus of some (homogeneous, in the projective case) polynomials. By *algebraic manifold* we mean a smooth irreducible algebraic variety.

Complex algebraic varieties have really nice topological properties, the most important for us being that they admit a canonical (finite) stratification, called the *singular stratification.*

Definition 3.2. A variety X is called *stratified* if it is the disjoint union of smooth varieties X_i (of possibly different dimensions). A stratification is called *even* if the strata are even dimensional.

Now the singular stratification of complex algebraic variety can be defined as follows: The smooth points form a smooth open subvariety (which will be the first stratum) while the singular points form a closed subvariety which then has at least one (complex) codimension and for which we can repeat the process (thus the strata are even dimensional smooth varieties).[9]

Proposition 3.3. *Let $A \subset X$ be a (possibly reducible) codimension k subvariety of a smooth variety X. Then $H^l(X|A) = 0$ for $l < 2k$ and $H^{2k}(X|A)$ is free with generators corresponding to the irreducible components A_i.*

Proof. Induction on the dimension of A. Denote by A° resp. Σ the smooth resp. singular locus of A, and by X° the set $X^\circ = X - \Sigma$. Write down the long exact sequence of the triple $(X, X - \Sigma, X - A)$:

$$\cdots \longrightarrow H^i(X|\Sigma) \longrightarrow H^i(X|A) \xrightarrow{\simeq} H^i(X^\circ|A^\circ) \longrightarrow H^{i+1}(X|\Sigma) \longrightarrow \cdots$$

The middle map is isomorphism if $i \leq 2k$ since the codimension of Σ is at least $2k+2$ (thus the outer groups are zero by the induction). On the other hand A° is smooth in X°, so by excision and the Thom isomorphism we have

$$H^i(X^\circ|A^\circ) \cong H^{i-2k}(A^\circ) \cong \begin{cases} 0 & i < 2k \\ \mathbb{R} & i = 2k \end{cases}$$

Finally, the connected components of A° are obviously in bijection with the irreducible components of A, which completes the proof. $\square$

Remark. Note that the same proof would work in the more general situation when A is a stratified subset with even dimensional strata (or with odd dimensional strata). We can define the "irreducible components" of a stratified subset as the closures of its top-dimensional strata.

[9] It is also true that a complex algebraic variety can be triangulated in such a way that respects this stratification, but we would like to avoid triangulations.

It will be useful to introduce the following notion, as it simplifies the language:

DEFINITION 3.4. An *algebraic cycle* of dimension d on a (smooth) variety X is just a (formal) integral linear combination of closed irreducible dimension d algebraic subvarieties of X.

Now we are ready to define the dual class represented by a subvariety.

DEFINITION 3.5. Let A be an irreducible codimension k subvariety of the smooth variety X. Use the notations above and denote by T° a tubular neighbourhood of A° in X°. Then the image of the Thom class u_{A° (as A° is complex, there is a canonical orientation, so the Thom class is well-defined, not just up to sign) at the isomorphisms

$$\begin{array}{ccccccc} H^{2k}(T^\circ|A^\circ) & \xrightarrow{\simeq} & H^{2k}(X^\circ|A^\circ) & \xrightarrow{\simeq} & H^{2k}(X|A) & \longrightarrow & H^{2k}(X) \\ u_{A^\circ} & \longmapsto & \tilde{u}_{A^\circ} & \longmapsto & \mu_{A,X} & \longmapsto & [A]_X \end{array}$$

is called the *refined class* of A and is denoted by $\mu_{A,X}$. Furthermore the image of this class at the restriction $H^{2k}(X|A) \to H^{2k}(X)$ is called the *dual class* of A and is denoted by $[A]_X$ or simply $[A]$.

Note that if A is smooth, then $[A]_X$ is just the Poincaré dual of A, hence the name 'dual class'. In fact, this is the only natural generalization of the notion of Poincaré dual to singular varieties.

REMARK. In the light of this definition we can name the generators of the group $H^{2k}(X|A)$ of Proposition 3.3: It is generated by the elements $j_i^*(\mu_{A_i,X})$ where j_i is the inclusion $j_i : (X, X - A) \hookrightarrow (X, X - A_i)$. It is then natural to extend the above definition to algebraic cycles in the following way:

$$\begin{aligned} \mu_{A,X} &:= \sum m_i \cdot j_i^*(\mu_{A_i,X}) \\ [A]_X &:= \sum m_i \cdot [A_i]_X \end{aligned}$$

where m_i is the multiplicity (coefficient in the formal linear combination) of the irreducible component A_i in A.

Again, the definition of the dual class would equally work for stratified subsets (with all strata being even dimensional, or all being odd dimensional). Note that the notion we get is *natural* in the following sense: If $A \subset X$ is an irreducible variety (or stratified subset), which is reducible with components $B_1, \ldots, B_k$ in the small neighbourhood $U \subset X$, then we have $j^*[A]_X = \sum m_i [B_i]_U$; this follows from the naturality of the Thom class. We also have the following:

PROPOSITION 3.6. *Let $f : Y \to X$ be a smooth map between the smooth manifolds X and Y, and let $A \subset X$ be a stratified subset with even dimensional strata. Suppose moreover that f is transversal to each strata of A. Then $f^{-1}(A)$ is again a stratified subset and $[f^{-1}(A)]_Y = f^*[A]_X$.*

Proof. Corollary of the same proposition with A being a smooth manifold (which is well-known), and the above (generalized) definition of dual class. $\square$

Time to return to algebraic varieties. From the definition is immediate that if A and B are cycles on the smooth varieties X and Y, then

$$\mu_{A\times B,X\times Y} = \mu_{A,X} \times \mu_{B,Y} \qquad \text{and} \qquad [A\times B]_{X\times Y} = [A]_X \times [B]_Y$$

where $\times$ is the (relative resp. absolute) cross product map in cohomology:

$$H^*(X|A) \otimes H^*(Y|B) \xrightarrow{\times} H^*(X\times Y|A\times B)$$
$$H^*(X) \otimes H^*(Y) \xrightarrow{\times} H^*(X\times Y)$$

As a corollary of [**Ful98**] Lemma 19.2 we have the following useful property of dual classes:

Proposition 3.7. *Let $f: Y \to X$ be a finite morphism (that is, $f^{-1}(x)$ is finite for all $x \in X$) between the smooth varieties X and Y, and $A \subset Y$ a subvariety, $B = f^{-1}(A)$. Then $[B]_Y = f^*[A]_X$.*

We could probably have a much more general proposition (without the assumption on the finiteness), but we don't need that. Also a more direct proof could be given using the definition of the dual class given above.

Definition 3.8. Two varieties (or cycles) A^k and B^l in the smooth variety X^n are said *intersecting properly* if every irreducible components of the intersection $A\cap B$ has the expected dimension, that is, $k+l-n$.

A very important proposition is the following (cf. [**Ful98**] Corollary 19.2).

Proposition 3.9 (about proper intersections). *Let A, B be subvarieties of codimensions k and l of the smooth variety X, intersecting properly in the variety Z. Then the relative cup product map $H^{2k}(X|A) \otimes H^{2l}(X|B) \to H^{2k+2l}(X|Z)$ takes*

$$\mu_{A,X} \smile \mu_{B,X} = m_1\cdot\mu_{Z_1,X} + m_2\cdot\mu_{Z_2,X} + \cdots + m_r\cdot\mu_{Z_r,X}$$

where Z_i are the irreducible components of Z and $m_i \in \mathbb{Z}$ are some integer numbers.

Definition 3.10. The numbers m_i are called the *intersection multiplicity of Z_i in $A\cap B$, and are denoted by $m_i = \mathrm{i.m.}(Z_i; A\cap B)$.*

Of course, the same is also true for the corresponding dual classes, from the commutativity of the diagram

$$\begin{array}{ccccc} H^*(X|A) & \otimes & H^*(X|B) & \xrightarrow{\smile} & H^*(X|A\cap B) \\ \downarrow & & \downarrow & & \downarrow \\ H^*(X) & \otimes & H^*(X) & \xrightarrow{\smile} & H^*(X) \end{array}$$

thus in the previous situation

$$[A]\smile[B] = m_1[Z_1] + m_2[Z_2] + \cdots + m_r[Z_r].$$

Remark. Intersection multiplicity in algebraic geometry is a highly nontrivial thing; for details we refer the reader to [**Ful98**]. For example the (irreducible) subvarieties of $\mathbb{A}^3$ defined by $x^3 + y^2 z = 0$ and $z = 0$ are intersecting in $x = z = 0$ and the i.m. is 3; but the i.m. of $y = z = 0$ in the intersection of $x^3 z + y^2 = 0$ and $z = 0$ is 2 (this is not a random example, we will use that in Example 4.11). However, one thing we know surely: If A and B are transversal smooth varieties, then they are intersecting properly and the intersection multiplicity is 1.

Part 2. Theory

"Symmetry, as wide or as narrow as you define its meaning, is one idea by which man through the ages has tried to comprehend and create order, beauty and perfection."
(Hermann Weyl)

4. Thom Polynomials

In this section we define our main object of study, the Thom polynomial, and discuss its basic properties. We use the unified framework introduced by M. É. Kazarian (see [**Kaz97**]): Let G be a Lie group, ρ an action of G on the contractible space V, and η a (closed) invariant subset. Then, to be cryptic, the Thom polynomial of η is just its G-equivariant cohomology class in V, living in $H^*_G(V) \cong H^*_G$. In this work we are interested only in the case when V is a complex vector space, and ρ is linear (that is, a representation), since this case is still general (and hard) enough, yet easier than the original one. In Section §6 we will describe the projective case, when instead of a vector space V we have the projective space $\mathbb{P}V$ (the action still being linear); this turns out not really different from the original problem. We will see in 5.2.1.1 how the 'classical' case fits in this framework.

Motivated by the work of R. Thom, the property we require from the Thom polynomial of η is that it is a class $\mathrm{Tp}(\eta) \in H^*(BG)$ such that for a ρ-bundle $\xi : V \to E \to M$ over a smooth manifold and a generic section $\sigma : M \to E$ the points $\sigma^{-1}(\eta)$ of M over which the section falls into η (we can define η in every fiber because its invariance) represents $f^*\mathrm{Tp}(\eta)$ where $f : M \to BG$ is the classifying map of the bundle ξ.[1] So one would define $\mathrm{Tp}(\eta)$ as the dual class to the fundamental class of $B_G\eta \subset B_GV$. Unfortunately there are some technical difficulties: First, η can be highly singular (so we can't use the theory of smooth manifolds directly); second, it is not compact (thus it does not possess a fundamental class in ordinary singular homology); third, the spaces we are talking about are infinite dimensional (one more reason to drop manifold theory). There are different ways to overcome these problems, all of them have their advantages and disadvantages. To sample the selection, we shall very briefly "describe" some of them:

For example, in [**BM60**] a homology theory for locally compact spaces (the so called *Borel-Moore homology*) and in [**BH61**] the fundamental classes of singular spaces using this homology theory is defined. We should note that for a space A possessing a *closed* embedding into a smooth manifold X^n the Borel-Moore homology group $\bar{H}_k(A)$ is isomorphic to the relative (singular) cohomology group $H^{n-k}(X, A)$ (thus the construction for the dual classes of algebraic varieties presented in Section §3 — not very surprisingly — coincides with that given by this theory). The Thom polynomial could be also defined as the *first obstruction* to the existence of a section avoiding η (see [**FR**]). In another approach the Thom polynomial is the

[1] This could be also taken as a definition, although then the proof of that such a cohomology class exists would be quite complicated, depending on how nice our invariant subset is; it may even happen that such a class does not exist at all.

edge homomorphism in a certain spectral sequence (see [**Kaz97**]).

We choose the following approach, which we believe is simpler than those above, however is also more restrictive: We require G to be a classical (complex or compact) Lie group, and η be an (invariant) complex algebraic subvariety of the (complex) vector space V (for example the orbit closures of actions of complex groups automatically satisfy this condition). We have seen in the previous section the complex algebraic varieties have really nice topological properties; particularly, the first two of the three problems mentioned above are solved there. The third one will be solved by approximating the infinite dimensional classifying space with finite dimensional (smooth) algebraic varieties.

PROPOSITION 4.1. *For a matrix groups G (algebraic subgroup of GL_n for some n) there exists an 'algebraic model' for the classifying space BG, that is, a series $\cdots \subset B_kG \subset B_{k+1}G \subset \cdots$ of smooth algebraic varieties and algebraic G-bundles $\gamma_k : G \to E_kG \to B_kG$ such that*

(i) *the union $\cup B_kG$ is the classifying space BG as a topological space;*
(ii) $\gamma_{k+1}\restriction_{B_kG} = \gamma_k$;
(iii) *the inclusion $B_kG \hookrightarrow BG$ induces isomorphisms $H^i(BG) \xrightarrow{\simeq} H^i(B_kG)$ for $i \leq k$.*

Sketch of the proof. For $G = \mathrm{GL}_n$ the standard choice $B_k = \mathrm{Gr}_n\mathbb{C}^{l(k)}$ and the frame bundle over $\mathrm{Gr}_n\mathbb{C}^{l(k)}$:

$$\begin{aligned} E_k = \{\ (x_1, \ldots, x_n, \Lambda) \in \mathbb{C}^{l(k)} \times \cdots \times \mathbb{C}^{l(k)} \times \mathrm{Gr}_n\mathbb{C}^{l(k)}\ :\ \\ x_1, \ldots, x_n \text{ are independent, and } \langle x_1, \ldots, x_n\rangle = \Lambda\} \end{aligned}$$

will do, where $l : k \mapsto l(k)$ is a series of integers growing fast enough; and for algebraic subgroups of $G \subset \mathrm{GL}_n$ we have G acting freely (from the right) on $E_k\mathrm{GL}_n$, thus we can choose

$$E_kG = E_k\mathrm{GL}_n \qquad \text{and} \qquad B_kG = (E_k\mathrm{GL}_n)/G$$

They are clearly algebraic varieties, and the projection map is regular. □

Note that we won't use really deeply that our objects of study are in the algebraic category — since we are interested only in the cohomology classes represented by them —; there is only two places where we need that. First, for the proof of that dual classes exist (but that could be done using general stratified sets instead of complex varieties); second, our main tool of calculation, the deformation lemma (Lemma 4.8) will use it.

Unfortunately, there is a subtle point here coming from our obsession with the algebraic category: Later we will have to work with actions of *compact* groups, which are definitely not *complex* varieties (for example they can have odd dimension). However we can proceed with the application of *Weyl's unitary trick.* The observation of Weyl was that every classical complex Lie group G has a compact subgroup K being homotopy equivalent with G; for example if $G = \mathrm{GL}_n$ then $K = \mathrm{U}_n$.[2] What

[2]This can be used for example to describe representations of complex groups, since for *compact* groups complete reducibility holds.

we really use is not that our principal bundles are algebraic bundles, but that the *associated vector bundles* with a given representation ρ are algebraic vector bundles. Because of the homotopy equivalence, principal G-bundles and K-bundles are just the same (at least from the viewpoint of topology); and if $\rho : K \to \mathrm{GL}(V)$ extends to a representation $\widetilde{\rho}$ of G (which will be always the case) then also ρ-bundles and $\widetilde{\rho}$-bundles are the same, so we can simply use the construction for G instead of K.

PROPOSITION 4.2. *In the above situation we have*

(i) $H^*(BG) = \varprojlim H^*(B_kG)$;

(ii) *for every G-bundle ξ over a compact manifold M there exist an number $N \in \mathbb{N}$ such that for every $k > N$ there is a classifying map $f_k : M \to B_kG$ with the property $\xi = f_k^*\gamma_k$.*

Proof. (i) is a corollary of property (iii) in the previous proposition.
(ii) holds because every (vector) bundle can be embedded into $\mathbb{C}^M$ for a large N. The groups we are talking about have a faithful representation (the embedding $G \hookrightarrow \mathrm{GL}_n$), thus principal G-bundles and vector bundles of rank n with structure group G are just equivalent things. Clearly every vector bundle of rank n embedded into $\mathbb{C}^M$ can be pulled back from γ_M, and that is we wanted to see. $\square$

For a G-representation V and an invariant subvariety $\eta \subset V$ introduce the notation $B_G^k\eta = E_kG \times_G \eta \subset E_kG \times_G V = B_G^kV$. Thus $B_G^k\eta$ is an algebraic subvariety of B_G^kV, having the same codimension as $\eta \subset V$. Now we are ready to define the Thom polynomial.

DEFINITION 4.3. Let ρ be a representation of the Lie group G on the complex vector space V, and η be an invariant subvariety. The *Thom polynomial* of η is defined as

$$\mathrm{Tp}_G(\eta) = \varprojlim_k [B_G^k\overline{\eta}]_{B_G^kV} \in H^{2d}(BG)$$

where d is the (complex) codimension of η in V (clearly, $j^*[B_G^{k+l}\overline{\eta}] = [B_G^k\overline{\eta}]$ for the inclusion $j : B_G^kV \hookrightarrow B_G^{k+l}V$, thus the inverse limit is meaningful). We will use the notations $\mathrm{Tp}_\rho(\eta)$, $\mathrm{Tp}_G(\eta)$ and $\mathrm{Tp}(\eta)$ interchangeably, depending on the context.

COROLLARY 4.4. *If $\xi : V \to E \to M$ is a ρ-bundle over a (compact) manifold M, then for a generic section $\sigma : M \to E$ we have $[\sigma^{-1}(\eta)]_M = f^*\mathrm{Tp}(\eta)$, where $f : M \to BG$ is the classifying map of ξ.*

Proof. We can choose an approximation $f_N : M \to B_NG$ for a large enough $N \in \mathbb{N}$. Using Proposition 3.6 first for the lifting $F_N : E \to B_G^NV$ of f_N gives that $[\xi(\eta)]_E = [F_N^{-1}(B_G^N\eta)]_E = F_N^*[B_G^N\eta]_{B_G^NV}$ (where $\xi(\eta) \subset E$ denotes η 'spreaded over' the bundle ξ), but since B_G^NV is homotopy equivalent to B_G^N and F_N is homotopic to f_n, we also have $F_N^*[B_G^N\eta]_{B_G^NV} = f^*\mathrm{Tp}(\eta)$. Then use Proposition 3.6 again, this time for the section σ. Note that we can give an exact meaning to the word 'generic' now: it should mean transversal to all strata of η ('spreaded over' ξ). $\square$

From the definition it follows that if we want to prove a property of the Thom polynomial, it is always enough to check it in the case of algebraic bundles over compact (algebraic) manifolds; that is we will do always. To ease the notation, we introduce the convention that bold symbols denote the 'twisted product' with a fixed (algebraic) G-bundle $G \to P \to M$; for example $\boldsymbol{\eta} = P \times_G \eta \subset P \times_G V = \boldsymbol{V}$.

PROPOSITION 4.5. *Let $j: H \to G$ be a morphism of Lie groups, $\rho: G \to \mathrm{GL}(V)$ a representation, $\eta \subset V$ a closed invariant subvariety. Then we have $j^*\mathrm{Tp}_\rho(\eta) = \mathrm{Tp}_{j^*\rho}(\eta)$, where $j^*\rho = (\rho \circ j) : H \to \mathrm{GL}(V)$ is the induced representation.*

Proof. Definition of the Thom polynomial and Bj. $\square$

PROPOSITION 4.6. *Let $\rho_1 : G_1 \to \mathrm{GL}(V_1)$ and $\rho_2 : G_2 \to \mathrm{GL}(V_2)$ be two representations, η_1 and η_2 two invariant subvarieties; then $\eta = \eta_1 \times \eta_2$ is invariant to the product representation $\rho_1 \times \rho_2 : G_1 \times G_2 \to \mathrm{GL}(V_1 \oplus V_2)$, and the Thom polynomial is*

$$\mathrm{Tp}_{\rho_1 \times \rho_2}(\eta) = \mathrm{Tp}_{\rho_1}(\eta) \times \mathrm{Tp}_{\rho_2}(\eta_2)$$

Moreover, if $G_1 = G_2 = G$, then η is also invariant to the direct sum representation $\rho_1 \oplus \rho_2 : G \to \mathrm{GL}(V_1 \oplus V_2)$; this time the Thom polynomial is

$$\mathrm{Tp}_{\rho_1 \oplus \rho_2}(\eta) = \mathrm{Tp}_{\rho_1}(\eta_1) \smile \mathrm{Tp}_{\rho_2}(\eta_2)$$

Proof. The first case is obvious from the definition of the Thom polynomial and the same property for the dual classes, since it is enough to check the equality for $G_1 \times G_2$-bundles in the form $P_1 \times P_2$ where P_1 is a G_1-bundle and P_2 is a G_2-bundle (as every $G_1 \times G_2$-bundle can be embedded in such a bundle). In that case $\boldsymbol{\eta} = \boldsymbol{\eta_1} \times \boldsymbol{\eta_2}$. The second case follows from the first one:

$$[\boldsymbol{\eta}] = [\Delta^{-1}(\boldsymbol{\eta_1} \times \boldsymbol{\eta_2})] = \Delta^*[\boldsymbol{\eta_1} \times \boldsymbol{\eta_2}] = \Delta^*([\boldsymbol{\eta_1}] \times [\boldsymbol{\eta_2}]) = [\boldsymbol{\eta_1}] \smile [\boldsymbol{\eta_2}]$$

where Δ denotes the diagonal map $\Delta : \boldsymbol{V} \to \boldsymbol{V} \times \boldsymbol{V}$. $\square$

There are a few cases when we can calculate the Thom polynomial directly. One such case is when η is a (linear) subspace:

PROPOSITION 4.7. *Let the torus $T^r = \mathrm{U}_1 \times \cdots \times \mathrm{U}_1$ act diagonally on the (complex) vector space $V^n = \langle x_1, \ldots, x_n \rangle$ and let $\eta \subset V$ be a linear subspace spanned by some coordinate axes $\{x_i : i \in I\}$; then η is of course T-invariant, and*

$$\mathrm{Tp}_T(\eta) = \prod_{i \notin I} w_i$$

where $w_i \in H^2(BT) = \mathbb{R}[\alpha_1, \ldots, \alpha_r]$ are weights of the representation of T on V corresponding to the coordinate axes.

Proof. According to Proposition 4.6, it is enough to prove the case $\dim(V) = 1$ and $\eta = \{0\}$. In this case there is only one weight, $w \in H^2_T$ and for a principal T-bundle $T \to P \to M$ induced by the classifying map $f : M \to BT$ we have $[\mathbf{0}] = [M] = c_1(P \times_\rho \mathbb{C}) = f^*w$ almost by definition. $\square$

We will sometimes need a slightly more general notation of the Thom polynomial: If $D \subset V$ is an open (open in the complex topology) ρ-invariant contractible subset, and $\eta \subset V$ is a closed ρ-invariant subvariety, then there is also an universal element $\mathrm{Tp}_G(\eta; D)$ such that for all principal G-bundles $G \to P \to M$ we have $f^*\mathrm{Tp}_G(\eta; D) = [P \times_\rho (\eta \cap D)]_{\boldsymbol{D}}$ for the classifying map $f : M \to BG$.

Now the Thom polynomial is local in the following sense. Fix a principal ρ-bundle $G \to P \to M$, and let $j : D \hookrightarrow V$ the inclusion map. Then we have the following commutative diagram:

$$\begin{array}{ccccc} H^*(\boldsymbol{V}|\boldsymbol{\eta}) & \longrightarrow & H^*(\boldsymbol{V}) & \xrightarrow{\simeq} & H^*(M) \\ \downarrow j^* & & \downarrow j^* & & \| \\ H^*(\boldsymbol{D}|\boldsymbol{\eta}) & \longrightarrow & H^*(\boldsymbol{D}) & \xrightarrow{\simeq} & H^*(M) \end{array}$$

Thus $j^*[\boldsymbol{\eta}]_{\boldsymbol{V}} = [\boldsymbol{D} \cap \boldsymbol{\eta}]_{\boldsymbol{D}}$ and then $\mathrm{Tp}(\eta) = \mathrm{Tp}(\eta; D)$; specifically $\mathrm{Tp}(\eta)$ is zero if η and D are disjoint.

Remark. We have a Zariski topology on D induced by that on V, so we can speak about irreducible and reducible algebraic subset of D; particularly we have $\mathrm{Tp}(\eta_1 \cup \eta_2; D) = \mathrm{Tp}(\eta_1; D) + \mathrm{Tp}(\eta_2; D)$. Or equivalently we could say that a stratified (closed) subset is irreducible (resp. reducible) if it has one (resp. more) top-dimensional strata; and its irreducible components are the closures of the top-dimensional strata.

This observation is important because often irreducible varieties becomes reducible in a small neighbourhood of some points; then the calculation of the Thom polynomial can be reduced to the calculation of those of the "local irreducible components". We will use this technique in Section 9.

One of our main tools of computation will be the following.

Lemma 4.8 (deformation lemma). *Let $\widetilde{\eta} = \{(x,t) \subset V \times \mathbb{A}^1 \ : \ x \subset \eta_t\}$ an irreducible family*[3] *of G-invariant affine algebraic varieties η_t of codimension d parameterized by the affine line $\mathbb{A}^1$. Let m_t denote the intersection multiplicity of η_t in $\widetilde{\eta} \cap V_t$. Then*

$$m_s \mathrm{Tp}(\eta_s) = m_t \mathrm{Tp}(\eta_t)$$

in $H^{2d}(BG)$ for all $s, t \in \mathbb{A}^1$.

Proof. Introduce the shorthand notations $W = V \times \mathbb{A}^1$, $V_t = V \times \{t\} \subset W$ (the action of G on $\mathbb{A}^1$ is the trivial one). Let $G \to P \to M$ a principal G-bundle over a (compact) algebraic manifold M. Consider the commutative diagram

$$\begin{array}{ccccccc} H^{2d}(\boldsymbol{W}|\widetilde{\boldsymbol{\eta}}) & \otimes H^2(\boldsymbol{W}|\boldsymbol{V_t}) & \xrightarrow{\smile} & H^{2d+2}(\boldsymbol{W}|\boldsymbol{\eta_t}) & \xleftarrow{\simeq} & H^{2d}(\boldsymbol{V_t}|\boldsymbol{\eta_t}) \\ \downarrow & \| & & \downarrow & & \downarrow i^* \\ H^{2d}(\boldsymbol{W}) & \otimes H^2(\boldsymbol{W}|\boldsymbol{V_t}) & \xrightarrow{\smile} & H^{2d+2}(\boldsymbol{W}|\boldsymbol{V_t}) & \xleftarrow{\simeq} & H^{2d}(\boldsymbol{V_t}) \end{array}$$

where the down arrows are restrictions, and the isomorphisms on the right are Thom isomorphisms: suppose $\xi : V^k \to E \to X$ is a vector bundle of rank k, $A \subset X$, then $H^*(X|A) \cong H^{*+k}(E|A)$ (here this is used for the trivial bundle $\boldsymbol{W} \to \boldsymbol{V_t}$). Let $f : M \to BG$ denote the classifying map of our bundle, then

$$m_t f^* \mathrm{Tp}(\eta_t) = m_t[\boldsymbol{\eta_t}] = i^*(\tau^{-1}(\mu_{\widetilde{\boldsymbol{\eta}},\boldsymbol{W}} \smile \mu_{\boldsymbol{V_t},\boldsymbol{W}}))$$

where τ is the Thom isomorphism $\tau : H^{2d}(\boldsymbol{V_t}|\boldsymbol{\eta_t}) \to H^{2d+2}(\boldsymbol{W}|\boldsymbol{\eta_t})$. Since the diagram is commutative, and the whole bottom line is independent of t (in the sense that there is a canonical isomorphism between the lines indexed by s and t

[3] That is, $\widetilde{\eta}$ is again an (irreducible) variety

induced by the translation by $s-t$), we got that particularly $m_t[\boldsymbol{\eta_t}]$ — thus also $m_t\mathrm{Tp}(\eta_t)$ — is independent of t. □

REMARK. We never really used in the proof that V is a vector space; thus we also have a "local deformation lemma" (local in the above sense), that is, if $D\subset V$ is an open contractible invariant subset, $E=D\times\mathbb{A}^1$, $D_t=D\times\{t\}\subset E$ and $D_t\cap\widetilde{\eta}$ is nonempty and of codimension d for all $t\subset\mathbb{A}^1$, the lemma still holds: we have $m_s\mathrm{Tp}(\eta_s;D)=m_t\mathrm{Tp}(\eta_t;D)$.

Although a trivial corollary of this lemma, but is useful to keep in mind, is that if $X\in\mathbb{C}^n$ is a G-invariant variety, and $\gamma:D\to(\mathbb{C}-\{0\})^n$ is a smooth analytic curve on the domain $D\subset\mathbb{C}$ such that

$$\vec{t}X=\{(t_1x_1,t_2x_2,\dots,t_nx_n)\ :\ (x_1,x_2,\dots,x_n)\in X\}$$

is still invariant to all $\vec{t}=(t_1,\dots,t_n)\in\mathrm{Im}(\gamma)\subset\mathbb{C}^n$ (for example if G is a torus acting diagonally on $\mathbb{C}^n$ then this holds for any curve) then we have

$$\mathrm{Tp}_G(\gamma(z_1)X)=\mathrm{Tp}_G(\gamma(z_2)X)$$

for all $z_1,z_2\in D$. To be down-to-earth, for example in the case of torus acting diagonally this means that we can 'scale' X with any $\vec{t}$ without changing its Thom polynomial, provided that $t_i\neq0$ for all i.

It will be useful to understand the relation of an affine variety and its cone. Let $\eta\subset V$ be a closed affine variety, and define the cone $c\eta\subset W=\mathbb{C}\times V$ to be $c\eta=\{(\tau,\tau x):x\in\eta,\tau\in\mathbb{C}\}$. If η is G-invariant, and we extend the action of G to W by letting G act on $\mathbb{C}$ trivially, $c\eta$ will be also G-invariant. Unfortunately, $c\eta$ won't be necessary closed; but the following proposition still holds.

PROPOSITION 4.9. $\mathrm{Tp}_G(\overline{c\eta})=\mathrm{Tp}_G(\eta)$.

Proof. Let U be then open subset $U=(\frac{1}{2},2)\times V\subset W$, and "localize" to U: $\mathrm{Tp}(\overline{c\eta};W)=\mathrm{Tp}(U\cap\overline{c\eta};U)$, but $U\cap\overline{c\eta}=U\cap c\eta$. Using Lemma 4.8 with

$$\eta_t=U\cap\big\{\big(t\tau+1-t,(t\tau+1-t)x\big)\in\mathbb{C}\times V\ :\ x\in\eta,\tau\in\mathbb{C}\big\}$$

(for $t=1$ and 0) shows that $\mathrm{Tp}(U\cap c\eta;U)=\mathrm{Tp}\big((\frac{1}{2},2)\times\eta;U\big)=\mathrm{Tp}(\eta)$. □

Finally let's see some examples.

EXAMPLE 4.10 (the rational normal curve). Consider the (affine) rational normal curve $\mathcal{A}_n\subset\mathbb{C}^n$, which is the image of the map $\nabla:y\mapsto(y,y^2,\dots,y^n)$. This is invariant to the action of GL_1 defined by $g(x_1,x_2,\dots,x_n):=(gx_1,g^2x_2,\dots,g^nx_n)$ for $g\in\mathrm{GL}_1$. We claim that $\mathrm{Tp}_{\mathrm{GL}_1}(\mathcal{A}_n)=n!\cdot\beta^{n-1}$, where β is the generator of $H^*_{\mathrm{GL}_1}=\mathbb{R}[\beta]$. We can prove that using Lemma 4.8 with $\mathrm{Im}(\widetilde{\nabla})$ playing the rôle of $\widetilde{\eta}$, where $\widetilde{\nabla}=(\nabla_t,t)$ and $\nabla_t:(y,t)\mapsto(y,y^2t,y^3t,\dots,y^nt)$. Observe that $\mathrm{Im}(\nabla_1)=\mathcal{A}_n$ and $\mathrm{Im}(\nabla_0)$ is the linear subspace $\langle*,0,\dots,0\rangle$, whose Thom polynomial we already calculated in Proposition 4.7. The intersection multiplicities are 1 by an easy transversality argument, thus we are ready.

EXAMPLE 4.11. Consider the (singular) subvariety $X\subset\mathbb{C}^2$ defined by $x^3+y^2=0$. This is clearly invariant to the action of GL_1 defined by $g(x,y)=(g^2x,g^3y)$; the

Thom polynomial is then $\mathrm{Tp}_{\mathrm{GL}_1}(X) = 6\beta$. We will (not very surprisingly) use Lemma 4.8 again, this time with $\widetilde{\eta}$ being the subvariety of $\mathbb{C}^3$ defined by the equation $x^3t + y^2 = 0$ (another possibility is $x^3 + y^2t = 0$). $\eta_0 = \{y = 0\}$ (or $\{x = 0\}$ in the second case), and in both cases $\eta_1 = X$. The intersection multiplicities are

$$m_t = \begin{cases} 1 & t \neq 0 \\ 2 & t = 0 \end{cases} \quad \text{or, if choose the second deformation: } m_t = \begin{cases} 1 & t \neq 0 \\ 3 & t = 0 \end{cases}$$

Thus, as the weights of our representation are $w_x = 2\beta$ and $w_y = 3\beta$, using Proposition 4.7 we get the desired result from both deformation.

5. Restriction Equations

5.1. Theory. In this section we present a method for calculating Thom polynomials, called the method of 'restriction equations'. The basic idea behind the calculations is of R. Rimányi; see [**FR**] for a general reference. Although the idea itself is really simple, it turns out to be quite powerful. It can be summarized as follows.

Instead of calculating the Thom polynomial $\mathrm{Tp}(\eta)$ directly, we can restrict it to some orbits of our representation ρ, calculate the restrictions (which may be easier), thus obtaining a system of (linear) equations on Tp, which we can solve then. Orbits are useful, because both the equivariant cohomology of them (Lemma 1.9) and the restriction maps are easily calculable: we have seen in Section §1 that the diagram

$$\begin{array}{ccccc} H^*_G(X) & = & H^*(B_G X) & \cong & H^*(BG_x) \\ \uparrow & & i^* \uparrow & & \uparrow j^* \\ H^*_G(V) & = & H^*(B_G V) & \cong & H^*(BG) \end{array}$$

is commutative, where $x \in X$ is an arbitrary point in the orbit X, G_x is its stabilizer, and the arrows are induced by the inclusions $i : X \hookrightarrow V$ and $j : G_x \hookrightarrow G$; and Lemma 1.10 tells us how to calculate j^*. In practice one further restricts to a (maximal) torus $T \subset G_x$, as the calculations become simpler and in fact there is no loss of information (this follows from Borel's theorem). The right-hand side of the restrictions is given by the following

THEOREM 5.1.1. *Let η be a closed G-invariant subvariety of V (being a representation of G), X a G-orbit, and $H \subset G_x$ a compact subgroup of the stabilizer of a point $x \in X$. Then*

$$i^*\mathrm{Tp}_G(\eta) = \begin{cases} 0 & X \cap \eta = \emptyset \\ e_H(N_x) & \overline{X} = \eta \\ \mathrm{Tp}_H((\eta - x) \cap N_x) & X \subset \eta \end{cases}$$

where i is the inclusion $i : H \hookrightarrow G$, $N_x < T_xV \cong V$ is a H-invariant complementary subspace to T_xX and $e_H(N_x)$ is the H-equivariant Euler class of the representation N_x (which is by definition that of the H-equivariant vector bundle $N_x \to \mathrm{pt}$).

Note that the first two case are in fact special cases of the third one.

Sketch of the proof. Assume that $x \in \eta$ (if this is not the case, then there is a small invariant neighbourhood of x disjoint from η, and the Thom polynomial is zero by an earlier remark). First, using Lemma 4.8 with $\eta_t = \eta - tx$ (here the minus sign means translation) we have $i^*\mathrm{Tp}_G(\eta) = \mathrm{Tp}_H(\eta) = \mathrm{Tp}_H(\eta - x)$; after that we have to (locally) deform $\eta - x$ to the product $((\eta - x) \cap N_x) \times T_x X$, and use Lemma 4.8 again (this can be done because X is smooth, and that's also why the intersection multiplicity will be 1). $\square$

At this point we would like to clarify the connection between the equivariant normal bundle ν_X and the 'normal slice representation' N_x we have seen above. Let's assume that G is a compact[4] Lie group acting (smoothly) on a smooth manifold Y, and $X \subset Y$ is an invariant submanifold. Denote by j the embedding $j : X \hookrightarrow Y$. Since G is compact, we can endow Y with a G-invariant (Riemannian) metric. The tangent bundle TX of X is of course an equivariant subbundle of the bundle $TY\restriction_X$; the action on the total space of TX is defined by $g\cdot v = (d\varphi_g)v$ $(v \in TY)$ where $\Phi : g \mapsto \varphi_g \in \mathrm{Diff}(Y)$ is the action (note that if Y is a vector space and the action is linear, then $d\varphi_g = \varphi_g$). The equivariant normal bundle ν_X is then defined as the subbundle of $TY\restriction_X$ orthogonal to TX. Exactly like in the non-equivariant setting, we have the following:

Theorem 5.1.2 (the equivariant tubular neighbourhood theorem). *In the previous situation there exists an open G-invariant neighbourhood $U \subset Y$ of X and equivariant diffeomorphism $\Phi : E_\nu \to U$ such that $\Phi \circ \sigma_0 = j$ where σ_0 is the zero section of ν.*

The point we should emphasize is that in some sense the G-equivariant normal bundle over X and the G_x-equivariant bundle N_x over the one-point space (where $x \in X$ is a point) are just the same. To make this claim exact, observe that the bundles $B_G(\nu_X)$ and $B_{G_x}(N_x \to \mathrm{pt})$ are the same (ordinary) bundles (if we fix a contractible space EG equipped with a free G-action, and let it play "double rôle", as G_x also acts freely on it). Particularly, the equivariant characteristic classes of ν_X and N_x are the same.

Definition 5.1.3. The G-representation V satisfies the *Euler condition* if there are finitely many orbits and for any orbit X the equivariant Euler class $e_G(\nu_X)$ of the normal bundle of X is not a zero divisor.

An important result is then the following.

Theorem 5.1.4 ([**FR**]). *Let's suppose that the representation V satisfies the Euler condition, and let η be an orbit. Then the equations*

$$j_\xi^*\mathrm{Tp}(\overline{\eta}) = \begin{cases} e_G(\nu_\eta) & \xi = \eta \quad \text{'principal equation'} \\ 0 & \xi \neq \eta \quad \text{'homogeneous equations'} \end{cases}$$

have a unique solution, where j_ξ is the inclusion map $j_\xi : \xi \hookrightarrow V$ and ξ runs over the orbits which satisfy $\mathrm{codim}(\xi) \leq \mathrm{codim}(\eta)$.

This theorem allows us to compute the Thom polynomials in the case there are finitely many orbits without understanding much geometry. We will give concrete examples in the next (sub)section.

[4] In fact, every classical (complex) Lie group G contains a maximal compact subgroup K being homotopy equivalent with G, so this assumption cause no problem for us.

To give a proof, first we have to build some more details of the theory. We have already mentioned that sections of a fiber bundle with fibers η associated to a principal G-bundle $G \to P \to M$ are in natural bijection with the G-equivariant maps $P \to \eta$, that is, for a section σ we have the commutative diagrams

$$\begin{tikzcd} & \eta \arrow[d, "\mathrm{pt}_\eta"] \\ P \arrow[ur, "\sigma"] \arrow[r, "\mathrm{pt}_P"'] & \mathrm{pt} \end{tikzcd} \qquad \begin{tikzcd} & H^*_G\eta \arrow[dl, "\sigma^*"'] \\ H^*(M) = H^*_G(P) & H^*_G \arrow[l, "\mathrm{pt}^*_P"] \arrow[u, "\mathrm{pt}^*_\eta"'] \end{tikzcd}$$

where pt_X denotes the collapse map $X \to \mathrm{pt}$. Thus existence of a section implies that $\mathrm{Ker}(\mathrm{pt}^*_\eta) \subset \mathrm{Ker}(\mathrm{pt}^*_P)$. Elements of H^*_G are called the *universal characteristic classes*, elements in $\mathrm{Im}(\mathrm{pt}^*_P)$ are called the *characteristic classes of* P, and finally $\mathcal{O}_\eta = \mathrm{Ker}(\mathrm{pt}^*_\eta)$ is called the *obstruction ideal of* η. If $\mathrm{pt}^*_P(\mathcal{O}_\eta) \subset H^*(M)$ is nonzero, then the bundle $P \times_G \eta$ can have no section. From this point we will assume that η is an irreducible algebraic variety of (complex) codimension d; we will see then that the Thom polynomial is the 'smallest' element of the obstruction ideal of $V - \overline{\eta}$, which we will call the *avoiding ideal* of η, and will denote by $\mathcal{A}_\eta$. Writing down the (equivariant) long exact sequence of the pair $(V, V - \overline{\eta})$

$$\cdots \longrightarrow H^k_G(V, V - \overline{\eta}) \xrightarrow{i^*} H^k_G(V) \xrightarrow{j^*} H^k_G(V - \overline{\eta}) \longrightarrow \cdots$$

we have $\mathrm{Ker}(\mathrm{pt}^*_{V-\overline{\eta}}) = \mathrm{Ker}(j^*) = \mathrm{Im}(i^*)$, as $\mathrm{pt}_{V-\overline{\eta}}$ is homotopic to $j : V - \overline{\eta} \hookrightarrow V$; and as a corollary of Proposition 3.3 we see that

$$H^k_G(V, V - \eta) \cong \begin{cases} 0 & k < 2d \\ \mathbb{R} & k = 2d \end{cases}$$

Thus we have $\mathcal{A}_\eta = \mathrm{Im}(i^*)$, and, using the notation $\mathcal{A}^i_\eta = H^i_G \cap \mathcal{A}_\eta$ (respectively $\mathcal{O}^i_\eta = H^i_G \cap \mathcal{O}_\eta$) for the degree i elements of the avoiding (resp. obstruction) ideal, $\mathcal{A}^{<2d}_\eta = 0$, moreover, if $\mathrm{Tp}(\eta) \neq 0$ then also $\mathcal{A}^{2d}_\eta = \mathbb{R} \cdot \mathrm{Tp}(\eta)$.

The proof of 5.1.4 will be based on the repeated use the following lemma.

LEMMA 5.1.5. *Let η and ξ be (invariant) complex manifolds such that $\eta \subset \overline{\xi}$ and $\eta \cup \xi$ is still a manifold. Suppose moreover that the (equivariant) Euler class $e(\eta)$ of the normal bundle of $\eta \hookrightarrow \eta \cup \xi$ is not a zero divisor. Then $\mathcal{O}_{\eta\cup\xi} = \mathcal{O}_\eta \cap \mathcal{O}_\xi$.*

Proof. This will follow from the (equivariant) Mayer-Vietoris sequence of $(\eta \cup \xi) = \xi \cup D\eta$, where $D\eta$ (resp. $S\eta$) denotes the disc (resp. sphere bundle) of the normal bundle of the embedding $\eta \hookrightarrow (\eta \cup \xi)$:[5]

$$\begin{tikzcd} & & H^*_G \arrow[d, "\mathrm{pt}^*_{\eta\cup\xi}"'] \arrow[drr, "\mathrm{pt}^*_\eta + \mathrm{pt}^*_\xi"] & & & \\ \cdots \arrow[r] & H^{k-1}_G(S\eta) \arrow[r, "\delta"] & H^k_G(\eta \cup \xi) \arrow[rr, "\alpha"] & & H^k_G(\eta) \oplus H^k_G(\xi) \arrow[r] & \cdots \end{tikzcd}$$

[5] $D\eta$ and $S\eta$ are not G-invariant; however, we can take a maximal compact subgroup $K \subset G$, being homotopy equivalent with G, and $D\eta$ and $S\eta$ will be invariant for K. Note that we could proceed without doing this, taking the disc (resp. sphere) bundle of the normal bundle of $B_G\eta \subset B_G(\eta \cup \xi)$ and 'ordinary' cohomology instead of equivariant. The two approach are of course equivalent; we chose the equivariant formalism because the notations are simpler.

We will show that $\delta = 0$, thus α is injective, what is exactly we would like too see. Consider the commutative diagram

$$\begin{tikzcd}
 & & H_G^{k-2d}(\eta) \arrow[r, "\smile e(\eta)"] & H_G^{k}(\eta) & \\
\cdots \arrow[r] & H_G^{k-1}(S\eta) \arrow[r, "d"] \arrow[drr] & H_G^{k}(D\eta, S\eta) \arrow[r, "i^*"] \arrow[u, "\tau", "\cong"'] & H_G^{k}(D\eta) \arrow[r] \arrow[u, "\cong"'] & \cdots \\
\cdots \arrow[r] & H_G^{k-1}(\xi) \arrow[r] \arrow[u] & H_G^{k}(\eta \cup \xi, \xi) \arrow[r] \arrow[u, "\cong"'] & H_G^{k}(\eta \cup \xi) \arrow[r] \arrow[u] & \cdots
\end{tikzcd}$$

where the long rows are from the long exact sequences of the pairs $(D\eta, S\eta)$ and $(\eta \cup \xi, \xi)$; and τ is the Thom isomorphism. The middle restriction map is an isomorphism because of the excision property of cohomology. Now, the condition of the lemma exactly says that $x \mapsto x \smile e(\eta)$ is an isomorphism; thus i^* is also an isomorphism from the commutativity, and then $d = 0$. But the δ map in the Mayer-Vietoris sequence is just the composition denoted by the diagonal arrow, thus it is zero too. $\square$

Proof of Theorem 5.1.4. Introduce the notation

$$U = V - \bigcup \{\xi \ : \ \xi \text{ is an orbit, } \operatorname{codim}(\xi) > d\}$$

Iterating the lemma we get that

$$\mathcal{O}^{2d}_{U-\eta} = \bigcap \{\mathcal{O}^{2d}_{\xi} \ : \ \xi \text{ is an orbit, } \operatorname{codim}(\xi) \leq d,\ \xi \neq \eta\}$$

On the other hand, we have

$$\mathcal{O}^{2d}_{U-\eta} = \mathcal{O}^{2d}_{V-\overline{\eta}} = \mathcal{A}^{2d}_{\eta} = \mathbb{R}\cdot \mathrm{Tp}(\eta)$$

since the inclusion $i : U - \eta \hookrightarrow V$ factors through $V - \overline{\eta}$, and from the long exact sequence of the pair $(V - \overline{\eta}, U - \eta)$

$$\longrightarrow H^{2d}_G(V - \overline{\eta}, U - \eta) \longrightarrow H^{2d}_G(V - \overline{\eta}) \xrightarrow{\sim} H^{2d}_G(U - \eta) \longrightarrow H^{2d+1}_G(V - \overline{\eta}, U - \eta) \longrightarrow$$

we see the $U - \eta \hookrightarrow V - \overline{\eta}$ induces isomorphism on H^2d_G, (the outer groups are zero from Proposition 3.3). Thus the solution of the homogeneous equations is the (at most) one dimensional (vector) space $\mathcal{A}^{2d}_{\eta}$, and the principal equation selects the Thom polynomial from this space. $\square$

Another uniqueness result can be proved for torus actions.

Definition 5.1.6. Let V be a vector space of dimension k. The hyperplanes $H_i = \operatorname{Ker}(\alpha_i)$, $0 \neq \alpha_i \in V^*$, $1 \leq i \leq m$ are said being in *general position* if for any k-tuple $1 \leq i_1 < \cdots < i_k \leq m$ the vectors $\alpha_{i_1}, \ldots, \alpha_{i_k}$ are linearly independent in V^*.

For a set of linear subspaces A_i and multiindex I let's introduce the short notation $A_I = \cap_{i \in I} A_i$. Note that if $A_1, \ldots, A_m$ are hyperplanes of V^k in general position, then for every nonempty subset $I \subset \{1, 2, \ldots, m\}$ the intersection A_I has dimension $\dim(A_I) = \max(0, k - |I|)$ where $|I|$ denotes the cardinality of the set I.

Lemma 5.1.7 (k-dimensional interpolation). *Let $W_0, \dots W_n < V$ be hyperplanes in the vector space V^k in general position, then the linear map*

$$\begin{aligned} \Psi : \mathrm{Sym}^n V^* &\longrightarrow \bigoplus_{i=0}^{n} \mathrm{Sym}^n W_i^* \\ p &\longmapsto (p\restriction_{W_0}, \dots, p\restriction_{W_n}) \end{aligned}$$

is injective, and its image is

$$U = \Big\{(p_0, \dots p_n) \in \bigoplus_{i=0}^{n} \mathrm{Sym}^n W_i^* \;:\; p_i\restriction_{W_{ij}} = p_j\restriction_{W_{ij}} \;\; \forall\, i, j\Big\}$$

(where W_{ij} denotes the intersection $W_i \cap W_j$).

Remark. There is (at least) two ways to think of an element $p \in \mathrm{Sym}^n V^* = (\mathrm{Sym}^n V)^*$: First, we can think that it is a polynomial (of degree n) on the vector space V; but we can also think that it is a linear form on the much bigger space $\mathrm{Sym}^n V$. The polynomial *function* can be restored from the linear form by composing the Veronese embedding $V \hookrightarrow \mathrm{Sym}^n V$ with it. We will of course use both viewpoints.

Proof. [6] First we will show that $\mathrm{Ker}(\Psi) = 0$. Suppose that $q \in \mathrm{Sym}^n V^*$ is polynomial such that $\Psi(q) = 0$, and let $\alpha_i \in V^*$ be linear forms such that $W_i = \mathrm{Ker}(\alpha_i)$; since $q\restriction_{W_i} = 0$ for all i, we have $q = \alpha_0 \alpha_1 \cdots \alpha_n r$, r also being a polynomial.[7] But the degree of q is n, thus $q = 0$; Ψ is injective.

Now let A_i be $A_i = \mathrm{Sym}^n W_i$ and $A_I = \cap_{i \in I} A_i = \mathrm{Sym}^n(W_I)$. Obviously, $\mathrm{Im}(\Psi) \subset U$; thus, to prove that image is U indeed, it is enough to see that $\dim(U) = \dim(\mathrm{Sym}^n V^*)$, which we can do with the help of the *sieve formula* or *inclusion-exclusion principle.* This says that

$$\dim(U) = \sum_{|J|>0} (-1)^{|J|} \dim(A_J) = \sum_{j>0} (-1)^j \binom{n+1}{j} \binom{n+k-1-j}{k-1-j}$$

But by easy combinatorics

$$\sum_{j>0} (-1)^j \binom{n+1}{j} \binom{n+k-1-j}{k-1-j} = \binom{n+k-1}{k-1} = \dim(\mathrm{Sym}^n V^*)$$

and we are ready. □

Remark. Note that the injectivity remains true if we drop the assumption on hyperplanes that the are in general position, and require only that they are different from each other.

Now consider the (almost) standard situation: We have a representation ρ of a torus T on the complex vector space V, and $\eta \subset$ is a (closed) invariant subvariety. This time, we will restrict the Thom polynomial to weight vectors (T-orbits of weight vectors, to be more precise). Since for a weight vector $x \in \mathcal{X}$ we have $tx = \exp(w_x)(t)x$ for all $t \in T$ (recall that w_x denotes the weight corresponding to x), the stabilizer of x equals to $T_x = \exp(w_x)^{-1}(1)$. The inclusion $j : T_x \hookrightarrow T$ induces the

[6] We would like to thank B. Csikós for the helpful discussion on this proof.

[7] This follows from Hilbert's Nullstellensatz, see eg. [**AM69**] or [**Sha94**].

restriction $j^* : H_T^* \to H_{T_x}^*$; we know that $H_T^* = \mathrm{Sym}^*\mathfrak{t}$ and $H_{T_x}^* = \mathrm{Sym}^*\mathfrak{t}_x^*$ and the restriction is of course just the projection $\mathrm{Sym}^*\mathfrak{t}^* \to \mathrm{Sym}^*\mathfrak{t}_x^*$. Thus, as a corollary[8] of Lemma 5.1.7, we have the following:

THEOREM 5.1.8. *Let η be a T-invariant subvariety of V. If we restrict to at least* $\mathrm{codim}(\eta)+1$ *weight vectors corresponding to different weights, then the restriction equations have a unique solution.*

For example, if all the weights of our representation are different, then restriction to the weight vectors is always enough, since the codimension is always[9] smaller than the dimension of the vector space, which equals to the number of the weights. An example for such a representation is $\mathrm{Sym}^n\rho_k$ or $\wedge^k\rho_k$, where ρ_k is the standard representation of GL_k.

REMARK. Of course we could simply restrict to any set of hyperplanes of $\mathfrak{t}$, but we can't compute the right hand sides in such generality; while restriction to the weight vectors can be worked out more often.

We will give an application of this theorem in the next section; and in essence, that is what we do in section §9, where the main result of this work is presented. Note also that this allows the computation of the Thom polynomials of the 'smallest' orbits of irreducible representations (of semisimple groups), since the right-hand sides of the restrictions are zero or the Euler class then. These are smooth, thus the theory briefly described in Section §7 is also applicable; the efficiency — we mean the efficiency of the resulting algorithms for concrete cases — of the two methods is roughly the same (very low, actually). For example, using this method we were able to calculate the Thom polynomials of the (cones of the) Grassmannians $C\mathrm{Gr}_k\mathbb{C}^n \subset \wedge^k\mathbb{C}^n$ (for the natural action of GL_n) — which have the nice geometric meaning that they are the cohomology classes dual to the locus where a (generic) k-form on a (compact) n-manifold is *totally decomponable*, that is, has the form $\alpha_1 \wedge \cdots \wedge \alpha_k$ for some linear forms α_i — for $n \le 7$ (with the help of a computer, of course); the results are shown in Table 1 (we omitted most terms from the $n = 7$, $k = 3, 4$ cases, because the full Thom polynomials would fill up the page). Note that the $k = 2$ case is known (for all n), see Section 5.2.2.

5.2. Examples. Here we would like to rederive some classical results to illustrate the usage of the methods presented in the previous section. These results first appeared in [**Por71**] (the first example) and in [**HT84**] and [**JLP82**] (the second example). The third example, although the most simple, is ours.

To state some of the results, let us introduce the notion of *Schur polynomials* $s_\lambda = \Delta_\lambda(c)$. For a partition $\lambda = (\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_k)$ let $\Delta_\lambda(c)$ be

$$\Delta_\lambda(c) = \det(c_{\lambda_i+j-i})_{k\times k} = \begin{vmatrix} c_{\lambda_1} & c_{\lambda_1+1} & \cdots & c_{\lambda_1+k-1} \\ c_{\lambda_2-1} & c_{\lambda_2} & \cdots & c_{\lambda_2+k-2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{\lambda_k-k+1} & c_{\lambda_k-k+2} & \cdots & c_{\lambda_k} \end{vmatrix}$$

[8]Although we proved Lemma 5.1.7 for *complex* vector spaces and *complex* polynomials, the injectivity clearly remains true if we consider *real* polynomials on *real* vector spaces.

[9]The zero orbit is an exception of course, but its Thom polynomial is just the equivariant Euler class $e_T(V)$; so we don't need any equations to calculate that...

TABLE 1. Thom polynomials of the Grassmannians $\mathrm{Gr}_k\mathbb{C}^n$ for $n \le 7$

n,k	$\mathrm{Tp}(C\mathrm{Gr}_k\mathbb{C}^n)$
$4,2$	c_1
$5,2$	$-c_3 + c_1c_2$
$5,3$	$c_3 + {c_1}^3$
$6,2$	$-{c_3}^2 + c_1c_5 + c_1c_2c_3 - {c_1}^2c_4$
$6,3$	${c_1}^2{c_2}^2c_4 + {c_1}^6c_4 + 2\,{c_1}^3c_2c_5 + 3\,{c_1}^2c_2{c_3}^2 - 2\,{c_1}^4c_2c_4 + 4\,c_1c_3c_6 - 4\,{c_1}^2c_3c_5 - 3\,{c_1}^3{c_2}^2c_3 + 4\,{c_1}^3c_3c_4 - 2\,c_2c_3c_5 + {c_1}^4{c_2}^3 + 2\,{c_1}^4c_6 - 8\,c_4c_6 + 2\,{c_2}^2c_6 - {c_1}^5c_5 - 2\,c_1c_2c_3c_4 - c_1{c_3}^3 + 2\,{c_3}^2c_4 - 2\,{c_1}^2{c_4}^2 + 2\,{c_5}^2 + c_1{c_2}^2c_5$
$6,4$	$-{c_3}^2 - 2\,c_1c_5 + 2\,c_1c_2c_3 - 2\,{c_1}^2c_4 + 2\,{c_1}^3c_3 + {c_1}^6$
$7,2$	$-c_1{c_2}^2c_5 + {c_1}^2c_2c_6 - c_1c_3c_6 + c_2c_3c_5 + c_1c_2c_3c_4 - {c_3}^2c_4 - {c_1}^2{c_4}^2 - {c_5}^2 + 2\,c_1c_4c_5 - c_1c_2c_7 + c_3c_7$
$7,3$	$5\,c_1{c_3}^2{c_5}^3 - 9\,{c_1}^4{c_3}^4c_6 - 8\,{c_1}^4{c_4}^3c_6 - 6\,c_1{c_3}^5c_6 - 4\,{c_1}^6{c_5}^2c_6 - 4\,{c_1}^2c_2{c_3}^6 + 6\,{c_1}^3{c_2}^2{c_3}^5 - 4\,{c_1}^4{c_2}^3{c_3}^4 + 3\,{c_1}^3{c_3}^5c_4 + 2\,{c_1}^9{c_4}^2c_5 + 4\,{c_3}^3{c_4}^2c_5 - 4\,{c_1}^8c_4{c_5}^2 + 2\,{c_1}^9{c_3}^2c_7 - 3\,{c_1}^2{c_3}^4{c_4}^2 + \cdots$
$7,4$	$-12\,c_1{c_3}^2{c_5}^3 - {c_1}^4{c_3}^4c_6 - 24\,{c_1}^4{c_4}^3c_6 + 2\,c_1{c_3}^5c_6 - 16\,{c_1}^6{c_5}^2c_6 - 5\,{c_1}^3{c_3}^5c_4 - 10\,{c_1}^9{c_4}^2c_5 + 4\,{c_3}^3{c_4}^2c_5 + 2\,{c_1}^8c_4{c_5}^2 - 18\,{c_1}^9{c_3}^2c_7 + 7\,{c_1}^2{c_3}^4{c_4}^2 - 29\,{c_1}^6{c_3}^3c_7 - 3\,{c_1}^7{c_2}^4c_7 - 8\,c_1{c_3}^3{c_6}^2 + \cdots$
$7,5$	$2\,c_1c_3c_6 + 10\,{c_1}^4c_6 + c_3c_7 - 2\,c_1c_4c_5 + 5\,{c_1}^3c_7 - {c_1}^4{c_3}^2 - {c_3}^2c_4 + 4\,{c_1}^5c_2c_3 - 5\,{c_1}^3c_2c_5 - 6\,{c_1}^2c_2{c_3}^2 - 5\,{c_1}^4c_2c_4 + 7\,{c_1}^2c_3c_5 + 5\,{c_1}^3{c_2}^2c_3 + {c_1}^3c_3c_4 + c_2c_3c_5 - 4\,{c_1}^6c_4 + {c_1}^5c_5 + 2\,c_1{c_3}^3 - {c_1}^2{c_4}^2 - {c_5}^2 + c_1c_2c_3c_4 + 3\,{c_1}^7c_3 + {c_1}^{10}$

REMARK. If $s_1, s_2, \dots$ is defined by the relation

$$(1 + c_1t + c_2t^2 + c_3t^3 + \dots)(1 - s_1t + s_2t^2 - s_3t^3 \pm \dots) = 1$$

(if c_i are the Chern classes of a vector bundle, then s_i are the *Segre classes*), then $\Delta_\lambda(c) = \Delta_\mu(s)$ where μ is the conjugate partition (see eg. [**Ful98**] Lemma A.9.2; although we won't use this in this text).

5.2.1. *The Giambelli-Thom-Porteous Formula.* Our first example is the classical case of the Thom polynomial, in the context where Thom originally defined them: singularities of smooth maps. This section is mainly based on [**FR**]. The original question is that given a generic smooth map $f : M^n \to N^{n+k}$ between the smooth manifolds M and N, what cohomology class the closures of the sets of singular points $\Sigma_s(f) = \{x \in M : \mathrm{corank}(d_xf) = s\}$ represent in $H^*(M)$? Thom proved in [**Tho56**] that this class depends only on the difference bundle $f^*TN \ominus TM$, and can be calculated by substituting the Chern classes of this difference bundle into a universal polynomial, which we call the Thom polynomial.

The differential df is a section of the bundle $\mathrm{Hom}(TM, f^*TN)$, so in our language we are looking for the Thom polynomials of invariant subsets of the representation $\mathrm{Hom}(\rho_n, \rho_{n+k}) = \rho_n^* \otimes \rho_{n+k}$ of the group $G = \mathrm{GL}_n \times \mathrm{GL}_{n+k}$ (where ρ_n and ρ_{n+k} are the standard representations of GL_n and GL_{n+k}, respectively). Fortunately, this representation has finitely many orbits, and they are given exactly by the (co)rank (an easy and well-known fact); so what we want are the Thom polynomials of the orbits Σ_s (whose closures are also called *(general) determinantal varieties*). These results was probably first proved by Porteous in [**Por71**], but apparently were already known by Giambelli.

The representation can be realized with the formula $(A,B)\cdot X = AXB^{-1}$ where $A \in \mathrm{GL}_n$, $B \in \mathrm{GL}_{n+k}$ and $X \in \mathrm{Mat}_{n\times(n+k)}$ are matrices. Define nice representative elements $X_s \in \Sigma_s$:

$$X_s = \begin{bmatrix} I_{n-s} & 0 \\ 0 & 0 \end{bmatrix} \in \mathrm{Mat}_{n\times(n+k)}$$

The tangent spaces $T_{X_s}\Sigma_s$ are easy to calculate with the help of Lemma 2.10; we get that

$$T_{X_s}\Sigma_s = \left\{ \begin{bmatrix} * & * \\ * & 0 \end{bmatrix} \ : \ * \text{ is anything} \right\} \subset \mathrm{Mat}_{n\times(n+k)}$$

where the upper left submatrix has size $(n-s)\times(n-s)$, as above. It is easy to see that a subgroup of the stabilizer of X_s being homotopy equivalent with it is given by

$$\left\{ \left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}, \begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix} \right) \ : \ A \in \mathrm{GL}_{(n-s)},\ B \in \mathrm{GL}_s,\ C \in \mathrm{GL}_{(s+k)} \right\}$$

thus a maximal torus of the stabilizer is

$$\begin{aligned} T_s = \big\{ \big(& \mathrm{diag}(\chi_1,\dots,\chi_{n-s},\varphi_{n-s+1},\dots,\varphi_n), \\ & \mathrm{diag}(\chi_1,\dots,\chi_{n-s},\psi_{n-s+1},\dots,\psi_{n+k}) \big) \in \mathrm{GL}_n \times \mathrm{GL}_{n+k} \ : \ \chi_i,\varphi_j,\psi_l \in U_1 \big\} \end{aligned}$$

The conditions of Theorem 5.1.4 hold in this case, so the restrictions equations

$$j_r^*\mathrm{Tp}(\Sigma_s) = \begin{cases} 0 & r < s \quad \text{‘homogeneous equations’} \\ e_{T_s}(N_s) & r = s \quad \text{‘principal equation’} \end{cases}$$

have a unique solution, where $j_r^* : H_G^* \to H_{T_r}^*$ is the restriction induced by $j_r : T_r \hookrightarrow G$ (of course, instead of G, we will work with the maximal torus $T \subset G$ containing the pairs of diagonal matrices, as $H_G^* = [H_T^*]^{\mathfrak{W}} \subset H_T^*$, where $\mathfrak{W} = \mathfrak{S}_n \times \mathfrak{S}_{n+k}$ is the Weyl group) and N_s is an invariant normal space of the orbit Σ_s at the point X_s. In our case, however, $G_s \subset G_{s+1}$ (where G_s is the stabilizer of X_s), which means that the principal equation *implies* the homogeneous equations, thus the principal equation alone has a unique solution.

Theorem 5.2.1.1 (Giambelli-Thom-Porteous formula). *The Thom polynomial of Σ_s is $\mathrm{Tp}(\Sigma_s) = \Delta_{(s,s,\dots,s)}(h) = \det(M)$ where*

$$M = \begin{bmatrix} h_s & h_{s-1} & \cdots & h_{1-k} \\ h_{s+1} & h_s & \cdots & h_{2-k} \\ \vdots & \vdots & \ddots & \vdots \\ h_{2s+k-1} & h_{2s+k-2} & \cdots & h_s \end{bmatrix}_{(s+k)\times(s+k)}$$

and h_i are the “Chern classes of the universal difference bundle”:

$$h = 1 + h_1 + h_2 + \cdots = c/d = \frac{1 + c_1 + c_2 + \cdots + c_n}{1 + d_1 + d_2 + \cdots + d_{n+k}}$$

where c_i, d_j are the generators of the cohomology ring

$$H_G^* = H^*_{\mathrm{GL}_n\times\mathrm{GL}_{n+k}} = H^*(B\mathrm{GL}_n \times B\mathrm{GL}_{n+k}) = \mathbb{R}[c_1,\dots,c_n,d_1,\dots,d_{n+k}]$$

finally $h_0 = 1$ and $h_{<0} = 0$.

Note that $d = 1 + d_1 + \cdots + d_{n+k}$ is an invertible element in the ring of formal power series $\mathbb{R}[[c_1,\dots,c_n,d_1,\dots,d_{n+k}]]$ so h_i is well-defined and lives in H_G^{2i}.

Proof. A T^s-invariant normal space of the orbit Σ^s at X_s is given by

$$N_s = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix} \ : \ * \text{ is anything} \right\} \subset \mathrm{Mat}_{n\times(n+k)}$$

where the upper left 'zero' means an $(n-s)\times(n-s)$ zero matrix, and the bottom right 'star' means an $s\times(s+k)$ arbitrary matrix. Thus the T^s-equivariant Euler class of this normal space is

$$e(N_s) = \prod_{i,j}(\varphi_i - \psi_j)$$

where with some abuse of notation we use the same greek letters to denote the corresponding generators of $H^*_{T_s}$ that we used to denote the coordinates of T_s above. The restriction map j_s^* is given by

$$\alpha_i \mapsto \begin{cases} \chi_i & 1\le i\le n-s \\ \varphi_i & n-s<i\le n \end{cases} \qquad \text{and} \qquad \beta_i \mapsto \begin{cases} \chi_i & 1\le i\le n-s \\ \psi_i & n-s<i\le n+k \end{cases}$$

where α_i and β_j are the Chern roots, that is, generators of $H^2(BT^n)$ and $H^2(BT^{n+k})$, respectively (thus the classes c_i and d_j are elementary symmetric polynomials of $\alpha_1,\dots,\alpha_n$ and $\beta_1,\dots,\beta_{n+k}$, respectively). Since we know that the equation $j_s^*\mathrm{Tp}(\Sigma^s) = e(N_s)$ has a unique solution in H^*_G, the only thing we have to prove that $\det(M)$ satisfies this equation (as $\det(M)$ is clearly symmetric to the action of the Weyl group). Observe that the image of h is

$$j_r^* h = j_r^* \frac{\prod(1+\alpha_i)}{\prod(1+\beta_i)} = \frac{\prod(1+\chi_i)\prod(1+\varphi_i)}{\prod(1+\chi_i)\prod(1+\psi_i)} = \frac{\prod(1+\varphi_i)}{\prod(1+\psi_i)}$$

Then Lemma A.2 tells us that $j^*\det(M) = R(p,q)$, the resultant of the polynomials $p=\prod(1+\varphi_i t)$ and $q=\prod(1+\psi_i t)$. On the other hand, Lemma A.3 shows that $R(p,q)=\prod(\varphi_i-\psi_j)$, which completes the proof. $\square$

To return the original questions, we proved that if we have a generic (complex) analytic map $f: M\to N$ between analytic manifolds, then

$$[\Sigma_s(f)]_M = \Delta_{s,s,\dots,s}(c(TM\ominus f^*TN))$$

5.2.2. *Symmetric and skew-symmetric determinantal varieties.* In this case we consider the vector space $V=\mathrm{Sym}^2\mathbb{C}^n$ (respectively $\wedge^2\mathbb{C}^n$) of symmetric (resp. anti-symmetric) matrices, and (the closures of the) subsets $\Sigma_s = \{X\in V : \mathrm{corank}(X) = s\}$, which are orbits of the natural GL_n action. See also [**HT84**] and [**JLP82**]. These actions can be realized with the formula $g: X\mapsto gXg^t$ where $g\in\mathrm{GL}_n$ is a nondegenerate and X is a symmetric (resp. antisymmetric) $n\times n$ matrix. The fact that there are finitely many orbits — classified by the (co)rank — and they are represented by the matrices X_s resp. Y_s (note that the form of X_s depends on whether $(n-s)$ is even or odd; while in the antisymmetric case $(n-s)$ is *always* even)

$$X_s^{\text{even}} = \begin{bmatrix} K & & & & \\ & \ddots & & & \\ & & K & & \\ & & & 0 & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix} \qquad X_s^{\text{odd}} = \begin{bmatrix} K & & & & & \\ & \ddots & & & & \\ & & K & & & \\ & & & 1 & & \\ & & & & 0 & \\ & & & & & \ddots & \\ & & & & & & 0 \end{bmatrix} \qquad Y_s = \begin{bmatrix} H & & & & \\ & \ddots & & & \\ & & H & & \\ & & & 0 & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}$$

— where H and K are the 2×2 matrices

$$H = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \text{and} \qquad K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and s denotes the number of zeros — is classical (in the symmetric case this is known as *Sylvester's law of inertia*). We have two reasons for choosing these somewhat esoteric representants; the first one is that we can choose maximal tori so they contain diagonal matrices; the second one is that, almost as in the previous example, we have $G_s \subset G_{s+2}$ (where $G_s = \mathrm{Stab}(X_s)$ resp. $\mathrm{Stab}(Y_s)$ is the stabilizer subgroup) thus the principal equation implies the homogeneous equations in the antisymmetric case; and in the symmetric case one homogeneous equation is enough (the conditions of Theorem 5.1.4 hold clearly).

THEOREM 5.2.2.1 ([**HT84**], [**JLP82**]).

$$\begin{aligned} \mathrm{Tp}_{\mathrm{Sym}^2\mathbb{C}^n}(\Sigma_s) &= 2^s\Delta_{(s,s-1,\dots,1)}(c) \\ \mathrm{Tp}_{\wedge^2\mathbb{C}^n}(\Sigma_s) &= \Delta_{(s-1,s-2,\dots,1)}(c) \end{aligned}$$

Proof. The proof is quite analogous to the previous example. It is easy to see that $\mathrm{O}_{(n-s)}\times \mathrm{GL}_s$ resp. $\mathrm{Sp}_{(n-s)}\times\mathrm{GL}_s$ are subgroups of the stabilizers being homotopy equivalent with them; thus as maximal tori of the stabilizers we can choose

$$\begin{aligned} T_s^{\mathrm{even}} &= \left\{\mathrm{diag}(\chi_1,\chi_1^{-1},\chi_2,\chi_2^{-1},\dots,\chi_{\frac{n-s}{2}},\chi_{\frac{n-s}{2}}^{-1},\varphi_1,\dots,\varphi_s) \;:\; \chi_i,\varphi_j\in \mathrm{U}_1\right\} \\ T_s^{\mathrm{odd}} &= \left\{\mathrm{diag}(\chi_1,\chi_1^{-1},\chi_2,\chi_2^{-1},\dots,\chi_{\frac{n-s}{2}},\chi_{\frac{n-s}{2}}^{-1},1,\varphi_1,\dots,\varphi_s) \;:\; \chi_i,\varphi_j\in \mathrm{U}_1\right\} \end{aligned}$$

in the symmetric ($n-s$ is even or odd) case; in the antisymmetric case, our maximal torus coincides with T_s^{even}. As always, we can compute the tangent spaces $T_{X_s}\Sigma_s$ with Lemma 2.10, and from that, invariant normal spaces N_s; in both cases we get

$$N_s = \left\{\begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix} \;:\; * \text{ is anything}\right\} \subset \mathrm{Mat}_{n\times n}$$

where the bottom right 'star' means an $s\times s$ arbitrary matrix. Thus the equivariant Euler class of this normal space is

$$e_T^{\mathrm{symm}}(N_s) = \prod_{i\le j}(\varphi_i+\varphi_j) \qquad \text{and} \qquad e_T^{\mathrm{anti}}(N_s) = \prod_{i<j}(\varphi_i+\varphi_j)$$

again with our "standard abuse of notation", denoting the generators of H_T^2 with the same letters as the coordinates of T. Note that by Lemma A.4 we have

$$e^{\mathrm{symm}}(N_s) = 2^s\Delta_{(s,s-1,\dots,1)}(\Phi) \qquad \text{and} \qquad e^{\mathrm{anti}}(N_s) = \Delta_{(s-1,s-2,\dots,1)}(\Phi)$$

where $\Phi = \prod(1+\varphi_i) = 1+\Phi_1+\dots+\Phi_s$. The restriction maps j_r^* are given by

$$\alpha_i \mapsto \begin{cases} (-1)^i\chi_{\lceil i/2\rceil} & 1\le i\le 2\lfloor (n-r)/2\rfloor \\ 0 & i = n-r \text{ and it is odd} \\ \varphi_{i-n+r} & n-r < i \le n \end{cases}$$

in both cases. Since $j_r^*\Delta_\lambda(c) = \Delta_\lambda(j_r^*c)$, we have to prove that

$$\left.\begin{array}{ll} 2^s\Delta_{(s,\dots,1)}(j_s^*c) & = 2^s\Delta_{(s,\dots,1)}(\Phi) \\ 2^s\Delta_{(s,\dots,1)}(j_{(s-1)}^*c) & = 0 \end{array}\right\} \text{ symmetric case}$$

$$\left.\Delta_{(s-1,\dots,1)}(j_s^*c) \;=\; \Delta_{(s-1,\dots,1)}(\Phi)\right\} \text{ antisymmetric case}$$

This is an not difficult, although a bit tricky. We will do that for the first case; the other two cases are completely analogous. Let's extend our standard notation σ_i for the elementary symmetric polynomials such that if i is not an integer, then define σ_i to be zero. Using the simple identity

$$\sigma_m(\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_k) = \sum_{i=0}^{m} \sigma_i(\alpha_1,\dots,\alpha_n)\sigma_{m-i}(\beta_1,\dots,\beta_k)$$

we have (with the short notation $q = \lfloor (n-s)/2 \rfloor$) that $j_s^*c_m$ equals to

$$\sigma_m(\chi_1,-\chi_1,\dots,\chi_q,-\chi_q,\varphi_1,\dots,\varphi_s) = \sum_{i\geq 0} \underbrace{(-1)^{i/2}\sigma_{i/2}(\chi_1^2,\dots,\chi_q^2)}_{X_{i/2}} \underbrace{\sigma_{m-i}(\varphi_1,\dots,\varphi_s)}_{\Phi_{m-i}}$$

and of course the same holds also if we have a zero in addition between the variables. Using the multilinearity of the determinant, we have then

$$\Delta_{s,\dots,1}(j_s^*c) = \begin{vmatrix} d_s & \cdots & d_{2s-1} \\ \vdots & & \vdots \\ d_{s-2k} & \cdots & d_{2s-2k-1} \\ \vdots & & \vdots \\ 0 & \cdots & d_1 \end{vmatrix} = \sum_{i\geq 0} X_{i/2} \cdot \begin{vmatrix} d_s & \cdots & d_{2s-1} \\ \vdots & & \vdots \\ e_{s-2k-i} & \cdots & e_{2s-2k-1-i} \\ \vdots & & \vdots \\ 0 & \cdots & d_1 \end{vmatrix}$$

where $d_i = j_s^*c_i$ and $e_i = \sigma_i(\varphi_1,\dots,\varphi_s)$ (we changed only the $(k+1)$-th row). Using this 'transformation' first to the last row, then to the row before the last, etc... we see always that only the $i = 0$ term survives, since for odd i we have $X_{i/2} = 0$, and for even i the determinant will have two identical row. Finally we get the desired result. $\square$

Note that the smallest projective orbit in $\mathbb{P}(\wedge^2\mathbb{C}^n)$ is the Grassmannian $\mathrm{Gr}_2\mathbb{C}^n$; but the Thom polynomials for general Grassmannians are not known (although we have already seen that we have an *algorithm* to calculate them; and there is another one described in Section §7).

5.2.3. *Veronese Varieties.* The homogeneous polynomials of degree n on the (complex) vector space V^{k+1} of dimension $k+1$ form the vector space $\mathrm{Sym}^n V^*$, whose dimension is $\binom{k+n}{n}$. There is natural map $V^* \to \mathrm{Sym}^n V^*$ associating to every linear form its n-th power. However, as it would be quite inconvenient to speak about the dual spaces all the time, define

$$\begin{aligned} v_n : \mathbb{P}V &\longrightarrow \mathbb{P}(\mathrm{Sym}^n V) \subset \mathbb{P}(\otimes^n V) \\ [v] &\longmapsto [v\otimes\cdots\otimes v] \end{aligned}$$

This is a regular map, an embedding in fact, and is called the n-th *Veronese embedding* of $\mathbb{P}V$.[10] Its image $\mathcal{V} = \mathcal{V}_{n,k} = v_n(\mathbb{P}V)$ is a smooth algebraic subvariety

[10] The significance of the Veronese embedding is that if $H \subset \mathbb{P}V$ is a hypersurface of degree n defined by $f = 0$ ($f \in \mathrm{Sym}^n V^*$), then $v_n(H) \subset \mathbb{P}W$ is the intersection of $\mathcal{V} \subset \mathbb{P}W$ with the

of $\mathbb{P}W$, called the *Veronese variety* (where $W = \mathrm{Sym}^n V$), and is obviously invariant to the natural action of $\mathrm{GL}(V)$ on $\mathbb{P}W$ (indeed, v_n is a $\mathrm{GL}(V)$-equivariant map). In fact, the cone $C\mathcal{V}$ is the smallest (nonzero) orbit of the representation $\mathrm{Sym}^n\rho_V : \mathrm{GL}(V) \to \mathrm{GL}(W)$, where ρ_V denotes the standard representation of $\mathrm{GL}(V)$ (see eg. [**Hrr92**]). Our goal is now to calculate the Thom polynomials of the (cones of the) Veronese varieties with the help of the restriction equations.

Fix a maximal torus $T \subset \mathrm{GL}(V)$ and let $x_0, \dots, x_k \in V$ be an eigenbasis of the T-action on V (thus T consists of diagonal matrices written in this basis). Then

$$\mathcal{X} = \Big\{ x_I = \prod_{j=1}^n x_{i_j} \in W = \mathrm{Sym}^d V \;:\; 0 \le i_1 \le i_2 \le \dots \le i_n \le k \Big\}$$

is a basis in W and these are also eigenvectors for T; let $w_I \in H^2_T$ denote the corresponding weights. The weights of the representation $\mathrm{Sym}^n\rho$ are

$$\mathcal{W} = \Big\{ \sum q_i\alpha_i : q_i \in \mathbb{N}, \sum q_i = n \Big\}$$

where $\alpha_1, \dots, \alpha_k \in \mathfrak{t}^* \cong H^2_T$ are the Chern roots. The highest weights $\mathcal{H} = \{n\alpha_i : 0 \le i \le k\}$ are exactly those whose weight vectors are in $C\mathcal{V}$, that is, $\{x \in \mathcal{X} : w_x \in \mathcal{H}\} = \mathcal{X} \cap C\mathcal{V}$. Let $T_I \subset T$ denote the stabilizer of $x_I \in W$; now we can restrict the Thom polynomial $\mathrm{Tp}(C\mathcal{V})$ to T_I: the inclusions $j_I : T_I \hookrightarrow T$ induces restrictions $j_I^* : H^*_T \to H^*_{T_I}$ on the cohomology rings, and these gives us equations

$$j_I^*\mathrm{Tp}(C\mathcal{V}_{n,k}) = \begin{cases} e(V)/e(T_{x_I}C\mathcal{V}) & x_I \in C\mathcal{V} \\ 0 & x_I \notin C\mathcal{V} \end{cases}$$

where $e(T_{x_I}C\mathcal{V})$ is the (equivariant) Euler class of the subrepresentation $T_{x_I}C\mathcal{V} < V$. As a subspace of W, it is easy to calculate the tangent spaces $T_{x_I}C\mathcal{V}$, with the help of Lemma 2.10; it turns out that

$$T_{x_i^n}C\mathcal{V} = \langle x_i^{n-1}x_j \;:\; 0 \le j \le k \rangle$$

thus the weights are

$$\mathcal{W}(T_{x_i^n}C\mathcal{V}) = \big\{ (n-1)\alpha_i + \alpha_j \;:\; 0 \le j \le k \big\}$$

THEOREM 5.2.3.1. *The Thom polynomial of the (cone of the) Veronese variety $C\mathcal{V}_{n,k}$ is*

$$\mathrm{Tp}(C\mathcal{V}_{n,k}) = n^k \prod_{w \in \mathcal{W}-\mathcal{H}} w$$

where $\mathcal{H}$ is the set of the highest weights.

Proof. As we know from Theorem 5.1.8 that the equations given by restricting to the weight vectors have a unique solution, the only thing we have to see is that the above polynomial satisfies these equations. As $T_I = \exp(w_I)^{-1}(1)$, the restriction $j_I^* : H^*_T = \mathrm{Sym}(\mathfrak{t}^*) \to H^*_{T_I} = \mathrm{Sym}(\mathfrak{t}_I^*)$ is given by the projection $\mathfrak{t} \to \mathfrak{t}_I = \mathrm{Ker}(w_I)$. From that, $j_I^*(\prod_{\mathcal{W}-\mathcal{H}} w) = 0$ holds obviously if $I \ne i^n$; and since $j_{i^n}^*\alpha_i = 0$,

$$j_{i^n}^* \prod_{w \in \mathcal{W}-\mathcal{H}} w = j_{i^n}^* \Big(\prod \Big\{ \sum q_l\alpha_l : 0 \le q_l < n, \sum q_l = n \Big\} \Big) = \prod \Big\{ \sum q_l\widetilde{\alpha}_l \Big\}$$

---

hyperplane $\mathrm{Ker}f$ (here we think of f as an element of $(\mathrm{Sym}^n V)^* = W^*$); this allows to reduce some problems concerning hypersurfaces to the case of hyperplanes.

where

$$\widetilde{\alpha}_m = \left\{ \begin{matrix} \alpha_m & m \neq i \\ 0 & m = i \end{matrix} \right\} \quad \Longrightarrow \quad j_{i^n}^* \prod_{m \neq i} ((n-1)\alpha_i + \alpha_m) = \prod_{m \neq i} \alpha_m$$

thus $j_{i^n}^*(\prod_{\mathcal{W}-\mathcal{H}} w) = e(V)/e(T_{x_I} C\mathcal{V})$. □

REMARK. Although this proof could be seen quite *ad hoc*, it isn't that at all. In the lowest dimension case, $k = 2$, the Thom polynomial is a homogeneous polynomial in two variables, thus it decomposes to the product of linear forms. If we want to calculate the Thom polynomial explicitly in this case, we get an interpolation problem, from which the formula can be seen easily. It is then quite natural to expect the same behaviour in higher dimension, which in fact happens, as the proof shows.

6. Projective Thom Polynomials and Degree

In this section we show how the 'projective Thom polynomial' is related to the affine one (which we called the Thom polynomial until this point), and how to calculate degrees of invariant projective varieties from their Thom polynomials. Most results of this section can be found in [**FNRa**].

Every representation $\rho: G \to \mathrm{GL}(V)$ determines a natural action on the space of linear subspaces, particularly on $\mathbb{P}V$; we will denote the latter action by $\mathbb{P}\rho : G \to \mathrm{PGL}(\mathbb{P}V)$. In the whole section we will suppose that we have a representation $\rho: G \to \mathrm{GL}(V^n)$ such that the image of ρ *contains the scalars*, that is,

$$\{\lambda \cdot \mathrm{id}_V \ : \ \lambda \in \mathrm{GL}_1\} \subset \mathrm{Im}(\rho).$$

Thus the nonzero orbits of ρ are in bijection with the orbits of $\mathbb{P}\rho$. Denote by $\varphi: \mathrm{GL}_1 \to G$ the "embedding of scalars", that is, a Lie group morphism such that $(\rho \circ \varphi)(\lambda) = \lambda^q \cdot \mathrm{id}_V$ for some integer $q \in \mathbb{Z}$ (note that the choice of φ is not unique).

DEFINITION 6.1. The *projective Thom polynomial* $\mathrm{Tp}(\mathbb{P}\eta)$ of an orbit of V is the dual class $\mathrm{Tp}(\mathbb{P}\eta) = [B_G \mathbb{P}\eta]_{B_G \mathbb{P}V}$.

So we have to understand the ring $H_G^*(\mathbb{P}V)$ first. As always, let's fix a (coordinatized) maximal torus $m : (\mathrm{U}_1)^r \hookrightarrow G$, and denote by T the torus itself $T = \mathrm{Im}(m) \subset G$. Suppose moreover that the "constants are in T", that is, $\mathrm{Im}(\varphi|_{\mathrm{U}_1}) \subset \mathrm{Im}(m)$ (this will be useful later).

THEOREM 6.2. *Let $w_1, \ldots, w_n \in H_T^2$ be the weights of the representation ρ, and $\xi = -c_1(\tau)$ be (minus one times) the first Chern class of the tautological line bundle over $B_G \mathbb{P}V$. Then*

$$H_G^*(\mathbb{P}V) = H_G^*[\xi]/\Gamma(\xi)$$

where $H_G^*[\xi]$ denotes the polynomial ring over $H_G^* = H^*(BG)$ generated by ξ and $\Gamma(\xi) = \prod(w_i + \xi)$.

Proof. From the Leray-Hirsch theorem, we know that $H_G^*(\mathbb{P}V)$ is a free module over H_G^*, and its generators correspond to the additive generators of $H^*(\mathbb{P}V)$. To be precise, there exists elements of $H_G^*(\mathbb{P}V)$ — $1, \xi, \xi^2, \ldots, \xi^{n-1}$ will do, actually — such that their restriction to any fiber is an additive generator system of the cohomology of that fiber, and $H^*_G(\mathbb{P}V)$ is a free module over H^*_G with these elements as additive generators, the module structure induced by π^*, where π is the projection map of the bundle $B_G\mathbb{P}V \to BG$. Consider the factor bundle $\mathcal{Q} = \pi^*V/\tau$; by definition and the Whitney product formula we have

$$\prod(1+\pi^*w_i) = c(\pi^*V) = c(\mathcal{Q}\oplus\tau) = (1+c_1(\mathcal{Q})+c_2(\mathcal{Q})+\cdots)(1-\xi)$$

Since $(1-\xi)^{-1} = 1+\xi+\xi^2+\xi^3+\cdots$, and $\dim(\mathcal{Q}) = n-1$, we have

$$0 = c_n(\mathcal{Q}) = \sum_{i=0}^{n} \xi^i \sigma_{n-i}(\pi^*w_1,\ldots,\pi^*w_n) = \Gamma(\xi)$$

Thus we found the desired relation, and the Leray-Hirsch theorem tells us that there are no other relations. □

Since we assumed that $\operatorname{Im}(\varphi\restriction_{\mathrm{U}_1}) \subset \operatorname{Im}(m)$, φ factors through $(\mathrm{U}_1)^r$, that is we have a morphism $\widetilde{\varphi}\colon \mathrm{U}_1 \to (\mathrm{U}_1)^r$ such that $\varphi\restriction_{\mathrm{U}_1} = m\circ\widetilde{\varphi}$. But every such morphism has the form $z \mapsto (z^{\omega_1}, z^{\omega_2},\ldots,z^{\omega_r})$ where $\omega_i \in \mathbb{Z}$ are integers. Now we can state the main result of the section:

THEOREM 6.3 ([**FNRa**]). *In the above situation, we have*

$$\operatorname{Tp}(\mathbb{P}\eta)(\xi,\alpha_1,\alpha_2,\ldots,\alpha_r) = \operatorname{Tp}(\eta)\big(\alpha_1+\frac{\omega_1}{q}\xi, \alpha_2+\frac{\omega_2}{q}\xi,\ldots,\alpha_r+\frac{\omega_r}{q}\xi\big)$$

where α_i are the generators of H^*_T (so elements of H^*_G are polynomials of $\alpha_1,\ldots,\alpha_r$ invariant to the action of the Weyl group)

Proof. Consider the bundle $\chi : \operatorname{Hom}(\mathbb{C},V) \to \operatorname{Hom}(\tau,\pi^*V) \to B_G\mathbb{P}V$; this bundle has $G_1 = \mathrm{GL}_1\times G$ as its structure group, which acts on $\operatorname{Hom}(\mathbb{C},V)$ by the representation $\rho_1 = \operatorname{Hom}(\mathrm{std}_1,\rho)$. Observe that $\operatorname{Hom}(\mathbb{C},V)$ is canonically isomorphic to V, moreover, the G_1-invariant subsets of $\operatorname{Hom}(\mathbb{C},V)$ are the same as the G-invariant subsets of V (using this isomorphism). The bundle χ has a canonical section σ, the "embedding of τ into V"; that is, $\sigma(x) = (\tau_x \hookrightarrow (\pi^*V)_x)$. By Corollary 4.4, $\sigma^{-1}(\eta) = B_G\mathbb{P}\eta$, thus if $f: B_G\mathbb{P}V \to BG_1$ denotes the classifying map of χ then we have $f^*\operatorname{Tp}_{\rho_1}(\eta) = \operatorname{Tp}_\rho(\mathbb{P}\eta)$. On the other hand, it is easy to see that $H^*_{G_1} = H^*_G[\xi_1]$ (where $\xi_1 = -c_1(\pi_1^*\tau_1)$, $\pi_1\colon BG_1 \to B\mathrm{GL}_1$ is the projection and τ_1 is the tautological line bundle over $B\mathrm{GL}_1$) and $f^*\colon H^*_G[\xi_1] \to H^*_G(\mathbb{P}V)$ is a H^*_G-module homomorphism mapping $\xi_1 \mapsto \xi$ thus in effect it is just the quotient by $\Gamma(\xi) = \prod(\xi+w_i)$. So we should calculate $\operatorname{Tp}_{\rho_1}(\eta)$. Choose the maximal torus $T_1 \subset G_1$ given by $T_1 = \operatorname{Im}(m_1)$ where $m_1 : (U_1)^{r+1} \to G_1 = \mathrm{GL}_1 \times G$ is given by $m_1(\lambda,t_1,\ldots,t_r) = (\lambda, m(t_1,\ldots,t_r))$; and let k be the mapping $k\colon (U_1)^{r+1} \to (U_1)^r$ given by $k(\lambda,t_1,\ldots,t_r) = (\lambda^{\omega_1}t_1^q,\ldots,\lambda^{\omega_r}t_r^q)$. Observe that $\rho\circ m\circ k = (\rho_1\circ m_1)^q$; from that, using Lemma 1.10 and Proposition 4.5 we have

$$\operatorname{Tp}_{\rho_1}(\eta)(q\xi,q\alpha_1,\ldots,q\alpha_r) = \operatorname{Tp}_\rho(\eta)(q\alpha_1+\omega_1\xi,\ldots,q\alpha_r+\omega_r\xi)$$

which completes the proof. □

DEFINITION 6.4. Let $X^k \subset \mathbb{P}^n$ be a (closed) projective variety of dimension k. Since $H^*(\mathbb{P}^n;\mathbb{Z}) = \mathbb{Z}[\xi]/(\xi^{n+1})$, there is an integer[11] number d such that the dual class

[11] In fact, it is a positive number, since we are working over the field of complex numbers, where everything has a canonical orientation.

$[X]_{\mathbb{P}V} = d{\cdot}\xi^{n-j}$. The number d is called the *degree* of the variety X, and is denoted by $\deg(X)$.

A more intuitive definition of the degree can be given as follows. Let $H_1, \dots, H_{n-k}$ be generic hyperplanes in $\mathbb{P}V$; then the set $X \cap H_1 \cap \dots \cap H_{n-k}$ contains finitely many points; the cardinality of this set is the degree. The equivalence of the two definition is easy to see, as $\xi = [H_i]_{\mathbb{P}V}$ and $[A \cap B]_{\mathbb{P}V} = [A]_{\mathbb{P}V} \smile [B]_{\mathbb{P}V}$ (at least for "generic" A, B).

COROLLARY 6.5 ([**FNRa**]). *In the above situation the degree of the projective variety $\mathbb{P}\eta$ can be obtained by the substitution*

$$\deg(\mathbb{P}\eta) = \mathrm{Tp}(\eta)\Big(\frac{\omega_1}{q}, \frac{\omega_2}{q}, \dots, \frac{\omega_r}{q}\Big)$$

We will illustrate the calculations on the examples we met in 5.2. For the determinantal varieties, see also [**HT84**].

6.1. *Degrees of the Veronese varieties.* This is really easy. We have seen that the Thom polynomial of the cones $C\mathcal{V}_{n,k}$ is

$$\mathrm{Tp}(C\mathcal{V}_{n,k}) = n^k \prod_{w \in \mathcal{W}-\mathcal{H}} w$$

To get the degree, we have to substitute $\alpha_i = 1/n$ for all i; but a weight has the form $w = \sum q_i \alpha_i$ where $\sum q_i = n$, thus the terms of the product are all 1 after the substitution. So we get that $\deg(\mathcal{V}_{n,k}) = n^k$ (which is of course well-known, and could be proven much easier).

6.2. *Degrees of determinantal varieties.* Here we apply the results of this section to calculate the degrees of determinantal varieties; we should note that we don't know any other derivation of the following formula (the well-known derivation in algebraic geometry — see eg. [**Ful98**] 14.4.14 — is the same in the heart).

THEOREM 6.2.1. *The degree of the determinantal variety $\mathbb{P}X_s \subset \mathbb{P}^{n(n+k)-1}$ where $X_s \subset \mathrm{Mat}_{n\times(n+k)}$ is defined by the vanishing of the $s \times s$ minors is*

$$\deg(\mathbb{P}X_s) = \prod_{i=0}^{s-1} \frac{i! \cdot (n+k+i)!}{(n-1-i)! \cdot (s+k+i)!}$$

Proof. Use the Giambelli-Thom-Porteous formula (Theorem 5.2.1.1), Corollary 6.5 and Lemma A.5. For the maximal torus T the natural choice is the set pairs of diagonal matrices in $\mathrm{GL}_n \times \mathrm{GL}_{n+k}$; the numbers q and ω_i can be then chosen to be $q = 1$, $\omega_1^L = \dots = \omega_n^L = 0$, $\omega_1^R = \dots = \omega_{n+k}^R = 1$ (or $\omega_i^L = -1$, $\omega_j^R = 0$) where the letters L and R stand for the words 'left' and 'right', respectively. It is practical to consider the other form of the Thom-Porteous formula: $\mathrm{Tp}(\Sigma_s) = \Delta_{(s+k^{(1)},\dots,s+k^{(s)})}(h')$ where $h' = 1/h = d/c$; the proof is the same. Then, after the substitution, we have to determine the determinant $\det \binom{n+k}{s+k+i-j}_{s\times s}$, and Lemma A.5 gives the degree formula immediately. $\square$

6.3. *Degrees of symmetrical and skew-symmetrical determinantal varieties.* This is quite analogous to the previous example. In both cases, we can choose $q = 2$, $\omega_1 = \cdots = \omega_n = 1$; thus for example $\deg(\mathbb{P}\Sigma_s^{\text{symm}}) = 2^s \det \binom{n}{s+j-i}_{s\times s}$. The only trick involved is that we use Lemma A.5 with the order of both the rows and the columns inverted (this of course does not change the determinant); then with $a_i = 2i-1$ we get

$$\deg(\mathbb{P}\Sigma_s^{\text{symm}}) = \prod_{i=0}^{s-1} \frac{(n+i)!\cdot i!}{(2i-1)!\cdot(n+s-2i)!}$$
$$\deg(\mathbb{P}\Sigma_s^{\text{anti}}) = \frac{1}{2^{s-1}} \prod_{i=0}^{s-2} \frac{(n+i)!\cdot i!}{(2i-1)!\cdot(n+s-2i-1)!}$$

observing that $\prod_{i<j}\big((2j-1)-(2i-1)\big) = 2^{\binom{s}{2}} \prod i!$.

7. The Duistermaat-Heckman Formula

In this section we will consider the following situation: Let the torus T act on the vector space V, and $X \subset V$ be an invariant cone, such that $\mathbb{P}X \subset \mathbb{P}V$ is a *smooth* projective variety. Furthermore let's suppose that the weights of V are different from each other and also that they are nonzero; thus the weights of V are in bijection with the (projective equivalence classes of) weight vectors.

Theorem 7.1 (the Duistermaat-Heckman formula). *Let's suppose that the weights of V are different from each other. Then*

$$\mathrm{Tp}_T(X) = \sum_{f\in \mathrm{Fix}} \frac{e_T(V)}{e_T(T_f\mathbb{P}X)w_f}$$

where $\mathrm{Fix} = \mathrm{Fix}_T\mathbb{P}X$ *is the set of fixed points in $\mathbb{P}X$ for the action of T, and w_f is weight corresponding to the weight vector (space) f.*

We don't give the proof as it would lead us far away from the main stream of this work. Unfortunately we don't know any appearance of this theorem in the literature; the formula was communicated to us by A. Knutson, and a proof was given by L. Fehér, using the Atiyah-Bott integral formula.

Remark. This formula is unfortunately not so useful as it may seem at first sight. The reason is that the right hand side of the formula lives in the field of fractions of the cohomology ring H_T^*, where the calculations are really cumbersome, even with the help of a computer. For example, we were not able to calculate the Thom polynomials of the Grassmannians $\mathrm{Gr}_k\mathbb{C}^n$ from this formula for $n \geq 7$ using the top personal computers of the day.

Nevertheless, if we would like to use the formula, we have to calculate the Euler classes of the representations $T_f\mathbb{P}X$. The following simple lemma (which is, even if not in this form, well-known) will do that for us.

Lemma 7.2. *Let $f \in \mathrm{Fix}_T\mathbb{P}X$, and denote by $F = \langle f\rangle$ the one-dimensional subrepresentation of V given by f (recall that f is a line in V, and is invariant since it is a fixed point of the projective representation). Then*

$$T_f\mathbb{P}X \cong \mathrm{Hom}(F, T_fX/F) = F^{-1} \otimes (T_fX/F).$$

Proof. It is enough to prove the $X = V$ case. Let τ denote the tautological line bundle over $\mathbb{P}V$; the tangent space $T_p\mathbb{P}V$ is canonically isomorphic to $\mathrm{Hom}(\tau_p, V/\tau_p)$. To see that, let $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{P}V$ be an analytic arc in $\mathbb{P}V$ such that $\gamma(0) = p$; fix a point $x \in V$ such that $[x] = p$ and choose a 'lifting' of this curve $\widetilde{\gamma} : (-\varepsilon, \varepsilon) \to V$, that is, $\widetilde{\gamma}(0) = x$ and $[\widetilde{\gamma}(t)] = \gamma(t)$ for each $t \in (-\varepsilon, \varepsilon)$; and consider the "map" $\varphi_x : \gamma \mapsto (\partial_t\widetilde{\gamma})(0)$. Of course, this is not well-defined; the difference of two such lifting is $t \mapsto t\delta(t)$ where $\delta : (-\varepsilon, \varepsilon) \to V$ is a curve satisfying $\delta(t) \in \tau_{\gamma(t)}$. But then for two lifting $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ the difference $\varphi(\widetilde{\gamma}_1) - \varphi(\widetilde{\gamma}_2) \in \tau_p$ thus $\varphi_x(\gamma)$ is well-defined in V/τ_p. Now

$$\begin{aligned} \Phi_p \;:\; T_p\mathbb{P}V &\longrightarrow \mathrm{Hom}(\tau_p, V/\tau_p) \\ (\partial_t\gamma)(0) &\longmapsto (x \mapsto \varphi_x(\gamma)) \end{aligned}$$

gives the desired isomorphism. Since this isomorphism clearly respects the action of T, we are ready. $\square$

Corollary 7.3. *Let $w_1, \dots, w_k$ be those weights of V whose weight vectors are in X, the first one corresponding to the fixed point f. Then the equivariant Euler class of the representation $T_f\mathbb{P}X$ is*

$$e_T(T_f\mathbb{P}X) = \prod_{i=2}^{k}(w_i - w_1).$$

Example 7.4 (Veronese varieties). We have already calculated these Thom polynomials in Section 5.2.3.1; here we reprove them using the Duistermaat-Heckman formula. Recall that $C\mathcal{V}_{n,k} \subset \mathrm{Sym}^n\mathbb{C}^{k+1}$ is the smallest orbit different from the zero orbit; we have seen that the weights are $\mathcal{W} = \{w_I = \sum q_i\alpha_i : \sum q_i = n\}$; the weight vectors in $C\mathcal{V}_{n,k}$ are $\mathcal{X} \cap C\mathcal{V}_{n,k} = \{x_i^n : 0 \le i \le k\}$. The affine tangent space at x_i^n is $T_{x_i^n} = \langle x_i^{n-1}x_j : 0 \le j \le k\rangle$. Thus the Thom polynomial is

$$\mathrm{Tp}(C\mathcal{V}_{n,k}) = \Big(\prod_I w_I\Big) \cdot \underbrace{\sum_i \frac{1}{(n\alpha_i)\prod_{j\neq i}(\alpha_j - \alpha_i)}}_{A}$$

Write the sum in common denominator form:

$$A = \Big[n\prod \alpha_i\Big]^{-1} \cdot \frac{\sum_i \Big(\prod_{j\neq i}\alpha_j\Big)\cdot(-1)^{(\dots)}\prod_{j,l\neq i}(\alpha_l - \alpha_j)}{\prod_{i<j}(\alpha_i - \alpha_j)}$$

The product in the denominator is a Vandermonde determinant; and observe that the nominator is just the expansion of this determinant by the first ("all 1") column. Thus $A = (n\prod\alpha_i)^{-1}$, and we get the same result as in 5.2.3.

Example 7.5 (Thom polynomials of the Grassmannians). The cone of the Grassmannian $X_{n,k} = C(\mathrm{Gr}_k\mathbb{C}^n)$ is a GL_n-invariant subset (the smallest orbit, actually) of the representation $V = \wedge^k\mathbb{C}^n$. The weights of this representation are indexed by the length k subsets of the set $\mathcal{N} = \{1, 2, \dots, n\}$: $w_I = \alpha_{i_1} + \dots + \alpha_{i_k}$ where $I = \{i_1, \dots, i_k\} \subset \mathcal{N}$ and $\alpha_1, \dots, \alpha_n$ are the generators of the cohomology ring H_T^*, T being the maximal torus of GL_n. All weight vectors are in $X_{n,k}$; the (affine) tangent space at a weight vector $x_I \in \mathcal{X}$ is generated by weight vectors x_J such that $|I \cap J| = k-1$ and of course x_I itself. So we have

$$\mathrm{Tp}(X_{n,k}) = \sum_{|I|=k} \frac{\prod_{J \neq I} w_J}{\prod_{i \in I, j \notin I} (\alpha_j - \alpha_i)}$$

where I and J runs over the length k subsets of $\mathcal{N}$.

If one would like to compute at least the degree (for example of the Grassmannian) from the Duistermaat-Heckman formula, the sad experience will be that after the necessary substitution divisions by zero will appear. However, with some tricks it is possible to extract the desired information from the formula. We would like to illustrate this on the example of the Grassmannian. We know *a priori* that the Duistermaat-Heckman formula gives a polynomial in the variables $\alpha_1, \dots, \alpha_n$, thus instead of substituting $\alpha_i = 1/k$ directly, we can first substitute for example $\alpha_i = (1 + i\varepsilon)/k$ and then take the limit $\varepsilon \to 0$. Unfortunately only the denominators of the terms in the formula tend to 0 as $\varepsilon \to 0$, so first we have to write the formula in common denominator form:

$$\deg(\mathrm{Gr}_k \mathbb{C}^n) = \frac{1}{k^c} \cdot \lim_{\varepsilon \to 0} \frac{\sum_{I \in \mathcal{I}} \left(\prod_{J \neq I} \widetilde{w}_J \right) \left((-1)^{(\dots)} \cdot \prod_{i \notin I \text{ or } j \in I} (j-i)\varepsilon \right)}{\prod_{i<j} (j-i)\varepsilon}$$

where $\mathcal{I}$ denotes the set of length k subsets of $\mathcal{N}$, c is the codimension of the Grassmannian $c = \binom{n}{k} - k(n-k) - 1$, and $\widetilde{w}_J = k + \varepsilon \sum_J j$. Now, since the denominator tends to 0 in the order of $\binom{n}{2}$, to nominator must do the same, thus we can use the l'Hospital rule $\binom{n}{2}$ times, then substituting $\varepsilon = 0$, and we get the pure combinatorial formula

$$\deg(\mathrm{Gr}_k \mathbb{C}^n) = \sum_{I \in \mathcal{I}} \frac{\sum_{\mathcal{J} \in \binom{\mathcal{I} - \{I\}}{k(n-k)}} \prod_{J \in \mathcal{J}} \left(\sum_{j \in J} j \right)}{\prod_{i \in I, j \notin I} (j-i)}$$

or, in a different form, with the short notation $d = k(n-k)$,

$$\deg(\mathrm{Gr}_k \mathbb{C}^n) = \sum_{\mathcal{J} \in \binom{\mathcal{I}}{d}} \left(\prod_{J \in \mathcal{J}} \left(\sum_{j \in J} j \right) \sum_{I \notin \mathcal{J}} \left(\prod_{i \in I, j \notin I} (j-i)^{-1} \right) \right)$$

Unfortunately, we don't know any simple (or complicated) combinatorial proof of the fact that this equals to the "more traditional formula"

$$\deg(\mathrm{Gr}_k \mathbb{C}^n) = \left[k(n-k) \right]! \prod_{i=0}^{k-1} \frac{i!}{(n-k+i)!}$$

However, if such combinatorial techniques exist, together with the Duistermaat-Heckman formula they could be use to calculate the (possibly yet unknown) degrees of other highly symmetric varieties. Recent[12] work of L. Fehér also generalizes the techniques presented here to the *duals* of smooth invariant varieties, which seems a more spectacular result, as calculating degrees of dual varieties is a notoriously hard task.

[12] That means in the days of writing this text.

Part 3. Practice

"We share a philosophy about linear algebra: we think basis-free, we write basis-free, but when the chips are down we close the office door and compute with matrices like fury."
(Irving Kaplansky on Halmos and himself)

At this point we would like to make some comments on the way we arrived to the results described in this work; particularly we would like to stress the extensive usage of computers. Traditionally, mathematics is the only natural science where scientist work from the pure nothing, not relying on the results of experiments. We think (among others) that this situation should change, as soon as possible. The reasons are numerous: First, we see from physics, particularly theoretical physics (which lies the most close to mathematics among all of the sciences) that they can advance much faster by building on the results of experiments and also by designing experiments to test theories. The second more important reason in our opinion is that in our days mathematics reached the complexity where the human instinct alone (based on some sense of 'universal beauty') is not enough to navigate in the forest of abstract concepts.

"Mathematics is the part of physics where experiments are cheap" — said Arnold[1], although we should say "cheaper" instead of "cheap", as for example in our case we quickly met the limits of the capacity of the personal computers of the day. Apart from the joke, oscillating between computer experiments and finding theoretical explanations for the phenomenons appearing in the experiments not just allows faster advance but also renders the research more fun.

8. Plane Cubics

In this section we give complete treatment of a "toy example" — the third symmetric tensor power of the standard representation of GL_3 — to get accustomed to the process of calculations.

The space $\mathbb{P}(\mathrm{Sym}^3\mathbb{C}^3)$ parameterizes *plane cubics*; the orbits of $\mathbb{P}\rho$ are in bijection with projective equivalence classes of plane cubics. The classification is well-known (see eg. [**Dlg03**] 10.3, or [**Hrr92**] p. 121) and can be summarized as in Table 2. There are 8 'simple' orbits and an infinite family of orbits (the nonsingular cubics), parameterized by $\mathbb{C}$. The so called *j-invariant* of a nonsingular cubic $y^2z + x^3 + Axz^2 + Bz^3 = 0$ can be defined as

$$j = 256\frac{27A^3}{4A^3 + 27B^2} \in \mathbb{C}$$

Two nonsingular cubics are projective equivalent (that is, are in the same orbit) if and only if their j-invariant are the same. Note that the two nonsingular cubics with $j = 0, 1728$ are exceptional ones (they correspond to the choices $A = 0$ and $B = 0$, respectively).

[1]V. I. Arnold: *On teaching mathematics*; an extended text of the address at the discussion on teaching of mathematics in Palais de Découverte in Paris on 7 March 1997.

TABLE 2. Classification of plane cubics — Orbits of $\mathbb{P}\mathrm{Sym}^3\mathbb{C}^3$

no.	**description of the orbit**	**codim**	**repr.**	**Stab.**
9	nonsingular cubic (family of orbits)	1	$x^3+y^2z+Axz^2+Bz^3$ $\Delta=4A^3+27B^2\neq 0$	$[1,1,1]$
8	nodal cubic	1	x^3+y^3+xyz	$[1,1,1]$
7	cuspidal cubic	2	x^3+y^2z	$[1,b,b^{-2}]$
6	an irred. conic and a line intersecting it in two distinct points	2	x^3+xyz	$[1,b,b^{-1}]$
5	an irred. conic and a tangent line	3	x^2y+y^2z	$[b,b^{-2},b^4]$
4	three nonconcurrent lines	3	xyz	$[b,c,b^{-1}c^{-1}]$
3	three concurrent lines	4	x^2y+xy^2	$[1,1,b]$
2	a double line and a single line	5	x^2y	$[b,b^{-2},c]$
1	a triple line	7	x^3	$[1,b,c]$

The hierarchy of the orbits is shown in Figure 1.[2] Arrows show that an orbit is in the closure of another; the dashed arrow means that the orbit of nodal cubics is in the closure of the union of orbits of nonsingular cubics, but not in the closure of the orbit of any particular nonsingular cubic.

In Table 2 there is also a representative element x_0 from each orbit and its "stabilizer" shown, in the following sense: The notation $[f_1(b,c),f_2(b,c),f_3(b,c)]$ means a Lie group morphism

$$\begin{aligned}\varphi: \mathrm{U}_1\times\mathrm{U}_1 &\longrightarrow T^3\subset \mathrm{GL}_3\\ (b\,,\,c) &\longmapsto \mathrm{diag}\big(f_1(b,c),f_2(b,c),f_3(b,c)\big)\end{aligned}$$

such that the induced representation $\rho\circ\varphi$ fixes the distinguished element p, and (its image) is maximal among such morphisms (of course, when there is no c we think it as a morphism $\mathrm{U}_1\to T^3$). Here T^3 denotes the maximal torus of diagonal matrices in GL_3; denote by $\alpha_1,\alpha_2,\alpha_3$ the generators of H_T^* corresponding to the standard coordinatization of T, and by β,γ the generators of $H^*_{\mathrm{U}_1\times\mathrm{U}_1}$. The Weyl group $\mathfrak{W}$ is the symmetric group $\mathfrak{S}_3$; as always, we will calculate the Thom polynomials for $\rho\restriction_T$, but we know *a priori* that they will be invariant to the action of the Weyl group (that is, they will be symmetric polynomials in the variables $\alpha_1,\alpha_2,\alpha_3$), since the (closures of the) orbits are GL_3-invariant by definition.

We choose the (most natural) basis consisting weight vectors; we can also think them as monomials. The following table shows the (quite obvious) correspondence between weights and weight vectors:

x^3	x^2y	xy^2	y^3	x^2z	xyz	y^2z	xz^2	yz^2	z^3
$3\alpha_1$	$2\alpha_1+\alpha_2$	$\alpha_1+2\alpha_2$	$3\alpha_2$	$2\alpha_1+\alpha_3$	$\alpha_1+\alpha_2+\alpha_3$	$2\alpha_2+\alpha_3$	$\alpha_1+2\alpha_3$	$\alpha_2+2\alpha_3$	$3\alpha_3$

We will use the notation $\mathcal{O}_i$ for the orbits, according to Table 2; for the nonsingular orbits, $\mathcal{O}_9$ will denote the case $j\neq 0,1728$, and $\mathcal{O}_{9a}$ resp. $\mathcal{O}_{9b}$ the exceptional cases

[2]The figure is based on [**Hrr92**]

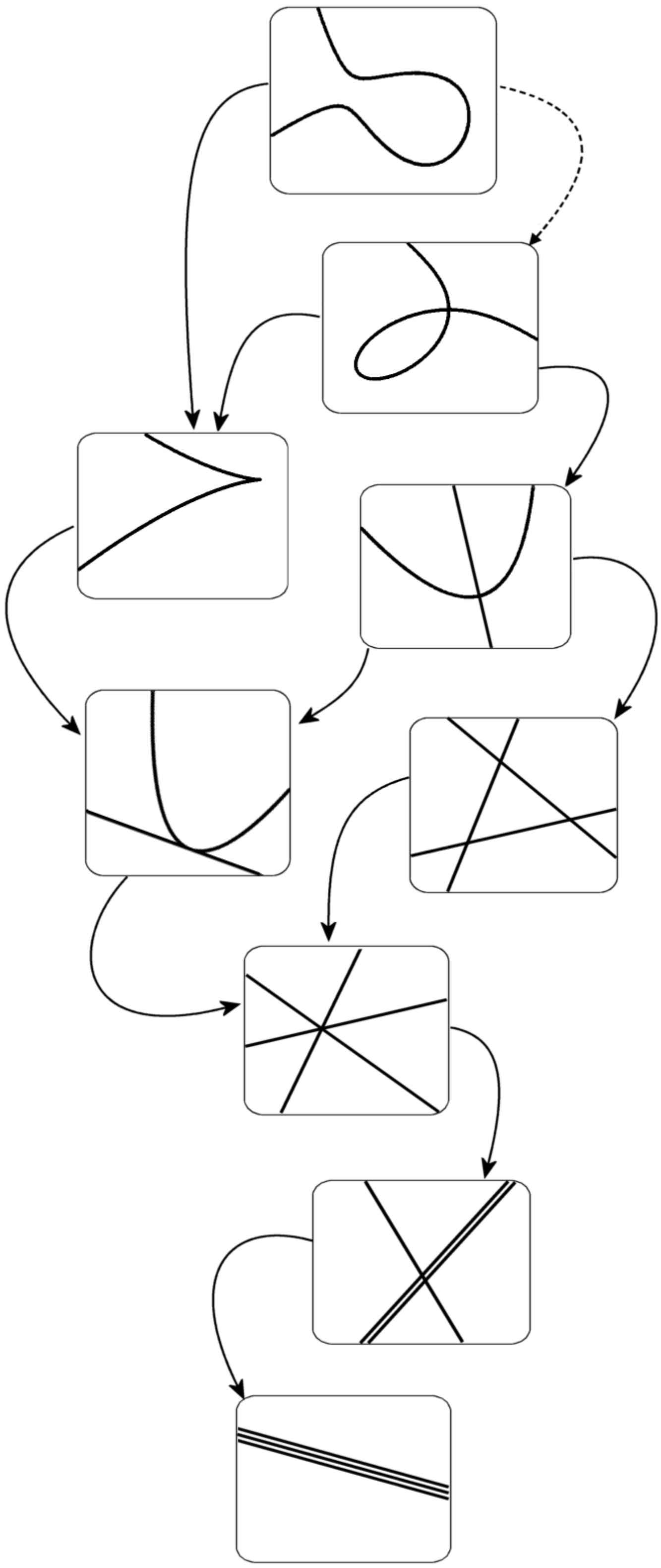


FIGURE 1. Hierarchy of the orbits

TABLE 3. Tangent and normal spaces of orbits in $\mathrm{Sym}^3\mathbb{C}^3$

orbit	codim	Tangent space at p_i	Invariant normal space at p_i
$\mathcal{O}_7$	2	$\langle x^2y, x^2z, xy^2, y^2z, yz^2, xyz, x^3, y^3\rangle$	$\langle xz^2, z^3\rangle$
$\mathcal{O}_6$	2	$\langle x^2y, x^2z, xy^2, xz^2, y^2z, yz^2, xyz, x^3\rangle$	$\langle y^3, z^3\rangle$
$\mathcal{O}_5$	3	$\langle x^2y, xy^2, y^2z, xyz, x^3, y^3, x^2z+2yz^2\rangle$	$\langle xz^2, yz^2, z^3\rangle$
$\mathcal{O}_4$	3	$\langle x^2y, x^2z, xy^2, xz^2, y^2z, yz^2, xyz\rangle$	$\langle x^3, y^3, z^3\rangle$
$\mathcal{O}_3$	4	$\langle x^2y, xy^2, x^3, y^3, 2xyz+x^2z, y^2z-x^2z\rangle$	$\langle xyz, xz^2, yz^2, z^3\rangle$
$\mathcal{O}_2$	5	$\langle x^3, x^2y, x^2z, xy^2, xyz\rangle$	$\langle xz^2, yz^2, y^2z, y^3, z^3\rangle$
$\mathcal{O}_1$	7	$\langle x^3, x^2y, x^2z\rangle$	$\langle xyz, xy^2, xz^2, yz^2, y^2z, y^3, z^3\rangle$

$j = 0$ resp. $j = 1728$. Similarly, φ_i will denote the 'stabilizer morphisms' of our distinguished points $p_i \in \mathcal{O}_i$, and N_i their $(\rho\circ\varphi_i)$-invariant normal spaces. Using Lemma 1.10 it is very easy to calculate the maps $\varphi_i^* : \mathbb{R}[\alpha_1,\alpha_2,\alpha_3] \to \mathbb{R}[\beta,\gamma]$; for example we have

$$\varphi_2^* = \begin{cases} \alpha_1 \mapsto \beta \\ \alpha_2 \mapsto -2\beta \\ \alpha_3 \mapsto \gamma \end{cases} \qquad \varphi_4^* = \begin{cases} \alpha_1 \mapsto \beta \\ \alpha_2 \mapsto \gamma \\ \alpha_3 \mapsto -\beta-\gamma \end{cases} \qquad \varphi_5^* = \begin{cases} \alpha_1 \mapsto \beta \\ \alpha_2 \mapsto -2\beta \\ \alpha_3 \mapsto 4\beta \end{cases}$$

As the are infinitely many orbits, the conditions of Theorem 5.1.4 fail; however we will see that solving the same equations — that is,

$$\varphi_j^*\mathrm{Tp}(\mathcal{O}_i) = \begin{cases} e(N_i) & i = j \quad \text{'principal equation'} \\ 0 & i \neq j \quad \text{'homogeneous equations'} \end{cases}$$

where j runs over the orbits satisfying $\mathrm{codim}(\mathcal{O}_j) \le \mathrm{codim}(\mathcal{O}_i)$ — will give us the Thom polynomials of the orbits $\mathcal{O}_1,\dots,\mathcal{O}_7$ (writing down the equations explicitly we will see that they have a unique solution).[3] Thus we have to calculate the normal spaces N_i first; we can do that by first calculating to tangent spaces $T_{p_i}\mathcal{O}_i$ of the orbits at the points p_i with the help of Lemma 2.10, and then it is easy to find a $\rho\circ\varphi_i$ invariant normal space by hand, as every linear subspace spanned by coordinate axes is even T-invariant; the results are summarized in Table 3 (we have omitted $\mathcal{O}_8, \mathcal{O}_9$ as we can't use that information in what follows). These calculations also gives us the (co)dimensions of the orbits.

So we are ready to calculate the $(\rho\circ\varphi_i)$-equivariant Euler classes $e_i = e(N_i)$ of normal spaces N_i. For example

$$e_7 = e(N_7) = \varphi_7^*\big(3\alpha_3(\alpha_1+2\alpha_3)\big) = \varphi_7^*(\alpha_1+2\alpha_3)\varphi_7^*(3\alpha_3) = (-4\beta)(-6\beta) = 24\beta^2$$

The results are summarized in Table 4. At this point, the only thing remained is writing down the equations and solving them. We will illustrate it on the example of the cuspidal cubic, $\mathcal{O}_7$. Since the codimension of $\mathcal{O}_7$ is 2, the Thom polynomial has the form $\mathrm{Tp}(\mathcal{O}_7) = \lambda_1c_1^2 + \lambda_2c_2$ (here c_i denotes the Chern classes, that is, generators of $H^*_{\mathrm{GL}_3} = \mathbb{R}[c_1,c_2,c_3]$). There are two (nontrivial) restriction whose right side we already know:

$$\varphi_6^*\mathrm{Tp}(\mathcal{O}_7) = 0 \quad \text{and} \quad \varphi_7^*\mathrm{Tp}(\mathcal{O}_7) = e_7 = 24\beta^2$$

[3]With some abuse of notation, we will usually write $\mathrm{Tp}(\mathcal{O}_i)$ instead of $\mathrm{Tp}(\overline{\mathcal{O}}_i)$; as we defined the Thom polynomial only for *closed* invariant varieties, this won't cause any misunderstanding.

TABLE 4. Euler classes of the 'normal space representations'

orbit	**Euler class** $e_i = e_{\mathrm{U}_1\times\mathrm{U}_1}(N_i)$
$\mathcal{O}_7$	$24\beta^2$
$\mathcal{O}_6$	$-9\beta^2$
$\mathcal{O}_5$	$648\beta^3$
$\mathcal{O}_4$	$-27\beta\gamma(\beta+\gamma)$
$\mathcal{O}_3$	$12\beta^4$
$\mathcal{O}_2$	$36\beta\gamma(\beta-\gamma)(\beta+2\gamma)(-4\beta+\gamma)$
$\mathcal{O}_1$	$36\beta^2\gamma^2(\beta+\gamma)(2\beta+\gamma)(\beta+2\gamma)$

Consider the second one first. We know that φ_7^* is given by

$$\alpha_1 \mapsto 0 \qquad \alpha_2 \mapsto \beta \qquad \alpha_3 \mapsto -2\beta$$

thus — since $c_i = \sigma_i(\alpha_1,\alpha_2,\alpha_3)$ is the i-th elementary symmetric polynomial in the three variables $\alpha_1,\alpha_2,\alpha_3$ — we have

$$\begin{aligned} c_1 &= \alpha_1+\alpha_2+\alpha_3 && \longmapsto -\beta \\ c_2 &= \alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1 && \longmapsto -2\beta^2 \end{aligned}$$

that is, we have the equation $j^*\mathrm{Tp}(\mathcal{O}_7) = (\lambda_1-2\lambda_2)\beta^2 = 24\beta^2$, so $\lambda_1-2\lambda_2 = 24$. Similarly, the other equation gives $\lambda_2 = 0$ (as $\varphi_6^*c_1 = 0$, $\varphi_6^*c_2 = -\beta^2$) thus the Thom polynomial must be $\mathrm{Tp}(\mathcal{O}_7) = 24c_1^2$. We have seen how easy the calculations are, although with the growth of the codimension the number of equations grows very fast. We can carry out these calculations for the orbits $\mathcal{O}_1,\dots,\mathcal{O}_6$ too; the results can be seen in the final Table 5. Note that we already knew $\mathrm{Tp}(\mathcal{O}_1)$, as it is a special case of Theorem 5.2.3.1.

We can't solve the remaining cases $\mathcal{O}_8,\mathcal{O}_9,\mathcal{O}_{9a},\mathcal{O}_{9b}$ be the method seen above since their stabilizers are discrete, do not contain U_1 as a subgroup. On the other hand, their Thom polynomial do not contain much information as their codimension is 1; thus we need to find only *one* nontrivial restriction whose right-hand side we are able to calculate.[4]

Until this point, the right-hand sides always followed from one of the first two cases in Theorem 5.1.1; now we will use the third case, where we have to understand some geometry. Consider for example the case of the nodal cubic, $\mathcal{O}_8$, and restrict it to $\mathcal{O}_7$; by Theorem 5.1.1 we have

$$\varphi_7^*\mathrm{Tp}(\overline{\mathcal{O}}_8) = \mathrm{Tp}_{\rho\circ\varphi_7}\big((\overline{\mathcal{O}}_8 - p_7)\cap N_7\big)$$

Thus we are interested in the set $(\overline{\mathcal{O}}_8 - p_7)\cap N_7 = -p_7 + \overline{\mathcal{O}}_8\cap(N_7+p_7)$; in coordinates $p_7 = x^3+y^2z$, $N_7 = \langle xz^2, z^3\rangle$, thus

$$p_7 + N_7 = \{x^3+y^2z+Axz^2+Bz^3 \;:\; A,B\in\mathbb{C}\}$$

[4]Since their codimension is 1, calculating the degree — which is could be done easily by hand — is equivalent to calculating the Thom polynomial; so this is really just an example to show how the calculations work.

which are just the *Weierstrass normal forms* of the irreducible cubics, and it is well-known (see eg. [**Dlg03**]) that

$$\text{the curve } x^3+y^2z+Axz^2+Bz^3=0 \text{ is } \begin{cases} \text{nonsingular, if} & \Delta=4A^3+27B^2\neq 0 \\ \text{nodal, if} & \Delta=0 \text{ but } AB\neq 0 \\ \text{cuspidal, if} & \Delta=A=B=0 \end{cases}$$

That is, we have $N_7 \cong \mathbb{C}\langle A,B\rangle$ and in these new coordinates

$$(\overline{\mathcal{O}}_8 - p_7)\cap N_7 = \{(A,B)\ :\ \Delta = 4A^3+27B^2=0\}$$

We need then the action of U_1 in these new coordinates, but that is easy:

$$(\rho\circ\varphi_7)(b)(A,B) = (b^{-4}A, b^{-6}B)$$

This Thom polynomial (to be correct, a very similar one, see Example 4.11) was among our basic examples: It is -12β. Since $\varphi_7^*(c_1) = -\beta$, we can conclude that $\mathrm{Tp}(\mathcal{O}_8) = 12c_1$.

The remaining three cases can be treated analogously. Using the same restriction, the intersection with N_7 is cut out by the equations $A=0$ (the case of $\mathcal{O}_{9a}$), $B=0$ (the case of $\mathcal{O}_{9b}$) and

$$j = 256\frac{27A^3}{4A^3+27B^2} = \lambda$$

(the case of $\mathcal{O}_9$) instead of $\Delta=0$, respectively, where $\lambda\neq 0, 1728$ is an otherwise arbitrary complex number. The results can be found in Table 5.

Using the Corollary 6.5, we can also calculate the degrees of the projective varieties $\mathbb{P}\overline{\mathcal{O}}_i$; these can be used for solving questions in enumerative algebraic geometry. As a much more general result, in [**AF00**] P. Aluffi and C. Faber gave an algorithm computing the degrees of orbits in $\mathbb{P}(\mathrm{Sym}^n\mathbb{C}^3)$, so we can use that to check our results[5] (not really surprisingly they agree). These degrees are also shown in Table 5.

9. Points in the Projective Line

"Rigour is to the mathematician what mortality is to men."
(after André Weil, who apparently said this with *morality* instead of *mortality*...)

In this section we apply the theory to a real problem. We will consider the n-th symmetric tensor product of the standard representation of $G=\mathrm{GL}_2$; that is, $V=\mathrm{Sym}^n\mathbb{C}^2$, the space of degree n homogeneous polynomials in two variables x,y, and the action of $g\in\mathrm{GL}_2$ on a polynomial P is given by the formula

$$(gP)(x,y) = P\big(g(x,y)\big).$$

A homogeneous polynomial has a well-defined zero locus in $\mathbb{P}^1$, which is a set of n points, at least if counted with multiplicities. It is then quite natural to introduce the stratification $\mathcal{S}_\nu$ of V indexed by partitions of n into positive integers $\nu=(\nu_1,\nu_2,\dots,\nu_d)$ where a polynomial belongs to S_ν if it has root distribution corresponding to ν, that is, it has a root with multiplicity ν_1, another root with

[5] Although our $n=3$ case is probably simple enough to calculate (most of) these degrees by hand, using naïve combinatorial count.

TABLE 5. The Thom polynomials $\mathrm{Tp}(\mathcal{O}_i)$ and degrees of $\mathbb{P}\overline{\mathcal{O}}_i$

orbit	**codim**	**degree**	**Thom polynomial**
$\mathcal{O}_9$	1	12	$12c_1$
$\mathcal{O}_{9a}$	1	4	$4c_1$
$\mathcal{O}_{9b}$	1	6	$6c_1$
$\mathcal{O}_8$	1	12	$12c_1$
$\mathcal{O}_7$	2	24	$24c_1^2$
$\mathcal{O}_6$	2	21	$18c_1^2 + 9c_2$
$\mathcal{O}_5$	3	42	$36c_1^3 + 18c_1c_2 = 2c_1\mathrm{Tp}(\mathcal{O}_6)$
$\mathcal{O}_4$	3	15	$12c_1^3 + 6c_1c_2 + 27c_3$
$\mathcal{O}_3$	4	15	$12c_1^4 + 6c_1^2c_2 + 27c_1c_3 = c_1\mathrm{Tp}(\mathcal{O}_4)$
$\mathcal{O}_2$	5	24	$72c_1^3c_2 + 36c_1c_2^2 - 108c_1^2c_3$
$\mathcal{O}_1$	7	9	$72c_1^3c_2^2 + 36c_1c_2^3 + 36c_1^4c_3 - 162c_1^2c_2c_3 + 243c_1c_3^2$

multiplicity ν_2, etc... There is one more stratum for $0 \in V$, denoted by $\mathcal{S}_0$. It is obvious that the strata are GL_2-invariant subsets, and their closures are GL_2-invariant affine varieties (cones, actually). The varieties $\overline{\mathcal{S}}_\nu$ are also called *coincident root loci.* Note that a stratum $\mathcal{S}_\nu$ is an orbit if and only if $|\nu| \le 3$ (since the action of GL_2 on $\mathbb{P}^1$ is triply transitive).

As the main result of this section we give closed formulas for (the coefficients of) the Thom polynomials of the closures of the strata $\overline{\mathcal{S}}_\nu$. We should note that parallel to our work, L. Fehér, A. Némethi and R. Rimányi calculated the same classes by a different method; see [**FNRb**]. Also the special cases $\mathrm{Tp}_{\mathrm{SL}_2}(\mathcal{S}_{1^k,j})$ (moreover $n = j + k$ is odd) were calculated by Kirwan, see [**Kir92**] (note that the SL_2 Thom polynomials can be obtained from the GL_2 ones by substituting $c_1 = 0$, thus they contain much less information).

9.1. Thom polynomial calculations. To calculate the Thom polynomials, we will restrict them the orbits of the weight vectors; thus these calculations can be considered as an application of Theorem 5.1.8, although it will be clear from the form of the equations that they have a unique solution. There are two quite different problems we will meet here. The first one is that we have to understand the highly nontrivial geometry of the restrictions; we will do that with the help of the techniques developed in Section §4, particularly the (local version) of the deformation lemma (Lemma 4.8). The second is that how to reconstruct the Thom polynomial from the restrictions. This is in fact an interpolation problem, the homogeneous version of the standard Newton interpolation: We have a homogeneous polynomial in two variables α_1, α_2, which we know restricted to different lines $t(i\alpha_1 + (n-i)\alpha_2)$, and we would like to know the original polynomial. Of course the solution is well-known, although we face a little problem at this point: we have more equations than degrees of freedom. To keep the symmetry of the situation, we introduce "virtual degrees of freedom" — we can do that because of our *a priori* knowledge on the existence of the solution — so we will be able to get nice formulas.

Through the whole section, ν will denote a partition of n with length $d=|\nu|$; for the subpartitions φ and ψ, the corresponding uppercase Greek letters Φ and Ψ denote the sum of these partitions, eg. if $\varphi\cup\psi=\nu$ then $\Phi+\Psi=n$. Often there are summations over the decompositions of the partition ν into two subpartitions φ and ψ, $\nu=\varphi\cup\psi$; in these cases empty subpartitions are *not allowed*, unless explicitly stated otherwise. $\sum_{\Phi=i}$ denotes summation over the decompositions $\nu=\varphi\cup\psi$ with $\Phi=i$.

We define another family of invariant subsets $\mathcal{A}_\nu$ (for an other representation) as they are intimately related to $\mathcal{S}_\nu$. Let Q be the polynomial

$$Q=\prod_{j=1}^{d}(x+a_j)^{\nu_j}$$

and Q_k be the coefficient of x^{n-k} in Q. Define the map

$$\nabla_\nu:(a_1,a_2,\dots,a_d)\longmapsto(Q_1,Q_2,\dots,Q_n)$$

(note that $Q_0=1$ is omitted), and let $\mathcal{A}_\nu$ be the image $\mathrm{Im}(\nabla_\nu)\subset\mathbb{A}^n$. The one-dimensional torus U_1 acts on $\mathbb{A}^n$ by the diagonal action

$$\begin{array}{rlcl}\rho_n: & \mathrm{U}_1\times\mathbb{A}^n & \longrightarrow & \mathbb{A}^n\\ & \big(\beta\ ,\ (x_1,\dots,x_n)\big) & \longmapsto & (\beta x_1,\beta^2x_2,\dots,\beta^nx_n)\end{array}$$

for which $\mathcal{A}_\nu$ is invariant. The Thom polynomial of $\mathcal{A}_\nu$ for this action will be denoted by $\mathrm{Tp}(\mathcal{A}_\nu)$. Note that if $d=1$, that is, $\nu=(n)$, then $\mathcal{A}_\nu=\mathcal{A}_n$ is just the rational normal curve, and that in general the closure of its cone $\overline{c\mathcal{A}_\nu}=\overline{\mathcal{S}}_\nu$ (thus $\dim(\mathcal{S}_\nu)=d+1$). Now we are ready to start the calculations.

Proposition 9.1.1. *For any partition ν, $|\nu|\ge2$, $1\le i\le q$,*

$$j_n^*\mathrm{Tp}(\mathcal{S}_\nu)=\mathrm{Tp}(\mathcal{A}_\nu)\tag{1}$$

$$j_{i,n-i}^*\mathrm{Tp}(\mathcal{S}_\nu)=n^{n-d}\sum_{\Phi=i}(-1)^{n+i+|\psi|}\cdot\mathrm{Tp}(\mathcal{A}_\varphi)\cdot\mathrm{Tp}(\mathcal{A}_\psi)\tag{2}$$

where $j_n,j_{i,n-i}:\mathrm{U}_1\to\mathrm{GL}_2$ are the following homomorphisms:

$$j_n:\beta\mapsto\begin{bmatrix}1&\\&\beta\end{bmatrix}\qquad\qquad j_{i,n-i}:\beta\mapsto\begin{bmatrix}\beta^i&\\&\beta^{i-n}\end{bmatrix}$$

Proof. Observe that the induced representations $j_n^*\rho$ and $j_{i,n-i}^*\rho$ has fixed points x^n and $x^{n-i}y^i$, respectively (here $\rho:\mathrm{GL}_2\to\mathrm{GL}(\mathrm{Sym}^n\mathbb{C}^2)$ stands for the n-th symmetric power of the standard representation).

Let's start with (1). We have just noted that $\overline{c\mathcal{A}_\nu}=\overline{\mathcal{S}}_\nu$, and as the action $j_n^*\rho$ also coincide with that in the definition of $\mathrm{Tp}(\mathcal{A}_\nu)$, by Proposition 4.9 we have

$$j_n^*\mathrm{Tp}(\mathcal{S}_\nu)=\mathrm{Tp}(\overline{c\mathcal{A}_\nu})=\mathrm{Tp}(\mathcal{A}_\nu).$$

(2) is more complicated. From the locality property of the Thom polynomial, we know that it is enough to understand $\overline{\mathcal{S}}_\nu$ only in a neighbourhood of $x^{n-i}y^i$. In the geometric picture (points in $\mathbb{P}^1$) that means we have some points near the north

pole $(1:0)$ with sum (weighted with the multiplicities) equal to i and the others are near the south pole $(0:1)$. Thus $\overline{\mathcal{S}}_\nu$ is not irreducible if we consider it only in a small neighbourhood of $x^{n-i}y^i$; its components correspond to the decompositions $\nu = \varphi \cup \psi$ where $\Phi = i$. For example, the (U_1-invariant) polycylinder

$$D_i = \{(\lambda_0, \ldots, \lambda_n) \in \mathrm{Sym}^n\mathbb{C}^2 \ : \ |\lambda_k - \delta_{i,k}| < \varepsilon \text{ for all } k\}$$

(where $\delta_{i,k}$ is the Kronecker-delta, and ε is a small positive number) will do for such a neighbourhood. The Thom polynomial is the sum of the Thom polynomials of the irreducible components, so we will treat a component $Y = Y_\varphi \subset D_i$ corresponding to a fixed decomposition $\nu = \varphi \cup \psi$. Note that action $j^*_{i,n-i}\rho$ of U_1 on $\mathrm{Sym}^n\mathbb{C}^2$ is exactly $(\rho_\Phi)^n \oplus 1 \oplus (\rho_\Psi)^{-n}$ (where ρ_Φ is the representation defined above for which $\mathcal{A}_\varphi$ was invariant, and 1 is the trivial representation), but of course Y is also invariant to the representation $\rho_\Phi \oplus 1 \oplus (\rho_\Psi)^{-1}$, and the Thom polynomials for this two actions differ by the factor n^{n-d} also appearing in (2), as $\mathrm{codim}(Y) = n - d$. So from now we will calculate with the latter representation. Now, $Y = D_i \cap \mathrm{Im}(\omega\restriction_C)$ (where $C = \{(\tau, \underline{a}, \underline{b}) \in \mathbb{C}\times\mathbb{C}^{|\varphi|}\times\mathbb{C}^{|\psi|} \ : \ |a_i| < 1, |b_j| < 1 \ \forall i, j\} \subset \mathbb{C}^{d+1}$ is another polycylinder) for the map $\omega : \mathbb{C}^{d+1} \to \mathrm{Sym}^n\mathbb{C}^2$ defined by

$$\omega : (\tau, a_1, \ldots, a_{|\varphi|}, b_1, \ldots, b_{|\psi|}) \longmapsto (Q_0, Q_1, \ldots, Q_n)$$

where

$$Q = \tau \cdot \prod_{j=1}^{|\varphi|} (a_j x + y)^{\varphi_j} \cdot \prod_{j=1}^{|\psi|} (x + b_j y)^{\psi_j}$$

Let $m_t : (\tau, \underline{a}, \underline{b}) \mapsto (\tau, t\underline{a}, t\underline{b})$ be the operator of multiplication by t and s_t the scaling operator

$$s_t : (z_0, z_1, \ldots, z_k, \ldots, z_n) \longmapsto (t^i z_0, t^{i-1} z_1, \ldots, t^{|i-k|} z_k, \ldots, t^{n-i} z_n)$$

finally let ω_t be the composition $\omega_t = s_{t^{-1}} \circ \omega \circ m_t$. Observe that ω_t is a polynomial in the variables $(t, \tau, \underline{a}, \underline{b})$, thus we can define ω_0 simply by substituting $t = 0$. Easy calculation shows that

$$\omega_0 : (\tau, a_1, \ldots, a_{|\varphi|}, b_1, \ldots, b_{|\psi|}) \longmapsto (\mathcal{L}_\Phi, \ldots, \mathcal{L}_1, \tau, \mathcal{R}_1, \ldots, \mathcal{R}_\Psi)$$

where

$$\begin{aligned} \mathcal{L} &= \tau \cdot \nabla_\varphi(a_1, \ldots, a_{|\varphi|}) \\ \mathcal{R} &= \tau \cdot \nabla_\psi(b_1, \ldots, b_{|\psi|}) \end{aligned}$$

We will use (the local version of) Lemma 4.8: the (local) image $(D_i \times \mathbb{A}^1) \cap \mathrm{Im}(\widetilde{\omega}\restriction_{C_1})$ of the map $\widetilde{\omega} = (\omega_t, t)$ will play the rôle of $\widetilde{\eta}$ (here C_1 stands for $C \times \mathbb{C}$); for $t = 1$ we have $\eta_1 = D_i \cap \mathrm{Im}(\omega\restriction_C) = Y$, and for $t = 0$ we have just seen that $\eta_0 = D_i \cap \mathrm{Im}(\omega_0\restriction_C) = D_i \cap c(\mathcal{A}_\varphi \times \mathcal{A}_\psi)$. Thus the only thing we have to prove is that the intersection multiplicities coincide: $m_0 = m_1$ (in fact both equals to 1); assuming that, we have

$$\mathrm{Tp}_{U_1}(Y_\varphi; D_i) = \mathrm{Tp}_{U_1}(D_i \cap c(\mathcal{A}_\varphi \times \mathcal{A}_\psi); D_i) = (-1)^{n-i-|\psi|}\mathrm{Tp}(\mathcal{A}_\varphi)\mathrm{Tp}(\mathcal{A}_\psi)$$

$$j^*_{i,n-i}\mathrm{Tp}(\mathcal{S}_\nu; V) = \mathrm{Tp}_{U_1}(\mathcal{S}_\nu; V) = \mathrm{Tp}_{U_1}(D_i \cap \mathcal{S}_\nu; D_i) = \sum_{\nu = \varphi\cup\psi} \mathrm{Tp}_{U_1}(Y_\varphi; D_i)$$

where the sign $(-1)^{n-i-|\psi|}$ arise from that $\mathrm{Tp}(\mathcal{A}_\psi)$ is the Thom polynomial for ρ_Ψ and not $(\rho_\Psi)^{-1}$, as $\mathrm{codim}(\mathcal{A}_\psi) = \Psi - |\psi| = (n-i) - |\psi|$. To prove that the intersection multiplicity m_t equals to 1 (for all t), we extend ω to

$$\Omega : (\tau, A_1, \ldots, A_\Phi, B_1, \ldots B_\Psi) \longmapsto (\widetilde{Q}_0, \ldots, \widetilde{Q}_n)$$

where

$$\widetilde{Q} = \tau \cdot \prod_{j=1}^{\Phi}(A_j x + y) \cdot \prod_{j=1}^{\Psi}(x + B_j y)$$

and as above, let $\Omega_t = s_{t^{-1}} \circ \Omega \circ m_t$ and $\widetilde{\Omega} = (\Omega_t, t)$. Observe that $\Omega_t \circ \iota = \omega_t$ where

$$\iota : (\tau, a_1, \ldots, a_k, b_1, \ldots, b_l) \longmapsto (\tau, \Delta^{\varphi_1} a_1, \ldots, \Delta^{\varphi_k} a_k, \Delta^{\psi_1} b_1, \ldots, \Delta^{\psi_l} b_l)$$

$$\Delta^m x = \underbrace{(x, x, \ldots, x)}_{m} \qquad k = |\varphi| \qquad l = |\psi|$$

Moreover, with the notation $Z = \mathrm{Im}(\iota\restriction_C)$ and

$$\widetilde{C} = \{(\tau, \underline{a}, \underline{b}) \in \mathbb{C} \times \mathbb{C}^\Phi \times \mathbb{C}^\Psi \;:\; |a_i| < 1,\ |b_j| < 1\ \forall i,j\} \subset \mathbb{C}^{n+1}$$

and $\widetilde{C}_1 = \widetilde{C} \times \mathbb{C}$ we have $Z = (\Omega_t \restriction_{\widetilde{C}})^{-1}(\eta_t)$; thus Proposition 3.7 implies (the conditions hold, since the image $X = \mathrm{Im}(\Omega_t \restriction_C)$ is open in $\mathrm{Sym}^n\mathbb{C}^2$, so it is smooth; and Ω_t (and also $\widetilde{\Omega}$) is clearly finite) we also have $[Z] = (\Omega_t\restriction_C)^*[\eta_t]$ and $[Z \times \mathbb{A}^1] = (\widetilde{\Omega} \restriction_{\widetilde{C}_1})^*[\widetilde{\eta}]$. Then the line of equalities (we have omitted the restrictions $\restriction_{\widetilde{C}_1}$ for brevity)

$$m_t \widetilde{\Omega}^*[\eta_t] = \widetilde{\Omega}^*\big([\widetilde{\eta}] \smile [V_t]\big) = \widetilde{\Omega}^*[\widetilde{\eta}] \smile \widetilde{\Omega}^*[V_t] = [Z \times \mathbb{A}^1] \smile [\widetilde{C} \times \{t\}] = [Z \times \{t\}] = \widetilde{\Omega}^*[\eta_t]$$

shows that $m_t = 1$ (for all t). $\square$

For the remaining part of the section, we shall introduce some notations. Let σ_k denote the k-th elementary symmetric polynomial in $q-1$ variables; the letters p and q stand for $p = \lfloor n/2 \rfloor$ and $q = \lfloor (n-1)/2 \rfloor$ respectively. Note that

$$q = \begin{cases} p - 1 & n \text{ is even} \\ p & n \text{ is odd} \end{cases}$$

and $p + q = n - 1$. Introduce the shorthand notations

$$\Gamma_k = \tfrac{k(k-n)}{(2k-n)^2}$$

$$\vartheta(j,k) = \sigma_{q-1-j}\big(\Gamma_1, \ldots, \widehat{\Gamma}_k, \ldots, \Gamma_q\big)$$

$$\Theta(j,k) = \Gamma_k \cdot \vartheta(j,k)$$

where $\widehat{\Gamma}_k$ means that the term Γ_k or Γ_{n-k} (!) must be omitted (this unusual notation is justified by the observation $\Gamma_k = \Gamma_{n-k}$). Note also that $\Gamma_k \neq \Gamma_l$ if $1 \le k < l \le q$. If $k = n/2$ then Γ_k and $\Theta(j,k)$ are somewhat ill-defined; define them, and also $\vartheta(j,k)$, to be zero in this case.

Define the integer-valued function F on the set of partitions as follows: let e_r denote the number of elements in the partition ν equal to r (thus $d = \sum e_r$ and $n = \sum r e_r$), then

$$F(\nu) := \prod_r e_r!$$

THEOREM 9.1.2. *For any partition $\nu \neq (p,p)$, $|\nu| \geq 2$,*

$$\mathrm{Tp}(\mathcal{A}_\nu) = \frac{n!}{F(\nu)}\beta^{(n-d)} \tag{3}$$

$$\mathrm{Tp}(\mathcal{S}_\nu) = \sum \lambda_j c_1^{(n-d-2j)} c_2^j \tag{4}$$

$$\lambda_j = \left(\frac{n}{2}\right)^{n-2q} \left[(n-2)!!\right]^2 \sum_{\nu=\varphi\cup\psi} \Theta(j,\Phi) \frac{(-1)^{n+p+j+|\psi|}}{F(\varphi)F(\psi)} \left[\frac{2\Phi-n}{n}\right]^{d-2}$$

where β (respectively c_1 and c_2) denotes the generator(s) of $H^*(BT) = \mathbb{R}[\beta]$ (resp. $H^*(B\mathrm{GL}_2) = \mathbb{R}[c_1, c_2]$).

We should remark that if one takes (3) granted (which could be probably proven directly, although we will prove it together with (4), using a tricky induction which we believe is quite nice) then the remaining problem is just a polynomial interpolation. Namely, the polynomial has the form

$$c_1^{n-d} \sum \lambda_j \left(\frac{c_2}{c_1^2}\right)^j$$

and after the substititutions $c_1 \mapsto (2i-n)$ and $c_2 \mapsto i(i-n)$ we get

$$j^*_{i,n-i}\mathrm{Tp}(\mathcal{S}_\nu) = (2i-n)^{n-d} \sum \lambda_j (\Gamma_i)^j = n^{n-d} \sum_{\Phi=i} (-1)^{n+i+|\psi|} \frac{i!(n-i)!}{F(\varphi)F(\psi)}$$

Thus we know the value of polynomial $\sum \lambda_i x^i$ at the points $x = \Gamma_i$ (for $1 \leq i \leq q$); and, for example from the classical Lagrange interpolation formula, we have

$$\mathrm{Tp}(\mathcal{S}_\nu) = c_1^{p-q+3-d} \sum_{i=1}^{q} \left(\frac{n}{2i-n}\right)^{n-d} \sum_{\Phi=i} (-1)^{n+i+|\psi|} \frac{i!(n-i)!}{F(\varphi)F(\psi)} \prod_{j\neq i} \frac{c_2 - \Gamma_j c_1^2}{\Gamma_i - \Gamma_j}$$

Thus in some sense the above theorem is just an expansion of this formula. However, at the time of proving the theorem we were not aware of this fact, so the proof given below could be seen quite complicated at first sight.

We will need the following combinatorial identity to prove the theorem (and it will be also useful when we will calculate the degrees):

LEMMA 9.1.3. *For any partition ν, $|\nu| \geq 2$*

$$\sum_{\nu=\varphi\cup\psi} (-1)^{|\psi|} \left(\frac{2\Phi-n}{2}\right)^h \frac{F(\nu)}{F(\varphi)F(\psi)} = \begin{cases} d! \cdot \prod \nu_k & h = d \\ 0 & h < d \end{cases}$$

where $n = \sum \nu_k$, and the summation runs over the decompositions of ν into two (possibly empty) subpartitions, φ and ψ.

Proof. Observe that a decomposition of $\nu = \varphi \cup \psi$ is given by deciding for each $r \in \nu$ how many (denote that by j_r) elements of ν equal to r will belong to φ (and then the rest $(e_r - j_r)$ of them will belong to ψ); that is, by giving a number $j_r \in \{0, 1, \ldots, e_r\}$ for each r. For such a decomposition $\Phi = \sum r j_r$, and

$$\frac{F(\nu)}{F(\varphi)F(\psi)} = \prod_r \binom{e_r}{j_r}$$

by the definition of F. That means that the left hand L of our identity equals to

$$L = \frac{(-1)^d}{2^h} \sum_{j_1=0}^{e_1} (-1)^{j_1} \binom{e_1}{j_1} \cdots \sum_{j_m=0}^{e_m} (-1)^{j_m} \binom{e_m}{j_m} \Big[-n + 2\sum_r r j_r \Big]^h$$

Expanding the innermost term

$$\Big[-n + 2\sum_r r j_r \Big]^h = \Big(\sum_r r(2j_r - e_r) \Big)^h = \sum_{|\alpha|=h} c_\alpha \cdot \prod_r r^{\alpha_r} (2j_r - e_r)^{\alpha_r}$$

where α runs over multiindices with $|\alpha| = \sum \alpha_r = h$, and $c_\alpha = h!/\prod \alpha_r!$ are constants. Thus

$$L = \frac{(-1)^d}{2^h} \sum_{|\alpha|=h} c_\alpha \cdot \prod_r \underbrace{\Big(\sum_{j_r=0}^{e_r} (-1)^{j_r} \binom{e_r}{j_r} r^{\alpha_r} (2j_r - e_r)^{\alpha_r} \Big)}_{A_{r,\alpha}}$$

If $h < d = \sum e_r$, there is always at least one r such that $\alpha_r < e_r$, and then $A_{r,\alpha} = 0$ follows. To see that, let's introduce the numbers

$$S(n,k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n$$

These numbers are called *Stirling numbers of the second kind*, and it is a well-known fact that $S(n,k) = 0$ if $k > n$ and $S(n,n) = 1$. Now, expand $A_{r,\alpha}$:

$$\begin{aligned} A_{r,\alpha} &= r^{\alpha_r} \sum_{j=0}^{e_r} (-1)^j \binom{e_r}{j} \sum_{i=0}^{\alpha_r} \binom{\alpha_r}{i} (2j)^i (-e_r)^{\alpha_r - i} = \\ &= r^{\alpha_r} \sum_{i=0}^{\alpha_r} \binom{\alpha_r}{i} 2^i (-e_r)^{\alpha_r - i} \Big((-1)^{e_r} \cdot e_r! \cdot S(i, e_r) \Big) = \\ &= \begin{cases} 0 & \alpha_r < e_r \\ (-2r)^{e_r} e_r! & \alpha_r = e_r \end{cases} \end{aligned}$$

thus we are ready if $h < d$. If $h = d$, then only the term with $a_r = e_r$ for all r survives from the above sum, and:

$$L = \frac{(-1)^d}{2^d} \cdot \underbrace{\frac{d!}{\prod e_r!}}_{c_\alpha} \cdot \prod_r (-2r)^{e_r} e_r! = d! \cdot \prod r^{e_r}$$

what is exactly what we proposed. $\square$

Observe that the contribution of the two degenerate decomposition — where φ or ψ is empty — to the above sum is $(n/2)^h(1 + (-1)^{d-h})$. From that

$$\frac{1}{F(\nu)} = \sum_{\nu = \varphi \cup \psi} \left(\frac{2\Phi - n}{n} \right)^{d-2} \frac{(-1)^{|\psi|+1}}{F(\varphi) F(\psi)}$$

follows immediately. This is the form of the identity we will use in the following proof.

Proof of Theorem 9.1.2. The proof is based on a tricky induction argument. Assume that the proposition is true for partitions of length less than d, and let ν be a partition with $|\nu| = d$. The restriction map $j^*_{i,n-i} : H^*(B\mathrm{GL}_2) \to H^*(B\mathrm{U}_1)$ is given by the substitution

$$\begin{aligned} c_1 &\mapsto (2i-n)\beta \\ c_2 &\mapsto i(i-n)\beta^2 \end{aligned}$$

The Thom polynomial $\mathrm{Tp}(\mathcal{S}_\nu) = \sum_{j=0}^{\tilde{q}-1} \lambda_j c_1^{n-d-2j} c_2^j$ has $\tilde{q} = 1 + \lfloor (n-d)/2 \rfloor$ coefficients, denoted $\lambda_0, \dots, \lambda_{\tilde{q}-1}$, but we introduce some more "virtual" coefficients, so we will have exactly q unknowns ($\tilde{q} > q$ may occur only when $d = 2$ and n is even, thus $\tilde{q} = q + 1$ and it is easy to see then that $\lambda_{\tilde{q}-1} = 0$ if $\nu \neq (p,p)$; finally we treat the remaining case $\nu = (p,p)$ as a special case). The restrictions $j^*_{i,n-i}$ for $1 \le i \le q$ gives q (independent) linear equations on the unknown variables $\lambda_0, \dots, \lambda_{\tilde{q}-1}$. The i-th equation has the form

$$(2i-n)^{n-d} \sum_{j=0}^{\tilde{q}-1} (\Gamma_k)^j \lambda_j = n^{(n-d)} i!(n-i)! \sum_{\Phi = i} \frac{(-1)^{n+i+|\psi|}}{F(\varphi)F(\psi)}$$

by Proposition 9.1.1 and induction. After dividing by $(2i-n)^{n-d}$ and extending the sums to run over $0 \le j < q$ instead of $\tilde{q}$ — we can do that without spoiling the solution, since the equations are independent and have at least one solution as the Thom polynomial exists; the extended equations have a unique solution thus it must be in the form $\lambda_0, \dots, \lambda_{\tilde{q}-1}, 0, \dots, 0$ — we get a system of linear equations the matrix of which is a square Vandermonde matrix, that is, we can solve it using the Cramer rule and the following fact about (generalized) Vandermonde determinants:

$$\begin{vmatrix} 1 & a_1 & \cdots & a_1^{j-1} & a_1^{j+1} & \cdots & a_1^{q-1} \\ 1 & a_2 & \cdots & a_2^{j-1} & a_2^{j+1} & \cdots & a_2^{q-1} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 1 & a_{q-1} & \cdots & a_{q-1}^{j-1} & a_{q-1}^{j+1} & \cdots & a_{q-1}^{q-1} \end{vmatrix} = \sigma_{q-1-j}(a_1, \dots, a_{q-1}) \cdot \prod_{1 \le i < k < q} (a_k - a_i)$$

The solution is then

$$\lambda_j = \sum_{i=1}^{q} (-1)^{i+j+1} B_i \frac{\vartheta(j,i) \prod_{e,f \neq i \,\&\, e<f} (\Gamma_f - \Gamma_e)}{\prod_{e<f} (\Gamma_f - \Gamma_e)}$$

where the B_i are the right hand of the equations:

$$B_i = \left(\frac{n}{2i-n} \right)^{n-d} i!(n-i)! \sum_{\Phi = i} \frac{(-1)^{n+i+|\psi|}}{F(\varphi)F(\psi)}$$

Simplifying these expression we get

$$\lambda_j = \sum_{i=1}^{q} (-1)^{i+j+1} \left(\frac{n}{2i-n} \right)^{n-d} \frac{\vartheta(j,i) \cdot i!(n-i)!}{(-1)^{i-1} \prod_{e \neq i} (\Gamma_e - \Gamma_i)} \sum_{\Phi = i} \frac{(-1)^{n+i+|\psi|}}{F(\varphi)F(\psi)}$$

Now,

$$\Gamma_e - \Gamma_i = \frac{e(e-n)}{(2e-n)^2} - \frac{i(i-n)}{(2i-n)^2} = \frac{n^2(e-i)(e+i-n)}{(2e-n)^2(2i-n)^2}$$

Thus

$$\prod_{e\neq i}(\Gamma_e-\Gamma_i) = \left(\frac{n}{2i-n}\right)^{2q-2}\cdot\frac{(2i-n)^2}{\prod_{k=1}^q(2k-n)^2}\cdot$$

$$\cdot\underbrace{(-1)^{i-1}(i-1)!(q-i)!}_{\prod(e-i)}\cdot\underbrace{\frac{(-1)^q(n-i-1)!}{(n-i-1-q)!(2i-n)}}_{\prod(e+i-n)} =$$

$$= (i-1)!(n-i-1)!\left(\frac{n}{2i-n}\right)^{2q-2}\cdot\frac{(-1)^{p+i+1}(2i-n)}{\prod_{k=1}^q(2k-n)^2}\cdot E$$

where

$$E=\left\{\begin{array}{ll}2/(2i-n) & n \text{ is even}\\ 1 & n\text{ is odd}\end{array}\right\}=\left(\frac{2}{2i-n}\right)^{n-2q-1}$$

Substituting this back, we get

$$\lambda_j=\underbrace{\prod_{k=1}^q(2k-n)^2}_{[(n-2)!!]^2}\sum_{i=1}^q i(i-n)\cdot\vartheta(j,i)\cdot n^{n-d-2q+2}(2i-n)^{d-4}\sum_{\Phi=i}\frac{(-1)^{n+p+j+|\psi|}}{F(\varphi)F(\psi)}$$

which obviously equals to what we proposed, that is, we proved (4).
Now let's see (3). The restriction map j_n^* is given by the substitutions

$$c1 \mapsto \beta$$
$$c2 \mapsto 0$$

thus $j_n^*\mathrm{Tp}(\mathcal{S}_\nu)=\lambda_0\beta^{(n-d)}$. From Proposition 9.1.1 then

$$\mathrm{Tp}(\mathcal{A}_\nu)=j_n^*\mathrm{Tp}(\mathcal{S}_\nu)=\lambda_0\beta^{(n-d)}$$

Observe that $\Theta(0,i)=\prod\Gamma_k$ for every i, and

$$(-1)^q\cdot\left[(n-2)!!\right]^2\cdot\Theta(0,i)=\left\{\begin{array}{ll}(n-1)!/p & n\text{ is even}\\ (n-1)! & n\text{ is odd}\end{array}\right\}=(n-1)!\cdot(2/n)^{n-2q-1}$$

Thus

$$\lambda_0=(-1)^q\frac{n}{2}(n-1)!\sum_{\nu=\varphi\cup\psi}\left(\frac{2\Phi-n}{n}\right)^{d-2}\frac{(-1)^{n+p+|\psi|}}{F(\varphi)F(\psi)}$$

and we are ready with the help of Lemma 9.1.3 (the signs also coincide since $p+q=n-1$).

The induction starts from Example 4.10. □

For the sake of completeness, we give the answer in the remaining cases, namely $\nu=(n)$ and $\nu=(p,p)$ too (although these are known, see [**FR**]).

Proposition 9.1.4.

$$\mathrm{Tp}(\mathcal{S}_n) = \left(n!/\prod_{i=1}^q\Gamma_i\right)\cdot\sum_{j=0}^q(-1)^j\cdot\sigma_{q-j}(\Gamma_1,\dots,\Gamma_q)\cdot c_1^{n-1-2j}c_2^j$$

$$\mathrm{Tp}(\mathcal{S}_{p,p}) = \frac{1}{4}[n!!]^2\cdot\sum_{j=0}^q(-1)^{q+j}\cdot\sigma_{q-j}(\Gamma_1,\dots,\Gamma_q)\cdot c_1^{n-2-2j}c_2^j$$

The calculations are easy, so we omit the proof (one should use Theorem 5.1.1; the right-hand sides of the restrictions are trivial — zero or the Euler class — in these two cases). Although these calculations, being very similar to that in Theorem 9.1.2, give the polynomials in this form, we can have much nicer formulas if we write them in factorized form (which is quite a natural thing to do; observe that the equations with their right side being zero mean exactly that the Thom polynomial is divisible with a linear form). Particularly we have the following:

$$\begin{aligned}
\mathrm{Tp}(\mathcal{S}_n) &= n\Big(\frac{nc_1}{2}\Big)^{p-q}\prod_{j=1}^{q}\big(j(n-j)c_1^2+(2j-n)^2c_2\big) = n\prod_{j=1}^{n-1}\big(j\alpha_1+(n-j)\alpha_2\big)\\
\mathrm{Tp}(\mathcal{S}_{p,p}) &= p^2\prod_{j=1}^{q}\big(j(n-j)c_1^2+(2j-n)^2c_2\big) = p^2\prod_{j\neq 0,p,n}\big(j\alpha_1+(n-j)\alpha_2\big)\\
\mathrm{Tp}(\mathcal{S}_{i,n-i}) &= 2i(n-i)\Big(\frac{nc_1}{2}\Big)^{p-q+1}\prod_{j\neq 0,i}^{q}\big(j(n-j)c_1^2+(2j-n)^2c_2\big) =\\
&= n\cdot i(n-i)\cdot(a+b)\prod_{j,n-j\neq 0,i}\big(j\alpha_1+(n-j)\alpha_2\big) \qquad (i<n/2)
\end{aligned}$$

Recall that $p-q$ is simply 1 if n is even and 0 if n is odd.

9.2. Degree calculations. We have seen in Section §6 that the degree of a projectivized invariant variety can be obtained from its Thom polynomial by a simple substitution. Applying this to the present situation, we can calculate the degrees of the projective varieties $\mathbb{P}\overline{\mathcal{S}}_\nu \subset \mathbb{P}(\mathrm{Sym}^n\mathbb{C}^2)$. These degrees was already known to Hilbert, so we can check our results by comparing the degrees derived from them to the known formula. A related result is presented in [**AF93**], where the degrees of the *orbit closures* were calculated.

According to Section §5, in our case the degree is obtained from the Thom polynomial by the substitution $\alpha_1 \mapsto 1/n$, $\alpha_2 \mapsto 1/n$ (here α_i are the Chern roots), that is

$$c_1 \mapsto 2/n \qquad \text{and} \qquad c_2 \mapsto 1/n^2$$

Thus

$$\deg \mathbb{P}\overline{\mathcal{S}}_\nu = \sum_{j=0}^{q-1}\lambda_j\left(\frac{2}{n}\right)^{n-d-2j}\left(\frac{1}{n^2}\right)^{j} = \frac{1}{n^{n-d}}\sum_{j=0}^{q-1}2^{n-d-2j}\lambda_j =$$

$$= \sum_{j=0}^{q-1}\frac{n^{2-2q}}{2^{d+2j-2q}}\big[(n-2)!!\big]^2\sum_{\nu=\varphi\cup\psi}\Theta(j,\Phi)\frac{(-1)^{n+p+j+|\psi|}}{F(\varphi)F(\psi)}(2\Phi-n)^{d-2}$$

Now, observe that

$$\Gamma_k = \frac{k(k-n)}{(2k-n)^2} = \frac{1}{4}-\frac{n^2}{4(2k-n)^2} = \frac{1}{4}(1-n^2\gamma_k)$$

(where $\gamma_k = 1/(2k-n)^2$), thus using the following simple identity

$$\sigma_k(A+x_1,\dots,A+x_m) = \sum_{i=0}^{k}\binom{m-i}{k-i}A^{k-i}\sigma_i(x_1,\dots,x_m)$$

we get

$$\vartheta(j,k) = \sigma_{q-1-j}\big(\Gamma_1,\dots,\widehat{\Gamma}_k,\dots,\Gamma_q\big) =$$

$$= \frac{1}{4^{(q-1-j)}}\sum_{l=0}^{q-1-j}\binom{q-1-l}{q-1-l-j}(-1)^l n^{2l}\underbrace{\sigma_l(\gamma_1,\dots,\widehat{\gamma}_k,\dots,\gamma_q)}_{\varepsilon(l,k)}$$

Substituting this back and interchanging the order of the sums we get that the degree equals to

$$[(n-2)!!]^2\cdot\frac{n^{2-2q}}{2^{d-2}}\cdot\sum_{\nu=\varphi\cup\psi}\frac{(-1)^{n+p+|\psi|}}{F(\varphi)F(\psi)}\cdot\frac{\Phi(\Phi-n)}{(2\Phi-n)^{4-d}}\sum_{l=0}^{q-1}\frac{n^{2l}\varepsilon(l,\Phi)}{(-1)^l}\sum_{j=0}^{q-1-l}\binom{q-1-l}{j}(-1)^j$$

Note that the last sum is zero if $l < q-1$ and equals to one if $l = q-1$, thus

$$\deg = \frac{n^{2-2q}}{2^{d-2}}\big[(n-2)!!\big]^2\cdot\sum_{\nu=\varphi\cup\psi}\frac{(-1)^{n+p+|\psi|}}{F(\varphi)F(\psi)}\cdot\frac{\Phi(\Phi-n)}{(2\Phi-n)^{4-d}}\cdot\frac{n^{2q-2}}{(-1)^{q-1}}\cdot\prod_{i\neq\Phi}\gamma_i =$$

$$= \sum_{\nu=\varphi\cup\psi}\frac{(-1)^{|\psi|}}{F(\varphi)F(\psi)}\Phi(\Phi-n)\left(\frac{2\Phi-n}{2}\right)^{d-2} =$$

$$= \sum_{\nu=\varphi\cup\psi}\frac{(-1)^{|\psi|}}{F(\varphi)F(\psi)}\left[\left(\frac{2\Phi-n}{2}\right)^{d} - \frac{n^2}{4}\left(\frac{2\Phi-n}{2}\right)^{d-2}\right]$$

and using Lemma 9.1.3 twice (for $h = d$ and $h = d-2$), we get Hilbert's formula:

$$\deg\mathbb{P}\overline{\mathcal{S}}_\nu = \frac{d!}{F(\nu)}\prod\nu_j$$

Remark. As one can easily check, this formula also holds for the two special cases.

Future Projects

In this section we list some questions fitting well in the context of this work we would like to study in the (near) future.

1. *Thom polynomials of the Grassmannians.* It is a really disturbing thing that up to the day we were not able to give a formula to the Thom polynomials of the Grassmannians $\mathrm{Gr}_k\mathbb{C}^n \subset \mathbb{P}(\wedge^k\mathbb{C}^n)$. It would be nice to change this situation (however, it is possible that the problem is hard; a sign that points in that direction is that the Grassmannian is the 'smallest' orbit, so in some sense the most complicated one).

2. *Degrees of duals of Grassmannians.* The *dual variety* $\check{X}$ of a projective variety $X \in \mathbb{P}V$ is defined to be the (closure of the) set of hyperplanes

$$\check{X} = \text{closure of } \{H \in \mathbb{P}V^* : \exists x \in X \text{ s.t. } T_xX \subset H\} \subset \mathbb{P}V^*$$

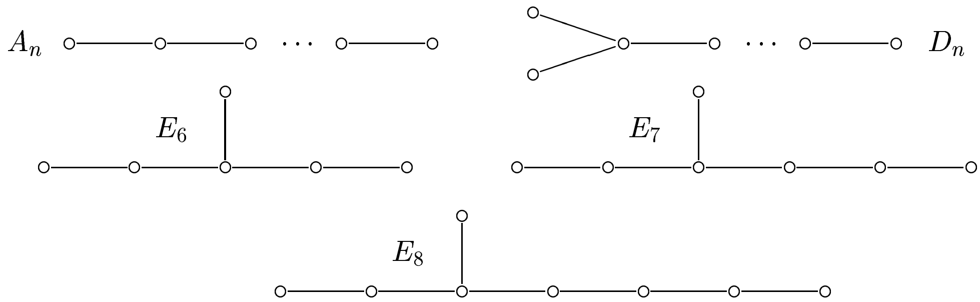
tangent to X. If X is G-invariant then $\check{X}$ is of course also invariant (to the dual representation of G). Calculating degrees of duals of known varieties is notoriously hard problem; although general formulas exists — the first ones due to Kleiman — they are effectively uncomputable. Although in the case of the Grassmannian the answer is known, see [**Las81**], the question is still interesting: First, one would like to see a much more nicer formula than Lascoux's; second, the dual of the Grassmannian $\mathrm{Gr}_k\mathbb{C}^n$ is known to have codimension 1 if $k > 2$ or n is even,[6] thus in this case calculating the Thom polynomial or calculating the degree is just the same thing.

3. *Singularities of families of hypersurfaces.* Degree n hypersurfaces in $\mathbb{P}^k$ are parameterized by $\mathbb{P}(\mathrm{Sym}^n\mathbb{C}^{k+1})$; it would be nice therefore to give some generalization of the results on $\mathrm{Sym}^n\mathbb{C}^2$. Although the questions are not so clear as in the $k = 1$ case, one comes in sight immediately: What is the Thom polynomial of the space of reducible hypersurfaces in $\mathbb{P}^k$?

4. *Thom-Boardman singularities.* R. Thom defined inductively a class of singularities of smooth maps $f : N^n \to P^p$ between smooth manifolds N and P as follows: $x \in N$ belongs to $\Sigma^I(f)$ — where $I = (i_1, i_2, \dots, i_k)$ is a sequence of non-increasing numbers $i_1 \geq i_2 \geq \cdots \geq i_k$ — if $x \in \Sigma^J(f)$ for $J = (i_1, \dots, i_{k-1})$ and $d_x(f\restriction_{\Sigma^J})$ has corank i_k. Thus $\Sigma^i(f)$ is just the set of points $x \in N$ such that d_xf has corank i. Later Boardman gave a more precise and also more general definition (as the definition of Thom implicitly requires $\Sigma^J(f)$ to be a submanifold). In our viewpoint, the classes Σ^I are subvarieties in the space $\oplus_{i\leq k}\mathrm{Hom}(\mathrm{Sym}^i\mathbb{C}^n, \mathbb{C}^p)$ of k-jets of germs of maps $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, invariant to the left-right action of k-jets of germs of diffeomorphisms $\widetilde{G} = j^k\mathrm{Diff}(\mathbb{C}^n, 0) \times j^k\mathrm{Diff}(\mathbb{C}^p, 0)$. However, this group is homotopy equivalent to the subgroup $G = \mathrm{GL}_n \times \mathrm{GL}_p$, so we can consider the representation of G on $\oplus_{i\leq k}\mathrm{Hom}(\mathrm{Sym}^i\mathbb{C}^n, \mathbb{C}^p)$. The $k = 1$ case is just the familiar Thom-Porteous situation we have considered in Section §5.2.1; for $k = 1$, the Thom polynomials of $\Sigma^{i,j}$ were in some sense calculated by Ronga in [**Ron72**]. For higher

[6]and codimension 3 if $k = 2$ and n is odd.

FIGURE 2. Dynkin graphs



k-s the question is totally open.[7] We should note that up to day we were not able to reproduce Ronga's results, although the $A_2 = \Sigma^{1,1}$, and recently in [**BFR02**] the $A_3 = \Sigma^{1,1,1}$ case were solved using the methods described in this work. It would be therefore a spectacular success if we were able giving some formulas or at least algorithms, even if only for the A_n case.

5. *Degeneracy of quivers.* Quiver representations is a surprisingly rich and interesting class of representations. A *quiver* is just a (finite) directed graph[8] $\Gamma = (\Gamma_0, \Gamma_1)$ — possibly with multiple arrows — and we associate a series of representations to it as follows:

$$G = \prod_{i\in\Gamma_0} \mathrm{GL}_{n_i} \qquad V = \bigoplus_{(j,k)\in\Gamma_1} \mathrm{Hom}(\mathbb{C}^{n_j}, \mathbb{C}^{n_k}) \qquad (\boldsymbol{g}\cdot\boldsymbol{\varphi})_{(j,k)} := g_j \circ \varphi_{(j,k)} \circ g_k^{-1}$$

where $\boldsymbol{g} = (g_i : i \in \Gamma_0) \in G$, $\boldsymbol{\varphi} = (\varphi_{(j,k)} : (j,k) \in \Gamma_1) \in V$ and $\boldsymbol{n} = (n_i : i \in \Gamma_0)$ is a *dimension vector*, a vector of positive integers, one for each vertex of Γ. For example the one-loop quiver (a quiver with one vertex and one loop) gives the adjoint representation of GL_n; and the quiver $\circ\!\longrightarrow\!\circ$ gives back the Thom-Porteous case (see eg. [**ARS95**] or [**Cra92**] for the theory of quiver representations). Orbits of quiver representations are in bijection with isomorphism classes of $\mathbb{C}\Gamma$-modules with given dimension vector, $\mathbb{C}\Gamma$ being the *path algebra* of Γ. A theorem of Gabriel classifies the representation-finite path algebras; from that we know that a quiver representation has finitely many orbits if and only if its underlying unoriented graph is a Dynkin graph (see Figure 2). Recently in [**BF99**] A. Buch and W. Fulton gave an algorithm and conjectured formulas for Thom polynomials of the orbits of representations associated to the equioriented A_n quivers (these are also connected to the so called *double Shur and Shubert polynomials*); the method of restriction equations is also readily applicable in the case of Dynkin quivers, see [**FR02**].

There are numerous problems appearing here: first, giving formulas instead of only algorithms; second, saying something about the non-Dynkin case. The other attackable class of quivers are the so called *Euclidean* (or extended Dynkin) quivers, the most simple of them being the *Kronecker quiver*: $\circ\!\rightrightarrows\!\circ$ which deserves further study, and does not seem hopeless at all. We should note that we have already done some sporadic calculations (that means tables filling dozens of pages) concerning the D_3 quiver, which is connected to the 3-subspace problem.

---

[7]Although there are results for special cases, for example if $n = p$.

[8]We use the notation Γ_0 for the set of vertices and Γ_1 for the set of edges.

Appendix

In this appendix we give proofs of some well-known and not so well-known facts from linear algebra and combinatorics we used in the text.

Definition A.1. The resultant of the polynomials

$$
\begin{aligned}
p &= a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n \\
q &= b_0 + b_1 t + b_2 t^2 + \cdots + b_k t^k
\end{aligned}
$$

is defined to be $R(p,q) = a_n^k b_k^n \prod(\alpha_i - \beta_j)$ where α_i and β_j are the roots of the polynomials p and q, respectively.

Lemma A.2 (Sylvester). *The resultant $R(p,q)$ of the polynomials*

$$
\begin{aligned}
p &= 1 + a_1 t + a_2 t^2 + \cdots + a_n t^n \\
q &= 1 + b_1 t + b_2 t^2 + \cdots + b_k t^k
\end{aligned}
$$

equals to the determinant of the $k \times k$ matrix

$$
\begin{bmatrix}
h_n & h_{n+1} & \cdots & h_{n+k-1} \\
h_{n-1} & h_n & \cdots & h_{n+k-2} \\
\vdots & \vdots & \ddots & \vdots \\
h_{n-k+1} & h_{n-k+2} & \cdots & h_n
\end{bmatrix}
$$

where h_i are the coefficients of the Taylor series of the quotient

$$
h = 1 + h_1 t + h_2 t^2 + \cdots = p/q = \frac{1 + a_1 t + a_2 t^2 + \cdots + a_n t^n}{1 + b_1 t + b_2 t^2 + \cdots + b_k t^k}
$$

and $h_0 = 1$, $h_{<0} = 0$.

Proof. Let $y = 1 + y_1 t + y_2 t^2 + \dots$ be the Taylor series of $1/q$. Now, multiply the $r \times r$ (here $r = n + k$) Sylvester matrix[9] S — whose determinant is the resultant $R(q,p)$ — from right by the following M matrix:

$$
\underbrace{\left[\begin{array}{cccccccccc}
1 & a_1 & \cdots & a_{k-1} & \cdots & a_n & & & & \\
 & 1 & a_1 & \cdots & a_{k-1} & \cdots & a_n & & & \\
 & & \ddots & \ddots & \vdots & \ddots & & \ddots & & \\
 & & & 1 & a_1 & \cdots & a_{k-1} & \cdots & a_n & \\
\hline
1 & b_1 & b_2 & \cdots & b_k & & & & & \\
 & 1 & b_1 & b_2 & \cdots & b_k & & & & \\
 & & \ddots & \ddots & \ddots & & \ddots & & & \\
 & & & 1 & b_1 & b_2 & \cdots & b_k & & \\
 & & & & 1 & b_1 & b_2 & \cdots & b_k &
\end{array}\right]}_{S}
\cdot
\underbrace{\left[\begin{array}{cccc|ccccc}
y_n & y_{n+1} & \cdots & y_{r-1} & 1 & & & & \\
\vdots & y_n & \cdots & y_{r-2} & & 1 & & & \\
y_{k-1} & \vdots & \ddots & \vdots & & & 1 & & \\
\vdots & y_{k-1} & & y_n & & & & \ddots & \\
y_1 & \vdots & \ddots & \vdots & & & & & 1 \\
\hline
1 & y_1 & \cdots & y_{k-1} & & & & & \\
 & 1 & \ddots & \vdots & & & & & \\
 & & \ddots & y_1 & & & & & \\
 & & & 1 & & & & &
\end{array}\right]}_{M}
$$

[9] This is probably not the standard form of the Sylvester matrix, since the coefficients are usually in the inverse order; in that case, the determinant is $R(p,q)$ instead of $R(q,p)$.

M is "almost" a block-upper-triangle matrix, from which $\det(M) = (-1)^{nk}$ easily follows, so $(-1)^{nk}R(q,p) = R(p,q) = \det(SM)$. On the other hand, easy calculation shows that SM equals

$$\left[\begin{array}{cccc|ccccc} h_n & h_{n+1} & \cdots & h_{r-1} & & & & & \\ h_{n-1} & h_n & \cdots & h_{r-2} & & & & & \\ \vdots & \vdots & \ddots & \vdots & & & * & & \\ h_{s+1} & h_{s+2} & \cdots & h_n & & & & & \\ \hline & & & & 1 & * & * & \cdots & * \\ & & & & & 1 & * & \cdots & * \\ & & & & & & 1 & & * \\ & & & & & & & \ddots & \vdots \\ & & & & & & & & 1 \end{array}\right]$$

where $s = n-k$. But the determinant of this matrix obviously equals to its upper left $k \times k$ minor, what is we wanted to prove. $\square$

LEMMA A.3. *The resultant $R(p,q)$ of the polynomials*

$$p = \prod_{i=1}^{n}(1+\alpha_i t) = \sum_{i=0}^{n} \sigma_i(\alpha_1,\dots,\alpha_n)t^i$$
$$q = \prod_{j=1}^{k}(1+\beta_j t) = \sum_{j=0}^{k} \sigma_j(\beta_1,\dots,\beta_k)t^j$$

is $R(p,q) = \prod(\alpha_i - \beta_j)$.

Proof. The roots of the polynomials are $-\frac{1}{\alpha_i}$ and $-\frac{1}{\beta_j}$, respectively, and the top coefficients are $p_n = \prod \alpha_i$ resp. $q_k = \prod \beta_j$; thus from the definition (A.1) of the resultant we have

$$R(p,q) = p_n^k \cdot q_k^n \cdot \prod_{i,j}\Big(\frac{1}{\beta_j} - \frac{1}{\alpha_i}\Big) = \Big(\prod_i \alpha_i\Big)^k \cdot \Big(\prod_j \beta_j\Big)^n \cdot \prod_{i,j} \frac{\alpha_i - \beta_j}{\alpha_i \beta_j} = \prod_{i,j}(\alpha_i - \beta_j)$$

which completes the proof. $\square$

Recall that the symbol $\Delta_{\lambda_1,\dots,\lambda_k}(c)$ denotes the determinant $\det(c_{\lambda_i+j-i})_{k\times k}$. These are usually called *Schur polynomials*, in the context when c_i is the i-th elementary symmetric polynomial in some variables $\alpha_1, \alpha_2, \dots$ There are also other well-known formulas for these polynomials, for example the following. For a *strict* partition $\mu = (\mu_1 > \mu_2 > \cdots > \mu_k)$, let a_μ denote the antisymmetric polynomial

$$a_\mu(\alpha) = \det(\alpha_j^{\mu_i})_{k\times k} = \begin{vmatrix} \alpha_1^{\mu_1} & \alpha_2^{\mu_1} & \cdots & \alpha_k^{\mu_1} \\ \alpha_1^{\mu_2} & \alpha_2^{\mu_2} & \cdots & \alpha_k^{\mu_2} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{\mu_k} & \alpha_2^{\mu_k} & \cdots & \alpha_k^{\mu_k} \end{vmatrix}$$

Then we have the following identity (for a proof, and more information on Schur polynomials, see [**Mac99**])

$$\Delta_\lambda(c) = \frac{a_{\delta+\lambda}(\alpha)}{a_\delta(\alpha)}$$

where $c_i = \sigma_i(\alpha_1,\dots,\alpha_k)$ and δ is the partition $\delta = (k-1, k-2, \dots, 1, 0)$.

LEMMA A.4. *Let c_i be the i-th elementary symmetric polynomial in the variables $\alpha_1, \ldots, \alpha_n$; then we have*

$$\prod_{i<j}(\alpha_i + \alpha_j) = \Delta_{n-1,n-2,\ldots,1}(c)$$
$$\prod_{i\leq j}(\alpha_i + \alpha_j) = 2^n \Delta_{n,n-1,\ldots,1}(c)$$

Proof. The first equality can be readily seen from the above alternative definition of Δ_λ. Since $\Delta_{n-1,n-2,\ldots,1} = \Delta_{n-1,n-2,\ldots,1,0}$, we have $\lambda = \delta$ and then a_δ and $a_{2\delta}$ are Vandermonde determinants for the variables $\alpha_1, \ldots, \alpha_n$ and $\alpha_1^2, \ldots, \alpha_n^2$, respectively; thus

$$\Delta_\delta = \frac{a_{2\delta}}{a_\delta} = \frac{\prod_{i<j}(\alpha_i^2 - \alpha_j^2)}{\prod_{i<j}(\alpha_i - \alpha_j)} = \prod_{i<j}(\alpha_i + \alpha_j)$$

The second equality follows trivially from the first one. □

LEMMA A.5 (see also [**HT84**]). *The determinant of the $(d+1) \times (d+1)$ matrix*

$$M_d = \begin{bmatrix} \binom{n}{a_0} & \binom{n}{a_0-1} & \cdots & \binom{n}{a_0-d} \\ \binom{n}{a_1} & \binom{n}{a_1-1} & \cdots & \binom{n}{a_1-d} \\ \vdots & \vdots & & \vdots \\ \binom{n}{a_d} & \binom{n}{a_d-1} & \cdots & \binom{n}{a_d-d} \end{bmatrix}$$

where $d \leq a_0, \ldots, a_d \leq n + d$ are integers is

$$\det(M_d) = \prod_{i=0}^{d} \frac{(n+i)!}{a_i!\,(n+d-a_i)!} \cdot \prod_{i<j}(a_j - a_i)$$

Proof. Induction on d. For $d = 0$, the proposition clearly holds. For $d > 0$, expanding the determinant by the rightmost column and using the proposition for $d - 1$ yields

$$\det(M_d) = \sum_{k=0}^{d}(-1)^{d+k}\binom{n}{a_k - d}\frac{\prod_{i=0}^{d-1}(n+i)! \cdot \prod_{i,j\neq k}(a_j - a_i)}{\prod_{i\neq k} a_i!\,(n+d-1-a_i)!} =$$

$$= \prod_{i=0}^{d} \frac{(n+i)!}{a_i!\,(n+d-a_i)!} \underbrace{\sum_{k=0}^{d} \frac{(-1)^{d+k} a_k!\,n!}{(a_k - d)!\,(n+d)!} \prod_{i\neq k}(n+d-a_i) \prod_{i,j\neq k}(a_j - a_i)}_{X}$$

Now, observe that the braced subexpression equals to

$$X = \frac{n!}{(n+d)!} \cdot \det(A)$$

where the matrix A is defined by

$$A_{ij} = \begin{cases} a_i^j \cdot (n+d-a_i) & 0 \leq j < d \\ a_i!\,/\,(a_i - d)! & j = d \end{cases}$$

(this can be seen by expanding X by its rightmost column). Introduce the notation $\mathcal{A}_j$ for the column A_{*j}. As $a_i^j \cdot (n+d-a_i) = (n+d)a_i^j - a_i^{j+1}$, by adding the linear combination of columns

$$(n+d)\mathcal{A}_{d-2} + (n+d)^2\mathcal{A}_{d-3} + \cdots + (n+d)^{d-2}\mathcal{A}_0$$

to the column $\mathcal{A}_{d-1}$, then

$$(n+d)\mathcal{A}_{d-3} + (n+d)^2\mathcal{A}_{d-4} + \cdots + (n+d)^{d-3}\mathcal{A}_0$$

to the column $\mathcal{A}_{d-2}$, etc... we get that

$$\det(A) = \begin{vmatrix} (n+d) - a_0 & (n+d)^2 - a_0^2 & \cdots & (n+d)^d - a_0^d & a_0!/(a_0-d)! \\ (n+d) - a_1 & (n+d)^2 - a_1^2 & \cdots & (n+d)^d - a_1^d & a_1!/(a_1-d)! \\ \vdots & \vdots & & \vdots & \vdots \\ (n+d) - a_d & (n+d)^2 - a_d^2 & \cdots & (n+d)^d - a_d^d & a_d!/(a_d-d)! \end{vmatrix}$$

Since $a!/(a-d)! = a(a-1)\cdots(a-d+1) = a^d + c_{d-1}a^{d-1} + \cdots + c_2a^2 + c_1a$ (by the way, $c_k = \sigma_{d-k}(1, 2, \ldots, d-1)$ where σ_k denotes the k-th elementary symmetric polynomial in $d-1$ variables), it is clear that by adding $\sum c_{j+1}\mathcal{A}_j$ to $\mathcal{A}_d$ we can replace the rightmost column to the constant $(n+d)!/n!$; thus after clearing the terms $(n-d)^j$ — which of course don't change the determinant — we get that X equals a Vandermonde determinant: $X = \prod(a_j - a_i)$ which is we wanted to prove. $\square$

Notations

We have tried to create a consequent and illuminating notation system for this work, which turned out to be a quite hard task...[10]

The reasons are numerous: First of all, the number of characters in different alphabets is finite; the second most important being the swamp of bad or simply not-so-good but wide-spread conventions (starting with that of Newton, that we write functions to the left side of their arguments — and because of that, we have to read compositions of maps from the right to the left). For example the universal principal G-bundle should be denoted by $E_G \to B_G$ instead of $EG \to BG$ to be consistent with the shorthand notation that we omit the one-point space from our formulas; but the latter one is so wide-spread that we resisted the temptation to use the first.

Nevertheless we hope that the text remains readable, at least with the help of this little table of notations.

Sets and maps

$\mathbb{N}, \mathbb{Z}$	the semigroup (resp. ring) of natural (resp. integer) numbers
$\mathbb{Q}, \mathbb{R}, \mathbb{C}$	the fields of rational, real and complex numbers
$A \cup B$, $A \cap B$	union / intersection of the sets A and B
$A - B$	difference of the sets A and B, ie. $\{a \in A : a \notin B\}$
$A \pm x$	translation of the set A with $\pm x$, ie. $\{a \pm x : a \in A\}$
$f\restriction_A$	restriction of the map f to the set A
V^*	dual space of the vector space V

Groups, Lie groups and Lie algebras

$\mathfrak{S}_n, \mathfrak{A}_n$	the n-th symmetric / alternating group
$G, H, \mathfrak{g}, \mathfrak{h}$	Lie groups and the corresponding Lie algebras
$T, \mathfrak{t}$	torus ($T \cong \mathrm{U}_1 \times \cdots \times \mathrm{U}_1$) and the corresp. Lie algebra
$\mathfrak{t}^*$	dual space of the Lie algebra $\mathfrak{t}$
$\exp : \mathfrak{g} \to G$	exponential map
GL_n, SL_n, O_n, Sp_{2n}	the classical complex matrix groups...
$\mathfrak{gl}_n$, $\mathfrak{sl}_n$, $\mathfrak{o}_n$, $\mathfrak{sp}_{2n}$	...and the corresponding Lie algebras
U_n, SU_n, $\mathfrak{u}_n$, $\mathfrak{su}_n$	the unitary groups and the corresp. Lie algebras
Mat_n, $\mathrm{Mat}_{n\times m}$	the space of $n \times n$ / $n \times m$ complex matrices
I_n	the $n \times n$ identity matrix

Algebraic Geometry

$\mathbb{A}^n, \mathbb{P}^n$	affine / projective space of dimension n
cX	cone of the aff. variety $X \subset V$; $cX = \{(tx, t) \in V \times \mathbb{C} : x \in X, t \in \mathbb{C}\}$
CX	cone of the proj. variety $X \subset \mathbb{P}V$; $CX = \cup\{L = \mathbb{C}x \subset V : L \in X\}$
$\mathbb{P}X$	projectivization of the aff. variety $X \subset V$; $\mathbb{P}X = \{L \in \mathbb{P}V : L \subset X\}$
$\deg(X)$	degree of the projective variety X
$\mathcal{V}_{n,k}$	Veronese variety, $\mathcal{V}_{n,k} \subset \mathbb{P}(\mathrm{Sym}^n \mathbb{C}^{k+1})$
$\mathrm{Gr}_n V$	Grassmannian: the variety of n-planes in V, subvariety of $\mathbb{P}(\wedge^k V)$

[10]...and we failed at this point probably.

Representations

$\rho : G \to \mathrm{GL}(V)$	representation of the (Lie) group G
$\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$	the adjoint repr. of G
$\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$	the adjoint repr. of $\mathfrak{g}$
V, W	representations of a Lie group (left $\mathbb{C}G$-modules, if G is the group)
$V \oplus W$, $V \otimes W$	direct sum / tensor product of representations
std_n	the standard representation of GL_n on $\mathbb{C}^n$
$\mathrm{Sym}^k V$, $\wedge^k V$	k-th symmetric / antisymmetric power of a representation
$\mathrm{Sym}^* V$, $\wedge^* V$	$\oplus_{i \geq 0} \mathrm{Sym}^i V$ / $\oplus_{i=0}^n \wedge^i V$ where $n = \dim(V)$
$\Gamma \subset \mathfrak{t}^*$	the weight lattice
$\mathcal{W}$, $\mathcal{X}$	set of weights / distinguished weight vectors of a representation
$w_x \in \mathcal{W}$	the weight corresponding to a weight vector $x \in \mathcal{X}$
$\mathcal{H} \subset \mathcal{W}$	the set of highest weights
$\mathfrak{W} = N_G T / T$	Weyl group
η	invariant affine subvariety

Bundles and homology

ξ, ζ	vector bundles
EG	a contractible space with a free (right) G-action
BG	the classifying space of G, $BG = EG/G$
$G \to EG \to BG$	the universal principal G-bundle
$E \times_G F$	'twisted product': $(E \times F)/G$
$B_\rho V = EG \times_\rho V$	the universal principal ρ-bundle, associated to the representation $\rho : G \to \mathrm{GL}(V)$; sometimes denoted by $B_G V$
$\varinjlim G_i$, $\varprojlim G_i$	direct / inverse limit of groups
$H_k X$, $H^k X$	k-th singular homology and cohomology groups of X with real coefficients
$H^*(X; R)$	cohomology ring of X with coefficients in the ring R
$H^*(X)$, $H^*(X, A)$	absolute resp. relative cohomology ring of X with real coefficients
$H^*(X \vert A)$	the relative cohomology $H^*(X, X - A)$
$\mu_{A,X}$	refined class of a subvariety $A^k \subset X^{k+c}$; $\mu_{A,X} \in H^c(X \vert A)$
$[A]_X$	dual class of a subvariety; $[A]_X \in H^c(X)$
pt	the one-point space (equipped with the trivial G-action)
$H_G^* = H^*(BG)$	G-equivariant cohomology ring of the one-point space
$H_G^* X = H^*(B_G X)$	G-equivariant cohomology ring of the G-space X

Miscellaneous

$g \circ f$, gf	composition of maps; $(g \circ f)(x) = (gf)(x) = g(f(x))$
$R[x_1, \ldots, x_k]$	polynomial ring over the coefficient ring R
$\sigma_i(x_1, \ldots, x_k)$	the i-th elementary symmetrical polynomial
$s_\lambda = \Delta_\lambda(c)$	Schur polynomial corresponding to the partition λ
$R(p, q)$	resultant of two univariate polynomials
$\langle x_1, \ldots, x_k \rangle$	the linear subspace generated by the vectors $x_1, \ldots x_k$

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