Weak conflict-free colorings of point sets and simple regions

Balázs Keszegh*

Abstract

In this paper we consider the *weak conflict-free colorings* of regions and points. This is a natural relaxation of *conflict-free coloring* [ELRS03]. One of the most interesting type of regions to consider for this problem is that of the *axis-parallel rectangles*. We completely solve the problem for a special case of them, for *bottomless rectangles*. We also give an almost complete answer for *half-planes* and pose several open problems. Moreover we give efficient algorithms for coloring with the needed number of colors.

1 Introduction

Motivated by a frequency assignment problem in cellular telephone networks, Even, Lotker, Ron and Smorodinsky [ELRS03] studied the following problem. Cellular networks facilitate communication between fixed *base stations* and moving *clients*. Fixed frequencies are assigned to base-stations to enable links to clients. Each client continuously scans frequencies in search of a base-station within its range with good reception. The fundamental problem of frequency assignment in cellular networks is to assign frequencies to base-stations such that every client is served by some base-station, i.e. it lies within the range of the station and no other station within its reception range has the same frequency. Given a fixed set of base-stations we want to minimize the number of assigned frequencies.

*Department of Mathematics and its Applications, Central European University, Budapest, Hungary, `tphkeb01@phd.ceu.hu` This work was done while visiting the School of Computing Science at Simon Fraser University.

First we assume that the ranges are determined by the clients, i.e. if a base-station is in the range of some client, then they can communicate. Let P be the set of base-stations and $\mathcal{F}$ the set of all possible ranges of any client. Given some set $\mathcal{F}$ of planar regions and a finite set of points P we define $cf(\mathcal{F}, P)$ as the smallest number of colors which are enough to color the points of P such that in every region of $\mathcal{F}$ containing at least one point, there is a point whose color is unique among the points in that region. The maximum over all point sets of size n is the so called **conflict-free coloring number** (cf-coloring in short), denoted by $cf(\mathcal{F}, n)$. Determining the cf-coloring number for different types of regions $\mathcal{F}$ is the main aim in this topic. Regions for which the problems has been studied include circles ([ELRS03], [PTT07], [Sm06], etc.) and axis-parallel rectangles ([ChSzPT], [PT03], [AEGR07], etc), for a brief list of some results see Chapter 4.
It is also a natural case to assume that the ranges are determined by the base-stations, i.e. if a client is in the range of some base-station, then they can communicate. For a *finite* set of planar regions $\mathcal{F}$ we define $\overline{cf}(\mathcal{F})$ as the smallest number of colors which is enough for coloring the regions of $\mathcal{F}$ such that for every point in $\cup\mathcal{F}$ there is a region whose color is unique among the colors of the regions covering it. For a (not necessarily finite) set $\mathcal{F}$ of planar regions let $\overline{cf}(\mathcal{F}, n)$, the **conflict-free region-coloring number of** $\mathcal{F}$ be the maximum of $\overline{cf}(\mathcal{F}')$ for $\mathcal{F}' \subseteq \mathcal{F}$, $|\mathcal{F}'| = n$.
In [Sm03] and then in [HS05] generalized versions of these notions are defined. A cf_k*-coloring* (they used the notation k-CF) of a point set is a coloring such that for each region F of $\mathcal{F}$ containing at least one point, there is a color which is assigned to at most k points covered by F. Note that a cf-coloring is actually a cf_1-coloring. The region-coloring version is defined similarly.

Modifying the definition of a conflict-free coloring, $wcf(\mathcal{F}, P)$ equals to the minimum number of colors needed to color the points of P such that whenever a region covers at least 2 of them, then there are 2 points with different colors covered by it (the region is *multicolored*). The maximum over all n element point set of size n is the **weak conflict-free coloring number** (wcf-coloring in short), denoted by $wcf(\mathcal{F}, n)$.

Observation 1. $cf(\mathcal{F}, n) \geq wcf(\mathcal{F}, n)$.

Generalizing our definition we can define $wcf_k(\mathcal{F}, P)$ as the minimum number of colors needed to color the points of P such that whenever a region covers at least k of them, then there are 2 points with different colors covered by it. The maximum of this value over all point sets of size n is denoted by $wcf_k(\mathcal{F}, n)$.

Observation 2. *$wcf(\mathcal{F}, n) = wcf_2(\mathcal{F}, n)$.*
$wcf_k(\mathcal{F}, n) \leq wcf_l(\mathcal{F}, n)$ if $k \geq l$.

A simple corollary of a theorem of [Sm06] (in [ELRS03] they already present a very similar version of the following algorithm) shows that the weak conflict-free coloring number gives a good upper bound to the conflict-free coloring number. More precisely, they present the following algorithm and prove that it gives a conflict-free coloring. In each step take a biggest color class in a weak conflict-free coloring of the point set. After coloring it to a new color, delete it and do the same for the new (smaller) point set. This algorithm gives the following bounds, stated in [HS05] in a slightly different way.

Lemma 3. *[HS05]*
(i) If $wcf(\mathcal{F}, n) \leq c$ for some constant c, then $cf(\mathcal{F}, n) \leq \frac{\log n}{\log(c/(c-1))} = O(\log n)$,
(ii) if $wcf(\mathcal{F}, n) = O(n^{\epsilon})$ for some $\epsilon > 0$, then $cf(\mathcal{F}, n) = O(n^{\epsilon})$.

Observation 1 and Lemma 3 show that wcf and cf are usually close to each other. Often the best known bound for cf is obtained from Lemma 3. This is the main motivation why we want to determine the weak conflict-free coloring number for different types of regions. Moreover, Lemma 3 holds even in the more general setting when we consider wcf_k-colorings and cf_{k-1}-colorings for some fixed k (the lemma considers the case $k = 2$). This motivates the investigation of wcf_k-colorings for $k > 2$.

Again we can define the dual version. For a finite $\mathcal{F}$, $\overline{wcf}(\mathcal{F})$ equals to the smallest number of colors which are enough to assign colors to the regions in $\mathcal{F}$ such that for every point covered by at least 2 regions in $\mathcal{F}$, there are two differently colored regions among the regions covering it (the point is *multicolored*). For a not necessarily finite $\mathcal{F}$ the maximum of this value over all n element subsets of $\mathcal{F}$ is the **weak conflict-free region-coloring number**, denoted by $\overline{wcf}(\mathcal{F}, n)$. Finally, we can again define $\overline{wcf_k}(\mathcal{F}, n)$ by restricting the condition only for points covered by at least k regions in $\mathcal{F}$. The dual version of Lemma 3 holds as well.

For a finite $\mathcal{F}$ following the notation of [Sm06] one can define the *geometric hypergraph* induced by $\mathcal{F}$. The base set is the set of regions F and for any point p covered by at least k regions there is a hyperedge r_p containing the regions covering p. A proper coloring of a hypergraph is a coloring of the points such that there are no monochromatic edges. Thus, determining the chromatic number of this hypergraph is clearly equivalent to determining $\overline{wcf_k}(\mathcal{F})$.

Following the notation of [PTT07] a partial *k-fold* covering of the plane with a set of regions $\mathcal{F}$ is *decomposable* if we can partition the set into two subsets such that for any point covered by $\mathcal{F}$ at least k times, there is a region in each part covering this point. Clearly, $\overline{wcf_k(\mathcal{F})} = 2$ is equivalent to this as a good weak conflict-free region-coloring of $\mathcal{F}$ gives a good partition and vice versa.

For the types of regions we study, the weak conflict-free coloring number can always be bounded from above by a constant not depending on n. Thus, we define $wcf_k(\mathcal{F}) = \max_n wcf_k(\mathcal{F}, n)$ and similarly $\overline{wcf_k}(\mathcal{F}) = \max_n \overline{wcf_k}(\mathcal{F}, n)$ if they exist. Our main aim is to determine these numbers and give coloring algorithms using this minimal number of colors.

Slightly modifying the notation of [HS05] we call a set of regions $\mathcal{F}$ **monotone** if for any finite P and $F \in \mathcal{F}$ and l positive integer if F covers at least l points of F then there exists $F' \in \mathcal{F}$, $F' \subseteq F$ covering exactly l points of P.

Observation 4. *For monotone set of regions in the definition of wcf-coloring it is enough to restrict our condition to regions covering exactly* 2 *points of the point set. Similarly, for the definition of* wcf_k *it is enough to restrict the condition to regions covering exactly* k *points.*

Note that monotonicity could be defined in the dual version as well, but none of the types of regions we study in this paper are monotone in that dual sense.

The paper is structured as follows. In Section 2 we prove theorems solving all cases for *bottomless rectangles*, a special case of axis-parallel rectangles, for the definition we refer to that section. In Section 3 we prove theorems which give an almost complete answer for *half-planes*. Finally, in Section 4 we summarize our results and pose several open problems.

2 Bottomless rectangles

A **bottomless rectangle** the set of points $(x, y)|a < x < b, y < c$ for some a, b and c. The set of all bottomless rectangles is denoted by $\mathcal{B}$. For our coloring purposes the family of bottomless rectangles is equivalent with the family of (ordinary) axis-parallel rectangles having their lower edge on a common horizontal *base-line*. We prove exact bounds for $wcf_k(\mathcal{B})$ and $\overline{wcf_k}(\mathcal{B})$ for all k.

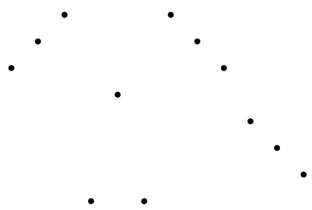
2.1 Coloring points

From now on we assume that there are no two points with the same x or y coordinate. It is easy to show that if this is not the case, then coloring the point set after a small perturbation gives a needed coloring for the original point set as well. In this section *upwards order* means the ordering of points according to their y coordinate starting with the point having the smallest y coordinate (the *lowest* point).

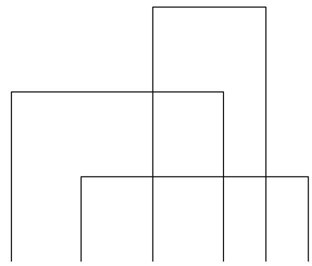
The proof of the following, rather trivial result is just presented for the sake of completeness.

Claim 5. *(folklore)* $wcf_2(\mathcal{B}) = 3$ *i.e. any set of points can be colored with* 3 *colors such that any bottomless rectangle covering at least* 2 *of them is multicolored.*

Proof. First we prove that $wcf_2(\mathcal{B}) \leq 3$. We want to color the points with 3 colors such that any bottomless rectangle covering at least 2 points covers two differently colored points.
First we color the lowest point of P arbitrarily with one of the three colors then we color the points one by one in upwards order. In each step we color the next point p with a color maintaining that in the x-coordinate order of the points already colored there are no two consecutive points with the same color.
In this way any bottomless rectangle B covering at least two points covers two differently colored ones. Indeed, when the highest point p in $P \cap B$ is considered, $B \cap P$ is an interval in the left to right order of the points considered so far. By the property maintained any such interval contains points of at least two colors.
The lower bound $wcf_2(\mathcal{B}) \geq 3$ follows from the fact that for example the



(a) Theorem 1(i)



(b) Theorem 2(i)

Figure 1: Lower bound constructions for bottomless rectangles

points with coordinates (0,0), (1,1) and (2,0) cannot be colored with 2 colors in a proper way. □

The following theorem shows that the smallest k for which $wcf_k(\mathcal{B}) = 2$ is 4 and so $wcf_k(\mathcal{B})$ is determined for every k as trivially $wcf_k(\mathcal{B}) \geq 2$ for any k.

Theorem 1.

(i) $wcf_3(\mathcal{B}) = 3$.

(ii) $wcf_4(\mathcal{B}) = 2$ *i.e. any set of points can be colored with* 2 *colors such that any bottomless rectangle covering at least* 4 *of them is multicolored.*

(iii) Such colorings can be found in $O(n \log n)$ *time.*

Proof. (i) Using Observation 2 with Claim 5 we got that $wcf_3(\mathcal{B}) \leq wcf_2(\mathcal{B}) = 3$. Thus, it is enough to prove that $wcf_3(\mathcal{B}) > 2$. For that we show that the 12 point construction on Figure 1(a) cannot be colored with 2 colors such that any bottomless rectangle covering at least 3 points covers two differently colored points. Suppose on the contrary that there is such a coloring. Denote the points ordered by their x coordinate from left to right by $p_1, p_2, \ldots, p_{12}$. Among the points p_4, p_5, p_6 there are two with the same color, wlog. assume that this color is red. If p_4 and p_5 are red, then all of p_1, p_2, p_3 are blue as there is a bottomless rectangle covering only p_4, p_5 and any one of these 3 points. This is a contradiction as there is a bottomless rectangle covering only these 3 points, all blue. If p_4 and p_6 are red then similar argument for the points p_{10}, p_{11}, p_{12}, if p_5 and p_6 are red then similar argument for the points p_7, p_8, p_9 yields to a contradiction.

(*ii*) We want to color the points red and blue such that any bottomless rectangle covering at least 4 points covers two differently colored points. First we color the lowest point of P red then we consider the points in upwards order. We do not color every vertex as soon as it is considered. We maintain that in the x-coordinate order of the points considered so far there are no two consecutive uncolored points and the colored points alternate in color. When a new point is considered we keep it uncolored unless it has an uncolored left or right neighbor in the x-coordinate order. In that case we color both in a way keeping the alternation. At the end we arbitrarily color the remaining points in P. Now we only need to prove that this coloring is good. We need to consider bottomless rectangles covering at least 4 points, so let B be such. Let p be the highest point covered by B. When p is considered $B \cap P$ is an interval in the left to right order of the points considered so far. By the properties maintained any such interval of at least 4 vertices contains both red and blue points as needed.

(*iii*) We need to prove that the algorithms presented in the proof of Claim 5 and Theorem 1 (*ii*) run in time $O(n \log n)$. Computing the upwards order of the points takes $O(n \log n)$ time, the rest of the algorithm has n steps in both cases, each computable in $O(\log n)$ time, in the latter algorithm there is a final coloring step that takes at most linear time, so the whole algorithm runs in $O(n \log n)$ time in both case. □

2.2 Coloring bottomless rectangles

In [Sm06] a very similar version is considered, namely coloring axis-parallel rectangles intersecting a common base-line. The proof of their result with a slight modification gives $\overline{wcf_2}(\mathcal{B}) \leq 4$. The following theorem determines $\overline{wcf_k}(\mathcal{B})$ for every k, also improving this bound to 3 colors, which is optimal.

From now on we assume that there are no two bottomless rectangles with overlapping sides. It is easy to show that if this is not the case, then coloring the rectangles after enlarging all of them with a very small amount such that afterwards there are no overlappings, gives a needed coloring for the original set of rectangles as well.

Theorem 2.

(i) $\overline{wcf_2}(\mathcal{B}) = 3$ *i.e. any set of bottomless rectangles can be colored with* 3 *colors such that any point covered by at least* 2 *of them is multicolored.*

(ii) $\overline{wcf_3}(\mathcal{B}) = 2$ *i.e. any set of bottomless rectangles can be colored with* 2 *colors such that any point covered by at least* 3 *of them is multicolored.*

(iii) Such colorings can be found in $O(n^2)$ *time.*

Proof. (i) For the lower bound, the arrangement of 3 rectangles on Figure 1(b) shows that 3 colors are sometimes needed. For the upper bound, given a set of rectangles with a common base line we want to color the rectangles red, blue and green such that any point covered by at least 2 rectangles is covered by two differently colored rectangles. We color the rectangles in downwards order according to their top edge's y-coordinate. We start with the empty set and reinsert the rectangles in this order. We color the first, i.e. the highest rectangle blue. After each step we have a proper coloring and we preserve the following additional assumption. *If a point on the base-line is covered by exactly* 1 *rectangle, then it is not red.*

In each step we insert the next rectangle B in downwards order, so it's top edge is below the top edge of all the rectangles already inserted. We color B red. We claim that this is again a proper coloring. Indeed, the condition for points not in B already holds. For any point covered by B the base-line point with the same x-coordinate is covered by the same rectangles as B is the lowest rectangle. Thus, it is enough to check the condition for base-line points in B. If a point is covered by at least 2 rectangles besides B then it is good by induction. Otherwise it is covered by B, which is red and exactly one more rectangle, which is not red by the assumption.

If there is no base-line point covered by only B, then the additional assumption holds too. If q is such a point then we need to do something else to maintain the validity of our assumption. If a base-line point is covered by only 1 rectangle then we say that the color of this point is the color of the rectangle covering it. It is easy to see that if there is such a point p and we switch the other two colors on the rectangles completely to the left (or to the right) to p, the coloring remains good. With only such 'divide and color' steps we will change the coloring such that there will be no point on the base-line covered by exactly 1 green rectangle. Finally we will switch the colors green and red on all the rectangles to have a good coloring satisfying the assumption. For an illustration of the rest of the proof see Figure 2.

In the current coloring all green base-line points are left or right to B as B is red. We will deal with the left side first, changing the colors only of rectangles strictly left from q and making a good coloring satisfying the condition for any base-line point left to q. For the right side we will do the mirrored

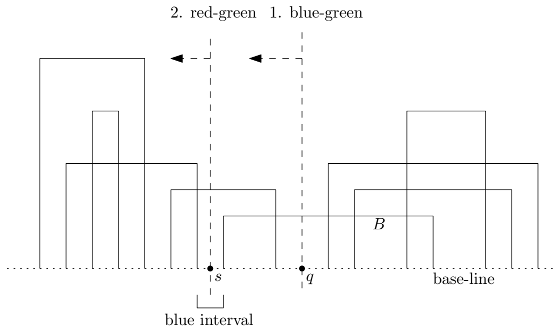


Figure 2: The color switches of the 'divide and color' method in Theorem 2(i).

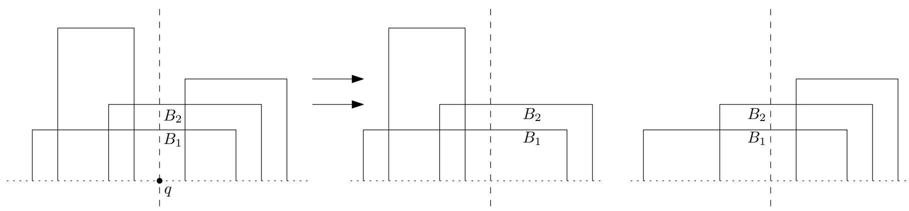


Figure 3: The division of the 'divide and color' method in Theorem 2(ii).

version of this algorithm changing the colors only of rectangles strictly right from q and making a good coloring satisfying the condition for any base-line point right to q. This way we get a good coloring satisfying the assumption for all base-line points.

On the base-line left to B there are some intervals of single colored points, all of them green or blue. If there is no green one then we are done. Otherwise we can suppose that the closest such interval to B is blue, otherwise switch colors blue and green on the rectangles strictly left from q, still having a good coloring. Now switch colors red and green on the rectangles strictly left from any point s of this blue interval, this way we got rid of all green points, making the assumption true for all points left to B.

(ii) Given a set of rectangles with a common base line we want to color them red and blue such that any point covered by at least 3 rectangles is covered by two differently colored rectangles. Moreover, the coloring we got will always satisfy the following additional assumption. *Any point on the base-line covered by exactly* 2 *rectangles is covered by a red and a blue rectangle.*

Now we need a different version of the 'divide and color' method. We will proceed by induction on the number of rectangles. A single rectangle is colored red. In a general step first assume that there is a point q on the base-line not covered by any rectangle and there are some rectangles strictly left and strictly right from that point too. Color the rectangles to the left of q and the ones to the right of q separately by induction, putting these together this is clearly a good coloring.

Now assume that there is a point q on the base-line covered by exactly 1 rectangle B and there are some rectangles strictly left and strictly right from that point too. Color first the rectangles to the left of q together with B then the rectangles to the right together with B. As there were some rectangles on both sides, this can be done by induction. By a possible switching of the colors in the left and right parts, B is red in both colorings. Putting together the two half-colorings we get a good coloring.

The next case is when we have a point q on the base-line covered by exactly two rectangles, B_1 and B_2 (see Figure 3 for an illustration). In this case color first by induction the rectangles strictly left from q together with these two rectangles. Using the assumption on q we see that B_1 and B_2 have different colors, after a possible switch of the two colors we can assume that B_1 is red and B_2 is blue. The same way we color the rectangles strictly right from q together with these two rectangles. This way the two rectangles are colored with the same colors in both colorings and so we can put together these two half-colorings (if none of them is empty) to have a coloring of the whole set of rectangles. This coloring is good by induction.

In the remaining case for any base-line point covered by exactly 1 or 2 rectangles, there is no rectangle strictly to the left or to the right to that point. The left and right sides of the rectangles divide the base-line into 2 half-lines and $2n-1$ intervals. It is easy to see that in this case the only base-line points covered by exactly 1 rectangle are the points of the leftmost L_1 and rightmost R_1 interval and the 2-covered points are the points of the second leftmost L_2 and second rightmost R_2 interval. Consider rectangle B, the one with the lowest top edge. It is easy to see that if it does not cover 1- or 2-covered base-line points then we can color the rest of the rectangles by induction and then color B with arbitrary color not ruining the coloring and the additional assumption. Otherwise B covers some intervals from L_1, L_2, R_1, R_2.

If it covers only some of L_1 and L_2 then color the rest of the rectangles by induction and then color B to a different color from the other rectangle covering L_2, this way we obtain a good coloring satisfying our additional assumption. If it covers only some of R_1 and R_2 then a symmetrical argument gives a good coloring.

It remains to deal with the case when B covers L_2 and R_2 as well. Consider now the rectangle B_2 with the second lowest top edge. First assume that B_2 does not cover any of L_1, L_2, R_1, R_2. In this case color the rest of the rectangles (including B) by induction and then color B_2 with the same color as B. We claim that this coloring is good. Any point ruining the condition must be in B_2. Assume on the contrary that there is a point p covered by at least 3 rectangles all having the same color.

If p is covered by 3 rectangles besides B_2 then by induction there are differently colored rectangles among these, a contradiction.

If p is covered by exactly 2 rectangles besides B_2 then take the base-line point p' having the same x-coordinate. If it is covered by the same rectangles as p, then by the additional assumption it is covered by two differently colored rectangles besides B_2. This holds for p as well, a contradiction. If p' is not covered by the same rectangles as p, the only possibility is that it is covered by B too (as only B is lower then B_2). By induction this point was covered by red and blue rectangles as well without considering B_2. As B has the same color as B_2, the same holds for p, a contradiction.

The additional assumption holds as well as it is enough to check the points of L_2 and R_2 and here the coloring is good by induction.

By dealing with the case when B_2 covers some of L_2, R_2 we exhaust all possibilities. By symmetry we can assume that B_2 covers L_2 (and maybe R_2 too). In this final case delete both B and B_2 and color the rest of the rectangles by induction. Now put back these two rectangles. If R_2 is covered by some rectangle besides B and B_2 then color B differently from the color of this rectangle. Otherwise color B arbitrarily. Finally, color B_2 differently from B. Any point ruining the condition must be in B or B_2. Again, suppose there is such a point p covered by at least 3 rectangles all having the same color.

If p is covered by both B and B_2 then its a contradiction as they are differently colored. If it is covered by at least 3 rectangles besides B and B_2 then again its a contradiction by induction.

If p is covered by one of B and B_2 and only two other rectangles then the base-line point p' with the same x-coordinate was covered by exactly two rectangles in the coloring without B and B_2. Thus, these rectangles have different colors by the assumption. As B and B_2 are the lowest rectangles, the point p is covered by these differently colored rectangles as well, a con-

tradiction. The additional assumption holds as well as it is enough to check the points of L_2 and R_2 and here the coloring is clearly good.
(iii) Finding the upwards order of the rectangles takes $O(n \log n)$ time. In each step we maintain an array of the intervals of the base line. If an interval is covered only by one rectangle, we keep its color as well.
In the algorithm of (i) in each step we search for some colored interval constant times and recolor some rectangles with a given property (left from a given interval, etc.) constant times. This takes $c \cdot k$ time if we have k rectangles at that step. We have n such steps and $k \leq n$ always, so the running time is $O(n^2)$.
In the algorithm of (ii) except the last case we always do the 'divide and color' step by cutting the set into two nontrivial parts and color separately. Finding whether there is such a cut, doing the cut (and maintaining the upwards order in the two parts) and the possible recolorings after the recursional colorings take $c_1 \cdot n$ time for n rectangles. The two recursional algorithms take at most $c_0 \cdot a^2$ and $c_0 \cdot b^2$ steps where $a + b \leq n + 2$. These altogether are less then $c_0 \cdot n^2$ for some c_0 big enough (depending on c_1). When we do a recursional step by deleting B or B_2 or both we can decide which kind of step is needed and color B and B_2 in $c_2 \cdot n$ steps, and we can do the recursion in $c_0 \cdot (n-1)^2$ steps. These altogether are less then $c_0 \cdot n^2$ for some c_0 big enough (depending on c_1 and c_2). $\square$

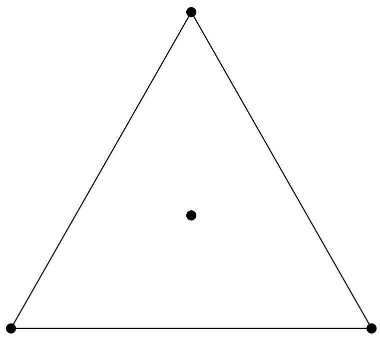
3 Half-planes

The set of all half-planes is denoted by $\mathcal{H}$. We prove exact bounds for $wcf_k(\mathcal{H})$ and almost exact bounds for $\overline{wcf_k}(\mathcal{H})$.
From now on we assume that there are no 3 points on one line. It is easy to show that if this is not the case, then coloring the point set after a small perturbation gives a needed coloring for the original point set as well. This way the vertices of the convex hull of a point set P are exactly the points of P being on the boundary of this convex hull.

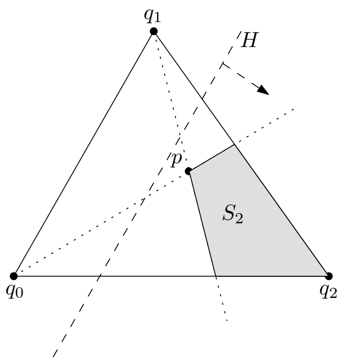
3.1 Coloring points

The following lemma follows easily from the definition of the convex hull.

Lemma 6. *Any half-plane H covering at least one point of P covers some vertex of the convex hull of P too. Moreover, the vertices of the convex hull*



(a) The exceptional case of Theorem 3(ii)



(b) The proof of Theorem 3(ii)

Figure 4: Theorem 3(ii)

of P covered by H are consecutive on the hull.

Theorem 3.

(i) $wcf_2(\mathcal{H}) = 4$ i.e. any set of points can be colored with 4 colors such that any half-plane covering at least 2 of them is multicolored, and 4 colors might be needed.

(ii) $wcf_2(\mathcal{H}, P) \leq 3$, except when P has 4 points, with one of them inside the triangle determined by the other 3 points (see Figure 4(a)), in which case $wcf_2(\mathcal{H}, P) = 4$.

(iii) $wcf_3(\mathcal{H}) = 2$ i.e. any set of points can be colored with 2 colors such that any half-plane covering at least 3 of them is multicolored.

(iv) Such colorings can be found in $O(n \log n)$ time.

Proof. (i) This follows from (ii), yet we give a short proof for the upper bound. Color the vertices of the convex hull of P with 3 colors such that there are no 2 vertices next to each other on the hull with the same color. Color all the remaining points with the 4th color. This coloring is good as by Lemma 6 any half-plane covering at least two points covers two neighboring vertices on the hull or one vertex on the hull and one point inside it.
(ii) Clearly, in the case mentioned in the lemma we need four colors to have a good coloring as any two points can be covered by some half-plane not covering the rest of the points.
As $\mathcal{H}$ is monotone, by Observation 4 it is enough to consider half-planes

covering exactly 2 points of P. We color the vertices of the convex hull with 3 colors as in (i). At this time the assumption already holds for every half-plane covering two vertices of the convex hull as by Lemma 6 they cover two neighboring ones which don't have the same color, as needed. Now we color the points inside the hull in a more clever way then in (i). Take an arbitrary point p inside the hull. The only case when the color of this inside point can ruin the coloring, is when there is a half-plane covering only this point and one vertex of the convex hull. If this can happen only with two vertices of the hull, then coloring p different from these, the coloring will be good for all half-planes covering p. Doing the same for every inside point we get a good coloring.

Denote the vertices of the convex hull by $q_0, ..., q_{k-1}$ in clockwise order (indexes are mod k). It is enough to prove that except the case mentioned in the lemma, there are no 3 such vertices on the hull corresponding to some p. For this, notice that if q_i and p can be covered by a half-plane not covering any other point, then p is inside $q_{i-1}qq_{i+1}\Delta$. It is easy to see that if the hull has more then 3 vertices, then there are no 3 such triangles having a common inner point. For the rest of the proof see Figure 4(b). If the hull has 3 vertices and p can be covered with any of these 3 vertices by some half-plane not covering any other point of P then the lines going through some q_i and p partition the triangle into 6 triangles. Denote for each vertex q_i the union of the two triangles having it as a vertex by S_i. Thus, we have three quadrangles, all of which must be empty. Indeed, for example by assumption there is some half-plane H covering p and q_2 not covering any other point of P. This half-plane always covers the quadrangle S_2 and so it must be empty. The same argument for the other two quadrangles shows that all of them are empty and so p is the only point in the triangle, which is the excluded case.

(iii) As $\mathcal{H}$ is monotone, it is enough to consider half-planes covering exactly 3 points of P. We color the points with colors red and blue. The points inside the convex hull of P are colored blue. The vertices of the convex hull of P are denoted by $q_0, \ldots, q_{k-1}$ in clockwise order. For each q_i we assign $T_i = q_{i-1}q_iq_{i+1}\Delta$, where indexes are modulo k. If T_i has some point of P inside it, then color q_i red.

If there are no nonempty T_i's then color the vertices of the convex hull with alternating colors, if its size is odd, then with the exception of two neighboring red points. If there is at least one nonempty T_i then these red points cut the boundary of the convex hull into chains. For each chain color its vertices with alternating colors, a chain with size one is colored blue.

Now we need to prove that this coloring is good. First observe that there are no 2 consecutive blue vertices on the convex hull. Take again an arbitrary half-plane H covering exactly 3 points. By Lemma 6 it covers some consecutive vertices of the convex hull of P. If it covers at least two consecutive vertices on the hull then it covers at least one red point. If it covers at least one point inside the hull, then it covers at least one blue point. If it covers three vertices of the convex hull but no points inside then it is easy to see that the triangle corresponding to the middle point in the ordering must be empty. So it belongs to some alternatingly colored chain. If any of its neighbors corresponds to the same chain, then H covers a red and a blue point too, if this point is a chain of size 1 then it is blue and its neighbors are red, again good. The only case remaining when H covers one vertex of the convex hull, q_i and two points of P in the inside of the convex hull. The latter points are blue and they must be in T_i, that is q_i is red, as needed.
(iv) The algorithm in (i) clearly works in $O(n \log n)$, the same as building the convex hull. For the other two algorithm we need the dynamic convex hull algorithm presented in [BJ02].
For the algorithm in (iii) we first compute a convex hull in $O(n \log n)$ amortized time and then we take its points one by one and do the following. Delete temporarily the convex hull vertex p, compute the new convex hull temporarily, if it has some new vertices on it, then the triangle corresponding to p is not empty. As any inner point has been added and deleted from the set of vertices of the hull at most two times and the convex hull algorithm makes a step in $O(\log n)$ amortized time, we could decide in $O(n \log n)$ time which vertices of the hull have empty triangles. After that the coloring of the vertices of the hull and the inside points takes $O(n)$ time, $O(n \log n)$ altogether.
For the algorithm in (ii) we do the same just when we temporarily delete p we assign to any additional convex hull vertex the point p, as this vertex can be cut out by a half-plane together with p. After these we simply color the vertices of the convex hull as needed and all the inner points with a color different from the color of the at most two convex hull vertex assigned to it. Altogether this is again $O(n \log n)$ time. □

Observation 7. *The algorithm in the proof of Theorem 3 (iii) gives a coloring which additionally guarantees that there are no half-planes covering exactly two points, both of them blue.*

3.2 Coloring half-planes

Theorem 4.

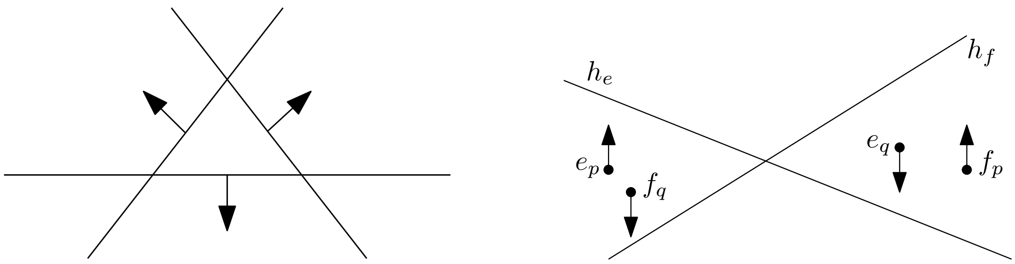
(i) $\overline{wcf_2}(\mathcal{H}) = 3$ *i.e. any set of half-planes can be colored with* 3 *colors such that any point covered by at least* 2 *of them is multicolored.*

(ii) $\overline{wcf_4}(\mathcal{H}) = 2$ *i.e. any set of half-planes can be colored with* 2 *colors such that any point covered by at least* 4 *of them is multicolored.*

(iii) Such colorings can be found in $O(n \log n)$ *time for* (ii) *and in* $O(n^2)$ *time for* (i).

Proof. We can assume that there are no half-planes with vertical boundary line. We dualize the half-planes and points of the plane S with the points (with an additional orientation) and lines of plane S', then we color the set of directed points corresponding to the half-planes which will give a good coloring of the original set of half-planes. The dualization is as follows. For a half-plane H with a boundary line given by the equality $y = ax + b$ the corresponding dual point h has coordinates (a, b). If this line is a lower boundary, then h has *orientation north*, otherwise it has orientation *south*. For an arbitrary point p with coordinates (c, d) the corresponding line P is given by $y = -cx + d$. Now it is easy to see that H contains p on the primal plane if and only if the vertical ray starting in h and going into its orientation meets line P (we say that h and P *see each other*). Indeed, for a half-plane with lower boundary both hold if and only if $d > ac + b$, for a plane with an upper boundary both hold if and only if $d < ac + b$. From this it follows that the wcf_k-coloring of half-planes is equivalent to a coloring of the dual set of oriented points such that any line with at least k points looking at it, there are at least two with different colors among these points.
All the proofs give colorings for directed points and from now on we assume that there are no 3 directed points on one line. It is easy to show that if this is not the case, then coloring the set of directed points after a small perturbation gives a needed coloring for the original set of directed points as well.

(i) For a construction proving that 3 colors might be needed, see Figure 5(a). For the upper bound given a set of directed points we will color them with 3 colors such that for any line seeing at least 2 points, not all of these points have the same color.
Take the lower boundary of the convex hull of the set of north-directed points and denote the vertices of it by $p_1, p_2, \ldots, p_k$ ordered by their x-coordinate.



(a) Construction for Theorem 4(i)

(b) Proof of Theorem 4(i)

Figure 5: Theorem 4(i)

Take the upper boundary of the convex hull of the set of south-directed points and denote the vertices of it by $q_1, q_2, \ldots, q_l$ ordered by their x-coordinate. The rest of the points we call *inner* points. Similarly to Lemma 6 any line seeing at least one north-directed point sees one p_i as well and any line seeing at least one south-directed point sees one q_j as well and the p_i's and q_i's seen by a line are consecutive. First we give a coloring of the p_i's and q_j's with 3 colors such that no two consecutive points have the same color and if for some p_i and q_j there is a line which sees exactly these two points, then these points have different colors. As a line seeing at least two points which does not see inner points sees either exactly one p_i and q_j or at least two consecutive ones of the same type, the coloring will be good for all such lines.
We define a graph on the points p_i and q_j. The consecutive points are connected forming a path of p's and a path of q's. Moreover, p_i and q_j are connected if there is a line which sees exactly these two points. Clearly, we need a proper 3-coloring of this graph. For algorithmic reasons we take a graph with more edges and prove that it can be 3-colored as well. In this graph p_i and q_j are connected if there is a line which sees no other points of the p-path and q-path. We claim that drawing the p-path and the q-path on two parallel straight lines, the q-path being on the higher line and in reverse order, and drawing all the edges with straight lines, we have a graph without intersecting edges. In other words the graph is a caterpillar-tree between two paths. Going from left to right on the trunk of the caterpillar-tree and at each trunk-point taking all the leaves in left to right order, it is easy to see that at any step the considered point has only at most 2 backwards edges (one path-edge and one caterpillar-edge), so we can color it properly with a third color.

So it is enough to prove that there are no intersecting edges. Without loss of generality such two edges e and f would correspond to points e_p, e_q, f_p and f_q with x-coordinates $e_p^x < f_p^x$ and $e_q^x < f_q^x$ (the points with index p are from the p-path and the points with index q are from the q-path). The line seeing only e_p and e_q is denoted by h_e, the line seeing only f_p and f_q is denoted by h_f. These two lines divide the plane into four parts, which can be defined as the north, south, west and east part. Clearly from e_p and f_p one must be in the west part and one in the east part. By $e_p^x < f_p^x$, e_p is in the west and f_p is in the east part. This means that h_e must be the line above the east and south parts and so e_q must be in the east part and f_q in the west, a contradiction together with $e_q^x < f_q^x$ (see Figure 5(b)).
Now we finish the coloring such that the condition will hold also for lines seeing inner points. As in Theorem 3 (*ii*) for any other north-directed point p there are two points p_i and p_{i+1} (the unique ones for which p_i has smaller and p_{i+1} has bigger x-coordinate then p) such that whenever a line h sees p then it sees p_i or p_{i+1} as well. Then coloring p differently from these points, guarantees that any h seeing p sees two differently colored points. Doing the same for the south-directed points we finished the coloring such that whenever a line sees some point which is not a p_i or q_j then it sees points with both colors.
(*ii*) Given a set of directed points we will color them with 2 colors such that for any line seeing at least 4 points, not all of these points have the same color. We color the north-directed points with the same algorithm as in Theorem 3 (*iii*). We color the south-directed points with the same algorithm as in Theorem 3 (*iii*) just with inverted colors. This guarantees that any line which sees at least 3 north-directed points, sees red and blue points as well. If a line sees exactly 2 points of each kind, then sees red and blue points as well of one kind or sees 2 red north-directed points and 2 blue south-directed points by Observation 7, again seeing points with both colors. There are no more cases for a line seeing at least 4 points, so the proof is complete.
(*iii*) The algorithm in (*ii*) clearly runs in time $O(n \log n)$ using Theorem 3 (*iv*). The algorithm in (*i*) can be made similarly to work in this time, only the building of the caterpillar tree might need $O(n^2)$ steps. Indeed, we just need to prove that deciding whether there is an edge between some p_i and q_j can be done in constant time. For that we just need to check whether the linear equations for a line assuring that is goes above q_{j-1}, q_{j+1} and below q_j, below p_{i-1}, p_{i+1} and above p_i have a solution. ☐

4 Discussion

In Table 1 we summarize the results presented in this paper, the bold ones are proved in Section 2 and Section 3, others come from monotonicity except for $wcf_2(\mathcal{B}) = 3$ which is folklore.

k	2	3	$\geq$4
$wcf_k(\mathcal{B})$	*3*	**3**	**2**
$\overline{wcf_k}(\mathcal{B})$	**3**	**2**	*2*
$wcf_k(\mathcal{H})$	**4**	**2**	*2*
$\overline{wcf_k}(\mathcal{H})$	**3**	*2 or 3*	**2**

Table 1: table of results

The case of half-planes is solved except to determine $\overline{wcf_3}(\mathcal{H})$.

Problem 8. *Determine the value of* $\overline{wcf_3}(\mathcal{H})$*, i.e. the lowest number of colors needed to color any finite set of half-planes such that if a point of the plane is covered by at least* 3 *of them then not all of the covering half-planes have the same color.*

Though the case of bottomless rectangles is completely solved, one can consider the aforementioned case of axis-parallel rectangles intersecting a common base-line (denoted by $\mathcal{B}'$). We start with the case of region coloring. For this the best upper bound is due to [Sm06], proving $\overline{wcf_2}(\mathcal{B}') \leq 8$, and for the case of $k > 2$ we can separately color the upper and lower parts (divided by the base-line) of the rectangles with 2 colors by Theorem 2 (ii) and then for a rectangle colored by a in the upper part and b in the lower part, we give the ordered pair (a, b) as a color. It is easy to see that this is a good $\overline{wcf_3}$-coloring of the rectangles, thus proving $\overline{wcf_3}(\mathcal{B}') \leq 4$.
The case of coloring points seems less natural for axis-parallel rectangles intersecting a common base-line, still it can be considered. Coloring the points in the lower and upper parts with different colors ensures that any rectangle covering one from both sides is multicolored. The two sides can be colored by Claim 5 with 3-3 colors, thus proving $wcf_2(\mathcal{B}') \leq 6$ (a rectangle either covers points from both sides or covers at least 2 points on one side). Further, the same claim implies $wcf_3(\mathcal{B}') \leq 3$. Indeed, color both sides with

the same 3 colors according to Claim 5, then any rectangle covering at least 3 points covers 2 point on one side, thus covering two differently colored ones as well. Finally, $wcf_7(\mathcal{B}') = 2$ as if we color both sides with the same two colors according to Theorem 1 (*ii*), then any rectangle covering at least 7 points covers 4 point on one side, thus covering a red and blue one as well.

Problem 9. *Give better bounds for* $\overline{wcf_k}(\mathcal{B}', n)$ *and* $wcf_k(\mathcal{B}', n)$.

The general case of axis-parallel rectangles (denoted by $\mathcal{R}$) is still far from being solved, the best bounds are $wcf(\mathcal{R}, n) = \Omega(\frac{\log n}{(\log\log n)^2})$ ([ChSzPT]) from below and recently $wcf(\mathcal{R}, n) = \tilde{O}(n^{.382+\epsilon})$ ([AEGR07]) from above, improving the previous bound $wcf(\mathcal{R}, n) = O(\sqrt{\frac{n \log\log n}{\log n}})$ ([PT03]). So probably one of the most interesting problems is still to give better bounds for $wcf(\mathcal{R}, n)$, i.e. the lowest number of colors needed to color any set of n points, such that if an axis-parallel rectangle covers at least two of them then not all of those covered by it have the same color.
For the dual case of coloring axis-parallel rectangles the proof of the upper bound $\overline{cf}(\mathcal{R}, n) = O(\log^2 n)$ ([HS05]) can be modified easily to give the upper bound $\overline{wcf}(\mathcal{R}, n) = O(\log n)$. There is a matching lower bound $\overline{wcf}(\mathcal{R}, n) = \Omega(\log n)$ [PTa]. This implies the same lower bound for $\overline{cf}(\mathcal{R}, n)$, thus in this case there is still a slight gap between the lower and upper bounds.

The case of discs (denoted by $\mathcal{D}$) in the plane is only partially solved. It is easy to see that a proper coloring of the Delaunay-triangulation of a point set gives a weak conflict-free coloring. From the four-color theorem we conclude that $wcf_2(\mathcal{D}) = 4$. Further, in [PTT07] it is shown that $wcf_k(\mathcal{D}) > 2$ for any k.

Problem 10. *Is it true for some* k *that* $wcf_k(\mathcal{D}) = 3$*? If yes, find the smallest such* k.

Answering the question if $\overline{wcf_k}(\mathcal{D}, n)$ exists for some k in [Sm06] it is shown that $\overline{wcf_2}(\mathcal{D}, n) = 4$.

Problem 11. *Give better bounds for* $\overline{wcf_k}(\mathcal{D}, n)$ *when* $k > 2$.

Acknowledgment. The author is grateful to Gábor Tardos for introducing this topic and for his many useful comments on this paper.

References

[AEGR07] D. Ajwani, K. Elbassioni, S. Govindarajan, S. Ray, Conflict-Free Coloring for Rectangle Ranges Using $\tilde{O}(n^{.382+\epsilon})$ Colors, *Proceedings of the 19th annual ACM symposium on Parallel algorithms and architectures* (2007), 181–187.

[BJ02] G. Brodal, R. Jacob, Dynamic planar convex hull, *Proc. 43rd IEEE Symp. on Foundations of Computer Science* (2002), 617–626.

[ChSzPT] X. Chen, J. Pach, M. Szegedy, G. Tardos, Delaunay graphs of point sets in the plane with respect to axis-parallel rectangles, *SODA '08: Proceedings of the nineteenth annual ACM-SIAM symposium on Discrete algorithms* (2008), 94–101.

[ELRS03] G. Even, Z. Lotker, D. Ron, S. Smorodinsky, Conflict-free colorings of simple geometric regions with applications to frequency assignment in cellular networks, *SIAM J. Comput.* **33(1)** (2003), 94–136.

[HS05] S. Har-Peled, S. Smorodinsky, On Conflict-free Coloring of Points and Simple Regions in the Plane, *Discrete & Comput. Geom (DCG)* **34** (2005), 47–70.

[PTa] J. Pach, G. Tardos, Coloring axis-parallel rectangles, *Computational Geometry and Graph Theory (KyotoCGGT2007)* (2007)

[PT03] J. Pach, G. Tóth, Conflict free colorings, *Discrete and Computational Geometry - The Goodman-Pollack Festschrift (S. Basu et al, eds.)*, Springer Verlag, Berlin (2003), 665–671.

[PTT07] J. Pach, G. Tardos and G. Tóth, Indecomposable coverings, *Discrete Geometry, Combinatorics and Graph Theory, The China–Japan Joint Conference (CJCDGCGT 2005), Lecture Notes in Computer Science*, Springer **4381** (2007), 135–148.

[Sm03] S. Smorodinsky, Combinatorial Problems in Computational Geometry, *PhD thesis, School of Computer Science, Tel-Aviv University, Tel-Aviv, Israel* (2003)

[Sm05] S. Smorodinsky, Combinatorial Problems in Computational Geometry, PhD thesis, School of Computer Science, Tel-Aviv University, Tel Aviv, Israel (2005)

[Sm06] S. Smorodinsky, On The Chromatic Number of Some Geometric Hypergraphs, *Proc. of 17th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)* (2006)