

# Improved Upper Bounds on the Reflexivity of Point Sets

Eyal Ackerman\*

Oswin Aichholzer†

Balázs Keszeg‡

April 22, 2008

## Abstract

Given a set  $S$  of  $n$  points in the plane, the *reflexivity* of  $S$ ,  $\rho(S)$ , is the minimum number of reflex vertices in a simple polygonalization of  $S$ . Arkin et al. [4] proved that  $\rho(S) \leq \lceil n/2 \rceil$  for any set  $S$ , and conjectured that the tight upper bound is  $\lfloor n/4 \rfloor$ . We show that the reflexivity of any set of  $n$  points is at most  $\frac{3}{7}n + O(1) \approx 0.4286n$ . Using computer-aided abstract order type extension the upper bound can be further improved to  $\frac{5}{12}n + O(1) \approx 0.4167n$ . We also present an algorithm to compute polygonalizations with at most this number of reflex vertices in  $O(n \log n)$  time.

## 1 Introduction

Given a set  $S$  of  $n \geq 3$  points in the plane, a *polygonalization* of  $S$  is a simple polygon  $P$  whose vertices are the points of  $S$ . Throughout this paper we assume that the points are in *general position*, that is, no three of them are collinear. A vertex of a simple polygon is *reflex* if the (interior) angle of the polygon at that vertex is greater than  $\pi$ . We denote by  $\rho(P)$  the number of reflex vertices of a polygon  $P$ . The *reflexivity* of a set of points  $S$ ,  $\rho(S)$ , is the smallest number of reflex vertices any polygonalization of  $S$  must have. Further, we denote by  $\rho(n)$  the maximum value  $\rho(S)$ , such that  $S$  is a set of  $n$  points. Table 1 lists  $\rho(n)$  for  $n \leq 10$ . These values were verified using a computer [2, 4].

$n$	3	4	5	6	7	8	9	10
$\rho(n)$	0	1	1	2	2	2	3	3

Table 1:  $\rho(n)$  for  $n \leq 10$

The notion of reflexivity was suggested by Arkin et al. [4] as a measure for the “goodness” of a polygonalization of a set of points. They showed that  $\lfloor n/4 \rfloor \leq \rho(n) \leq \lceil n/2 \rceil$  and conjectured that the lower bound is tight – see also Conjecture 7 in Chapter 8.5 of [5]. Settling this

---

\*School of Computing Science, Simon Fraser University, Burnaby, BC, Canada, [eyal@cs.sfu.ca](mailto:eyal@cs.sfu.ca)

†Institute for Software Technology, Graz University of Technology, Austria, [oaich@ist.tugraz.at](mailto:oaich@ist.tugraz.at) Supported by the Austrian FWF Joint Research Project ‘Industrial Geometry’ S9205-N12.

‡Department of Mathematics and its Applications, Central European University, Budapest, Hungary, [tphkeb01@phd.ceu.hu](mailto:tphkeb01@phd.ceu.hu) This work was done while visiting the School of Computing Science at Simon Fraser University.

conjecture is one of the open problems listed in *The Open Problems Project* [7]. We refer the reader to [4] and [5] for a more detailed discussion on the notion of reflexivity, its applications, and related problems.

Our main result is the following improvement for the upper bound of  $\rho(n)$ .

**Theorem 1.**  $\rho(n) \leq 3\lfloor \frac{n-2}{7} \rfloor + 2$ .

The result will be obtained by considering a slightly modified version of reflexivity, namely to force a given convex hull edge to be part of the polygonalization. The main ingredient is an iterative subdivision of the point set, together with a good polygonalization of sets of constant size. Theorem 1 then directly follows from Theorem 5 below.

Utilizing a computer-aided abstract order type extension [3] we will further improve the upper bound to

**Theorem 2.**  $\rho(n) \leq 5\lfloor \frac{n-2}{12} \rfloor + 4$ .

## 2 Modified Reflexivity and Iterative Subdivision

Recall that the convex hull of a finite set  $S$  of points,  $\mathcal{CH}(S)$ , is composed of the boundary and the interior of a convex polygon. A *boundary edge* of  $\mathcal{CH}(S)$  is an edge of that polygon. To prove a stronger variant of Theorem 1 we first introduce some notation. Let  $S$  be a set of points and let  $e$  be a boundary edge of  $\mathcal{CH}(S)$ . We denote by  $\rho_e(S)$  the minimum number of reflex vertices in any polygonalization  $P$  of  $S$ , such that  $e$  is an edge of  $P$ . Similarly, let  $\bar{\rho}(S)$  be the maximum value of  $\rho_e(S)$  taken over all the edges  $e$  of the boundary of  $\mathcal{CH}(S)$ , and let  $\bar{\rho}(n)$  be the maximum value of  $\bar{\rho}(S)$  taken over all sets  $S$  of size  $n$ . The definition of  $\bar{\rho}(\cdot)$  is perhaps a bit counter-intuitive (one might expect to take the minimum over all edges), however, it is crucial for our purposes.

Obviously  $\rho(n) \leq \bar{\rho}(n)$ , so our goal is to derive good upper bounds for  $\bar{\rho}(n)$ . To this end we first provide a central lemma, which allows us to subdivide a point set in a way that we can consider the polygonalizations of the subsets rather independently.

**Lemma 3.** *Given an integer  $k > 2$ , a set  $S$  of  $n > k$  points, and two points  $p, q \in S$ , such that  $pq$  is a boundary edge of  $\mathcal{CH}(S)$ , then there exists a point  $t \in S \setminus \{p, q\}$  and two sets  $L, R \subset S$  such that:*

1.  $L \cup R = S$ ,  $L \cap R = \{t\}$ ,  $q \in R$ , and  $p \in L$ ;
2. The triangle  $\Delta pqt$  contains no other points from  $S$ ;
3.  $\mathcal{CH}(R) \cap \mathcal{CH}(L) = \{t\}$ ; and
4.  $|R| = k$ .

*Proof.* Assume, w.l.o.g., that  $p$  and  $q$  lie on the  $x$ -axis, such that  $p$  is to the left of  $q$  and all the remaining points are above the  $x$ -axis. Let  $t_1$  be the point of  $S$  such that the angle  $\angle t_1pq$  is the smallest. Let  $e_1$  be the line determined by  $q$  and  $t_1$ , and let  $H_1$  be the closed half-plane to right of  $e_1$ . Let  $S_1$  be the subset of points of  $S$  contained in  $H_1$ . If  $|S_1| > k$ , then define  $r_1 \in S_1 \setminus \{q, t_1\}$  to be the point creating the  $(k - 1)$ st smallest angle  $\angle r_1t_1q$ ,

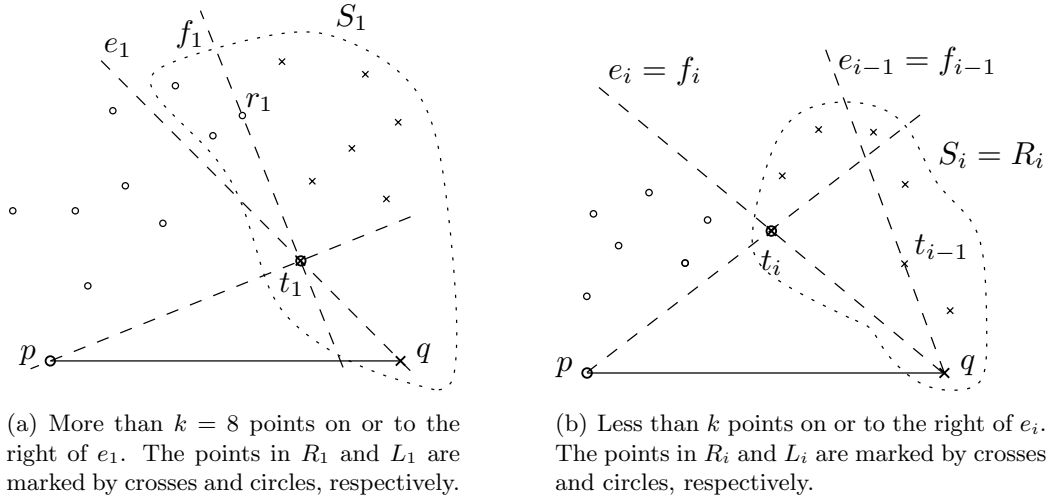


Figure 1: Illustrations for the proof of Lemma 3

and denote by  $f_1$  the line through  $t_1$  and  $r_1$ . Otherwise, if  $|S_1| \leq k$  let  $f_1 = e_1$ . Set  $R_1 = \{q, t_1\} \cup \{p' \in S_1 \mid p' \text{ is to the right of } f_1\}$  and  $L_1 = (S \setminus R_1) \cup \{t_1\}$ . Note that  $r_1$ , if defined, is in  $L_1$ . We claim that  $t_1$ ,  $R_1$ , and  $L_1$  satisfy properties (1)–(3) of the lemma: (1) This property holds by the definition of  $R_1$  and  $L_1$ ; (2) By the choice of  $t_1$  the triangle  $\triangle pqt_1$  is empty; (3) All the points in  $R_1$  are to the right of  $f_1$ , except for  $t_1$  and possibly  $q$ . All the points in  $L_1$  are to the left of  $f_1$ , except for  $t_1$  and possibly  $r_1$ . However  $q$  and  $r_1$  cannot both lie on  $f_1$ . If  $|S_1| \geq k$ , then we also have that  $|R_1| = k$  (either by the choice of  $r_1$  or because  $|S_1| = k$ , see Figure 1(a) for an illustration of the former case).

Suppose now that  $|S_1| < k$ . We define  $t_i$ ,  $e_i$ , and  $S_i$  for  $i > 1$  and  $|S_{i-1}| < k$  recursively. Let  $t_i$  be the point that minimizes the angle  $\angle t_i p q$  among the points in  $L_{i-1} \setminus \{t_{i-1}\}$  (note that this set of points is not empty since  $|S_{i-1}| < k$  and we show below that  $S_1 \subset \cdots \subset S_{i-1}$ ). Let  $e_i$  be the line through  $q$  and  $t_i$ , let  $H_i$  be the closed half-plane to the right of  $e_i$ , and let  $S_i$  be the set of points contained in  $H_i$ . Next, we define  $r_i$ ,  $f_i$ ,  $R_i$ , and  $L_i$ . If  $|S_i| > k$  define  $r_i \in S_i \setminus \{q, t_i\}$  to be the point creating the  $(k-1)$ st smallest angle  $\angle r_i t_i q$ , and denote by  $f_i$  the line through  $t_i$  and  $r_i$ . Otherwise, if  $|S_i| \leq k$  set  $f_i = e_i$ . Set  $R_i = \{q, t_i\} \cup \{p' \in S_i \mid p' \text{ is to the right of } f_i\}$  and  $L_i = (S \setminus R_i) \cup \{t_i\}$ . See Figure 1(b) for an example where  $|S_i| < k$ . The existence of a point  $t$  and sets  $R, L \subset S$  as required, will follow from the next claim.

**Proposition 2.1.** *Set  $S_0 = \emptyset$ . Then, for every  $i \geq 1$  such that  $|S_{i-1}| < k$ ,  $t_i$ ,  $R_i$ , and  $L_i$  satisfy properties (1)–(3) of Lemma 3, and  $S_{i-1} \subsetneq S_i$ .*

*Proof.* By induction on  $i$ . For  $i = 1$  the claim holds by the discussion above. Assume that  $i > 1$  and  $|S_{i-1}| < k$ . Property (1) holds by the definition of  $R_i$  and  $L_i$ . The triangle  $\triangle t_i p q$  is empty since:  $\triangle t_{i-1} p q$  is empty;  $t_i$  is to the left of  $f_{i-1}$  and therefore  $\triangle t_i p q$  does not contain any point from  $R_{i-1}$ ; and by the choice of  $t_i$ . Thus, Property (2) holds. Property (3) clearly holds if  $f_i = e_i$ . Otherwise, if  $r_i$  is defined, denote by  $C_i$  the cone whose apex is at  $p$  and is bounded by the line through  $p$  and  $t_{i-1}$  and the line through  $p$  and  $t_i$ . By the choice of  $t_i$  all the points in  $C_i$  are in  $S_{i-1}$ . Since  $|S_{i-1}| < k$  it follows that  $r_i$  is to the left of the line through  $p$  and  $t_i$ . Recall that  $r_i$  is to the right of  $e_i$ , since  $r_i \in S_i$ . Therefore,  $f_i$  is tangent

to both  $\mathcal{CH}(R_i)$  and  $\mathcal{CH}(L_i)$  and separates them, except for the point  $t_i$ . Thus, Property (3) holds. Finally, since  $t_i$  is to the left of  $e_{i-1}$  we have  $S_{i-1} \subseteq S_i$ . However  $t_i \in S_i \setminus S_{i-1}$ , thus,  $S_{i-1} \subsetneq S_i$ .  $\square$

Since  $|S_i| > |S_{i-1}|$  there is an integer  $j$  such that  $|S_{j-1}| < k$  and  $|S_j| \geq k$ . It follows from Proposition 2.1 and the definition of  $R_j$  that  $t_j$ ,  $R_j$ , and  $L_j$  satisfy the required properties.  $\square$

Note that Lemma 3 implies that  $pt$  is a boundary edge of  $\mathcal{CH}(L)$  and  $tq$  is a boundary edge of  $\mathcal{CH}(R)$ , respectively. Using this fact we will apply the suggested subdivision in the next section in order to obtain our first main result.

### 3 A New Upper Bound

Figure 2 illustrates the subdivision obtained in the previous section. The idea to prove an upper bound on  $\bar{\rho}(n)$  is to iteratively split a set into subsets of constant size, to obtain good polygonalizations for these sets, and then to combine them based on Lemma 3. The base case is covered by the following result.

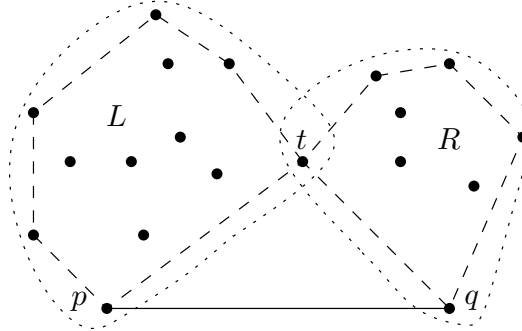


Figure 2: The subdivision of  $S$  guaranteed by Lemma 3

**Lemma 4.** *Let  $S$  be a set of at most 8 points in the plane. Then  $\bar{\rho}(S) \leq 2$ .*

*Proof.* The claim is clearly true for  $n \leq 5$  since any vertex on the boundary of  $\mathcal{CH}(S)$  is a convex vertex of any polygonalization of  $S$ . For  $6 \leq n \leq 8$  we prove the statement by a case-analysis over the size of the onion layers of  $S$ ; see Appendix A for details. The correctness of the statement was also verified using a computer by checking all possible configurations of at most 8 points in general position.  $\square$

We are now ready for a first upper bound on  $\bar{\rho}(n)$ .

**Theorem 5.**  $\bar{\rho}(n) \leq 3\lfloor \frac{n-2}{7} \rfloor + 2$ .

*Proof.* We prove the claim by induction on  $n$ . For  $n \leq 8$  we directly get the result from Lemma 4.

For  $n > 8$  we apply Lemma 3 on the set  $S$  with  $k = 8$  and some edge  $pq$  of the boundary of  $\mathcal{CH}(S)$ , and obtain the point  $t$  and the subsets  $L$  and  $R$ . Now according to Lemma 4 there is a polygonalization of  $R$  containing the edge  $qt$  with at most two reflex vertices (note that  $qt$  is a boundary edge of  $\mathcal{CH}(R)$ ). By induction,  $L$  has a polygonalization containing the edge  $pt$  (which is a boundary edge of  $\mathcal{CH}(L)$ ) with at most  $3\lfloor \frac{n-9}{7} \rfloor + 2 = 3\lfloor \frac{n-2}{7} \rfloor - 1$  reflex vertices. By removing the edge  $qt$  from the first polygonalization and the edge  $pt$  from the second, the remaining polygonal chains, along with the edge  $pq$ , form a proper polygonalization of  $S$  with at most  $2 + 3\lfloor \frac{n-2}{7} \rfloor - 1 + 1 = 3\lfloor \frac{n-2}{7} \rfloor + 2$  reflex vertices (note that  $t$  may be a reflex vertex in the resulting polygon).  $\square$

## 4 Improving the Constant

Generalizing the approach used to prove Theorem 5 to arbitrary  $k \geq 2$  we get

**Corollary 4.1.** *If for some  $k \geq 2$  we have  $\bar{\rho}(k) \leq l$ , then  $\rho(n) \leq (l+1)\lfloor \frac{n-2}{k-1} \rfloor + k \leq \frac{l+1}{k-1}n + k$ . If, additionally, for any  $k' \leq k$  we have  $\bar{\rho}(k') \leq l$ , then  $\rho(n) \leq \frac{l+1}{k-1}n + l$ .*

Improved bounds for  $\bar{\rho}(n)$  for small, constant values of  $n$  thus yield a better bound on the reflexivity of arbitrarily large sets of points. From Lemma 4 together with an extension to  $n = 9, 10$  by using the point set order type data base [2] we observe that  $\bar{\rho}(n) = \rho(n)$  for  $n \leq 10$ , see Table 1. Therefore our next goal is to determine good bounds on  $\bar{\rho}(n)$  for  $n \geq 11$ . To this end, we use the following observation which is implied by Lemma 3 and the discussion in the previous section.

**Observation 4.2.** *For any integers  $2 < k < n$ , we have  $\bar{\rho}(n) \leq \bar{\rho}(n - k + 1) + \bar{\rho}(k) + 1$ . Moreover, for every set of  $n$  points,  $S$ , there is a subset  $L \subset S$ , such that  $|L| = n - k + 1$  and  $\bar{\rho}(S) \leq \bar{\rho}(L) + \bar{\rho}(k) + 1$ .*

Using the values of Table 1 for  $k = 3$  and  $k = 8$  we get

$$\begin{aligned} \bar{\rho}(n) &\leq \bar{\rho}(n - 2) + 1 \\ \bar{\rho}(n) &\leq \bar{\rho}(n - 7) + 3 \end{aligned} \tag{1}$$

Applying these two relations we obtain the upper bounds on  $\bar{\rho}(n)$  shown in Table 2 with an exception for  $n = 13$ .

$n$	11	12	13	14	15	16
$\rho(n)$	3	3..4	3..4	4..5	4..5	4..6
$\bar{\rho}(n)$	4	4	4	4..5	4..5	4..6

Table 2:  $\rho(n)$  and  $\bar{\rho}(n)$  for  $n = 11 \dots 15$

By using the point set order type data base for  $n = 11$  points it turned out that  $\rho(11) = 3$  whereas  $\bar{\rho}(11) = 4$ . Interestingly, only for 36 of the 2 334 512 907 existing order types, the best polygonalization required 4 reflex vertices. In all these sets the boundary of the convex hull was a triangle, and only for one (out of three) convex hull edge  $e$  we obtained  $\rho_e(S) = 4$ . This has to be seen in contrast to the worst case examples for  $\rho(S)$  obtained in [4], which are

so-called double circles. There half of the vertices are on the convex hull, and the remaining vertices form a second onion layer, each point lying close to the middle of one edge of the convex hull.

We have extended examples providing  $\bar{\rho}(11) = 4$  to verify that 4 reflex vertices are necessary for polygonalizing certain point sets of size  $n = 12, \dots, 16$ , as is listed in Table 2. Thus, for  $n = 12$ , together with Equation 1 we have  $\bar{\rho}(12) = 4$ . So we will have to look for values of  $k > 12$  in order to benefit from Corollary 4.1. Thus we aim to show that  $\bar{\rho}(13) = 4$ .

From Equation 1 we already know that  $\bar{\rho}(13) \leq 5$ . So assume that there exists a set  $S$ ,  $|S| = 13$ , with  $\bar{\rho}(S) = 5$ . By Observation 4.2  $S$  contains a subset  $L$  of 11 points with  $\bar{\rho}(L) = 4$ . We now apply abstract order type extension, which is a tool that can be used to generate all (abstract) point sets containing a given class of sets of smaller cardinality, see [3] for details. Applying this method to the 36 sets of  $n = 11$  points which require 4 reflex vertices, we obtain all sets for  $n = 13$  which might require 5 reflex vertices. Our computations show that all obtained sets contain a polygonalization with at most 4 reflex vertices, contradicting our assumption, and we conclude that  $\bar{\rho}(13) = 4$ .

By Corollary 4.1 we therefore get

**Corollary 4.3.**  $\bar{\rho}(n) \leq 5 \lfloor \frac{n-2}{12} \rfloor + 4$

which implies Theorem 2. Obviously determining  $\bar{\rho}(n)$  for  $n \geq 14$  could further improve the constant of Corollary 4.3, and we leave this for future research.

## 5 An Algorithm

After establishing the existence of a polygonalization with few reflex vertices we describe an efficient way to find one.

**Theorem 6.** *Given a set of  $n$  points  $S$  and two points  $p, q \in S$  such that  $pq$  is a boundary edge of  $\mathcal{CH}(S)$ , a polygonalization  $P$  of  $S$  such that  $pq$  is an edge of  $P$  and  $\rho(P) \leq 5 \lfloor \frac{n-2}{12} \rfloor + 4$  can be found in  $O(n \log n)$  time.*

*Proof.* The proof of Theorem 5 and the discussion in Section 4 yield an algorithm for computing a polygonalization with at most  $5 \lfloor \frac{n-2}{12} \rfloor + 4$  reflex vertices, based on the subdivision of the set  $S$  into (not necessarily disjoint) subsets of size 13 (apart from one subset of size at most 13). Set  $t'_0 = q$ ,  $L'_0 = S$ , and  $M = \lfloor (n-2)/12 \rfloor$ . Define  $t'_l$ ,  $R'_l$ , and  $L'_l$ , recursively, to be the point  $t$  and the sets  $R$  and  $L$ , respectively, guaranteed by applying Lemma 3 on the set  $L'_{l-1}$  and the edge  $pt'_{l-1}$ ,  $1 \leq l \leq M$ . Once the points  $t'_l$  and the sets  $R'_l$  have been computed it is easy to compute in linear time the polygonalization of  $S$  with the stated number of reflex vertices: For each  $1 \leq l \leq M$  we computed (in constant time) a polygonalization  $P_l$  of  $R'_l$ , such that  $P_l$  contains the edge  $t'_l t'_{l-1}$  and has at most four reflex vertices. By removing the edge  $t'_l t'_{l-1}$  from every polygon  $P_l$ , we get a polygonal chain  $P$  starting at  $t'_0 = q$  and ending at  $t'_M$ . If  $|L'_M| = 2$ , that is  $L'_M = \{p, t'_M\}$ , then we obtain the desired polygonalization of  $S$  by concatenating to  $P$  the edges  $pt'_M$  and  $pq$ . Otherwise, one can compute in constant time a polygonalization  $P_{M+1}$  of  $L'_M$  containing the edge  $pt'_M$  and having at most four reflex vertices

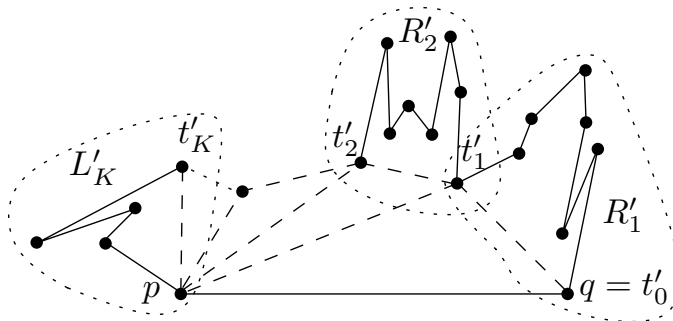


Figure 3: The subdivision of the point set into constant-size subsets

(note that  $|L'_M| \leq 13$ ). By removing the edge  $pt'_M$  from  $P_{M+1}$  and concatenating the resulting chain to  $P$  and the edge  $pq$  we obtain the desired polygonalization of  $S$  (see Figure 3 for an illustration).

Therefore, it remains to depict the details of the subdivision described in Lemma 3. Algorithm 1 describes the implementation of this subdivision. It uses the dynamic planar convex hull of Brodal and Jacob [6]. This data structure, denoted by DCH, maintains the convex hull of a set of points and supports, among other things, the following operations:

- `DCH.INSERT( $v$ )`: insert a new point  $v$ ;
- `DCH.DELETE( $v$ )`: remove the point  $v$ ; and
- `DCH.CCW( $v$ )`: get the counter-clockwise neighbor in the convex hull of the point  $v$ , where  $v$  is a vertex of the convex hull;

Insertions and deletions are performed in  $O(\log n)$  amortized time, while the counter-clockwise neighbor query takes  $O(\log n)$  worst-case time.

Next, we explain Algorithm 1, using the notation of Lemma 3. The algorithm begins by removing the vertex  $q$  (line 5). Now,  $\text{CCW}(p)$  is  $t_1$ . Then, we remove  $\text{CCW}(t_1)$  repeatedly at most  $k - 2$  times while it is to the right of  $e_1$  (lines 15–23). If  $k - 2$  times  $\text{CCW}(t_1)$  was to the right of  $e_1$ , then all those points that were removed along with  $t_1$  constitute the set  $R$ , with  $t = t_1$  (lines 24–27). Otherwise, by deleting  $t_1$  (line 5),  $\text{CCW}(p)$  is the point  $t_2$  that forms the smallest angle  $\angle t_2pq$  among the points to the left of  $e_1$ . The algorithm proceeds by re-inserting all the points that were deleted in the previous iteration and are to the left of the line determined by  $p$  and  $t_2$  (lines 9–12). This step is performed since it is possible that these points will not be among the set of points  $p' \in S_2$  creating the  $(k - 2)$ nd smallest angles  $p't_iq$  (whereas the points that are to the right of the line through  $p$  and  $t_2$  must be in this set, and thus, remain in  $R$ ). Next, we remove  $\text{CCW}(t_2)$  repeatedly (as long as it is to the right of  $e_2$ ), this time  $k - 1 - |R|$  times (lines 15–23). As before, if  $k - 1 - |R|$  times  $\text{CCW}(t_2)$  was to the right of  $e_2$ , then we are done. Otherwise, we proceed to the next point  $t_3$ .

For the same arguments used to claim that  $|S_i| > |S_{i-1}|$  in the proof of Lemma 3 it follows that the size of  $R$  at, say, line 24 grows along the iterations of the main loop (lines 4–28). Therefore, the main loop is executed  $O(k)$  times, and thus, the run-time of the procedure

```

Require: A set of  $n$  points  $S$ ;  $\text{DCH}(S)$ ;  $p, q \in S$  s.t.  $pq$  is a boundary edge of  $\text{CH}(S)$ ; an
integer  $k > 2$ 
Ensure: The set  $R$  and the point  $t$  as described in Lemma 3.
 $R \leftarrow \emptyset$ ;
 $i \leftarrow 0$ ;
 $t_i \leftarrow q$ ;
while  $|R| < k$  do
   $\text{DCH.DELETE}(t_i)$ ;
   $R \leftarrow R \cup \{t_i\}$ ;
   $i \leftarrow i + 1$ ;
   $t_i \leftarrow \text{DCH.CCW}(p)$ ;
  for all  $\{r \in R : r \text{ is to the left of the line through } p \text{ and } t_i\}$  do
     $R \leftarrow R \setminus \{r\}$ ;
     $\text{DCH.INSERT}(r)$ ;
  end for
   $e_i \leftarrow$  the line through  $q$  and  $t_i$ ;
   $m \leftarrow (k - 1 - |R|)$ ; /* The number of points missing in  $R$  */
  for  $j = 1$  to  $m$  do
     $s \leftarrow \text{DCH.CCW}(t_i)$ ;
    if  $s$  is to the right of  $e_i$  then
       $\text{DCH.DELETE}(s)$ ;
       $R \leftarrow R \cup \{s\}$ ;
    else
      quit the for-loop;
    end if
  end for
  if  $s$  is to the right of  $e_i$  then
     $R \leftarrow R \cup \{t_i\}$ ; /*  $|R| = k$  */
     $t \leftarrow t_i$ ;
  end if
end while

```

**Algorithm 1:** Generating the subdivision of Lemma 3

described in Algorithm 1 is  $O(k^2 \log n)$  amortized time. The number of times this procedure is executed is  $O(n/k)$ . Thus, the overall run-time, including the initialization of  $\text{DCH}$ , is  $O(nk \log n)$ . As in our case  $k = 13$ , the run-time is  $O(n \log n)$   $\square$

## 6 Discussion and Open Problems

We showed that for every set  $S$  of  $n$  points in general position in the plane there is a polygonalization of  $S$  with at most  $5 \lfloor \frac{n-2}{12} \rfloor + 4$  reflex vertices, and such a polygonalization can be found in  $O(n \log n)$  time. The basic idea of the proof is that by Lemma 3 we can subdivide  $S$  into some fixed-size parts and use a stronger result on each of these parts. It would be interesting to find other applications of the subdivision suggested in Lemma 3.

Conjecture 3.4 in [4] states that  $\rho(n) = \lfloor \frac{n}{4} \rfloor$ . Considering the values for  $\rho(n)$  in Tables 1 and 2 the conjecture has to be modified to

**Conjecture 6.1.**  $\lfloor \frac{n}{4} \rfloor \leq \rho(n) \leq \lceil \frac{n}{4} \rceil$ .

It is challenging to determine the structure of sets maximizing the reflexivity for fixed cardinality. On the one hand we have the sets used in [4] to provide the bound of  $\rho(n) \geq \lfloor \frac{n}{4} \rfloor$ , which have half of their vertices on the boundary of the convex hull. This so-called double circle configuration is also conjectured to minimize the number of triangulations [1], and therefore seems to be a promising extremal example, supporting Conjecture 6.1. On the other hand all maximizing examples for  $\bar{\rho}(11)$  have a triangular convex hull, so it could be that for larger cardinality  $\bar{\rho}(n)$  is more than a constant additive factor larger than  $\rho(n)$ , contradicting Conjecture 6.2.

It would be interesting to bound  $\bar{\rho}(n)$  in terms of  $\rho(n)$ .

**Conjecture 6.2.** *There is a constant  $c_0$  such that  $\bar{\rho}(n) \leq \rho(n) + c_0$ .*

Note that the stronger statement that  $\bar{\rho}(S) \leq \rho(S) + O(1)$  for any set  $S$  might also hold.

Conjecture 6.2, if true, would mean that it is possible (although not necessarily practical) to get arbitrarily close to the best possible linear upper bound by checking only finitely many small cases. In other words, suppose the conjecture holds and  $c$  is a constant such that  $\rho(n) \leq cn$ . Then, for any  $\epsilon > 0$  there is  $k = k(\epsilon)$  such that if we verify that  $\rho(k) \leq ck$ , then for  $n > k$  we have  $\rho(n) \leq (c + \epsilon)n + O(1)$ . Indeed,  $k$  large enough such that  $\frac{ck + c_0 + 1}{k-1} \leq (c + \epsilon)$  holds, would do. Moreover, the discussion above is still valid if we replace  $c_0$  in Conjecture 6.2 by some function  $f(n)$  such that  $f(n) \in o(n)$ .

Conjecture 6.2 is true when we consider reflexivity in the presence of *Steiner points*. Following the notation of [4], a *Steiner point* is a point  $q \notin S$  that may be added to  $S$  in order to improve some structure. For example, we define the *Steiner reflexivity* of  $S$ ,  $\rho'(S)$ , to be the minimum number of reflex vertices of any simple polygon with vertex set  $V \supseteq S$ . Similarly,  $\rho'(n) = \max_{|S|=n} \rho'(S)$ . The (stronger statement) of Conjecture 6.2 can be easily proved if we allow Steiner points.

**Lemma 7.** *Let  $S$  be a set of  $n$  points and let  $pq$  be a boundary edge of  $\mathcal{CH}(S)$ . Then, there are points  $p', q'$  (inside  $\mathcal{CH}(S)$ ) such that  $S \cup \{p', q'\}$  has a polygonalization containing the edge  $pq$  and having at most  $\rho(S) + 1$  reflex vertices.*

*Proof.* We assume, w.l.o.g., that the fixed edge  $pq$  is horizontal,  $p$  is left to  $q$ , and the remaining points  $S \setminus \{p, q\}$  are above the line through  $p$  and  $q$ . Let  $P$  be a polygonalization of  $S$ , such that  $P$  does not contain the edge  $pq$ . We show that  $P$  can be modified into a polygonalization  $P'$  of a set  $V \supset S$  such that  $P'$  contains the edge  $pq$  and  $\rho(P') \leq \rho(P) + 1$ .

Let  $p_1$  be the counter-clockwise neighbor of  $p$  in  $P$ , and let  $q_1$  and  $q_2$  be the counter-clockwise and clockwise neighbors of  $q$  in  $P$ , respectively. Fix  $p'$  slightly to the right and above  $p$ , and  $q'$  slightly to the left and above  $q$ . Now by replacing the chain  $q_1 q q_2$  with the chain  $q_1 q' q_2$ , and the edge  $p p_1$  with the chain  $p q p' p_1$ , one obtains the desired polygonalization  $P'$  (see Figure 4 for an illustration). Note that the only reflex vertex that might be introduced in these steps is  $p'$ .  $\square$

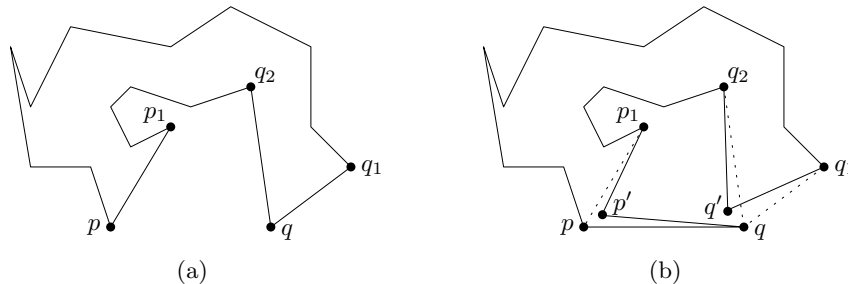


Figure 4: Illustrations for the proof of Lemma 7

As before, this implies that we can get arbitrarily close to any linear upper bound on  $\rho'(n)$  by checking only finitely many small cases. Note that it is important here that the Steiner points we add lie inside the convex hull of the original set of points.

Adding a (Steiner) point to a set of points might result in a set of points whose reflexivity is smaller than this of the original set (see [4] for examples). However, we are confident, although we were not able to prove, that for every set of points one can add some point that will not reduce the reflexivity. This would imply

**Conjecture 6.3.**  $\rho(n + 1) \geq \rho(n)$

A similar statement should hold for restricted reflexivity.

**Conjecture 6.4.**  $\bar{\rho}(n + 1) \geq \bar{\rho}(n)$

If this conjecture is true, then the last inequality of Corollary 4.1 always holds.

**Acknowledgments.** We thank anonymous referees for suggestions that helped improving the presentation of the paper.

## References

- [1] O. Aichholzer, F. Hurtado, and M. Noy, A lower bound on the number of triangulations of planar point sets. *Computational Geometry: Theory and Applications*, 29(2):135–145, 2004.
- [2] O. Aichholzer and H. Krasser, The point set order type data base: A collection of applications and results, *Proc. 13th Canadian Conf. on Computational Geometry*, Waterloo, Ontario, Canada, August 2001, 17–20.
- [3] O. Aichholzer and H. Krasser, Abstract order type extension and new results on the rectilinear crossing number, *Computational Geometry: Theory and Applications*, 36(1):2–15, 2006.
- [4] E. M. Arkin, S. P. Fekete, F. Hurtado, J. S. B. Mitchell, M. Noy, V. Sacristán, and S. Sethia, On the reflexivity of point sets, In B. Aronov, S. Basu, J. Pach, and M. Sharir, editors, *Discrete and Computational Geometry: The Goodman-Pollack Festschrift*, 139–156, Springer, 2003.
- [5] P. Brass, W. Moser, J. Pach, *Research Problems in Discrete Geometry*, Springer, 2005.
- [6] G. Brodal and R. Jacob, Dynamic planar convex hull, *Proc. 43rd IEEE Symp. on Foundations of Computer Science*, Vancouver, BC, Canada, November 2002, 617–626.
- [7] E. Demaine, J. S. B. Mitchell, and J. O’Rourke, Problem 66: Reflexivity of Point Sets, *The Open Problems Project*, <http://maven.smith.edu/~orourke/TOPP/P66.html>

## A Proof of Lemma 4

**Lemma 4.** *Let  $S$  be a set of at most 8 points in the plane. Then  $\bar{\rho}(S) \leq 2$ .*

*Proof.* Given a set of  $n$  points  $S'$ , the algorithm in the proof of Theorem 3.1 in [4] generates a polygonalization of  $S'$  with at most  $\lceil n_I/2 \rceil$  reflex vertices, where  $n_I$  is the number of points in  $S'$  that are internal points of  $\mathcal{CH}(S')$ . Moreover, this algorithm begins with fixing one edge of the boundary of  $\mathcal{CH}(S')$  (the edge  $p_0p_1$  in [4]'s notation), and one can observe that this edge is an edge of the resulting polygonalization when the algorithm terminates. Therefore, it is enough to consider the case in which  $|S| = 8$  and the boundary of  $\mathcal{CH}(S)$  is a triangle.

Let  $\mathcal{CH}_i(S)$  denote the  $i$ th layer in the ‘‘onion peeling’’ of  $S$ . More precisely, set  $\mathcal{CH}_0(S) = \mathcal{CH}(S)$ , and let  $\mathcal{CH}_i(S)$  be the convex hull of  $S \setminus \{p \in S \mid p \text{ is a vertex of the boundary of } \mathcal{CH}_j(S), 0 \leq j < i\}$ . We say that a point  $p$  outside of  $\mathcal{CH}_i(S)$  *sees* a vertex  $q$  of the boundary of  $\mathcal{CH}_i(S)$  if the segment  $pq$  does not cross  $\mathcal{CH}_i(S)$ .

Assume that the fixed edge of the boundary of  $\mathcal{CH}(S)$  is  $e = (A, B)$ , such that  $e$  is on the  $x$ -axis,  $A$  is left of  $B$ , and let  $C$  be the third vertex of the boundary of  $\mathcal{CH}(S)$ . Consider the lines determined by  $C$  and each of the internal points. Let  $p_0$  be the point that determines the line with smallest slope. Clearly,  $p_0$  is a vertex of the boundary of  $\mathcal{CH}_1(S)$ . Let  $p_1, p_2, \dots, p_k$  be the remaining vertices of the boundary of  $\mathcal{CH}_1(S)$  in a clockwise order around  $\mathcal{CH}_1(S)$ . Denote by  $p_i$  the point that determines (along with  $C$ ) the line with the largest slope, and by  $p_l$  the lowest point in  $\mathcal{CH}_1(S)$ . (Note that it is possible that  $p_l = p_0$  or  $p_l = p_i$ .) The following two observations are easy.

**Observation A.1.** *The point  $A$  (resp.,  $B$ ) sees all the vertices on the boundary of  $\mathcal{CH}_1(S)$  along the clockwise (resp., counter-clockwise) chain from  $p_l$  to  $p_i$  (resp.,  $p_0$ ).*

*Proof.* Follows from convexity. □

**Observation A.2.** *If  $i \leq 3$  then  $A$  sees  $p_1$  or  $B$  sees  $p_{i-1}$ .*

*Proof.* Since  $i \leq 3$  we have  $l \in \{0, 1, i-1, i\}$ . If  $l \in \{1, i-1\}$ , then, since both  $A$  and  $B$  see  $p_l$  we are done. Otherwise, suppose that  $l = 0$ . Then, by the previous observation,  $A$  sees all the vertices on the clockwise chain from  $p_0$  to  $p_i$ . Similarly,  $B$  sees this chain in case  $l = i$ . □

We proceed proving Lemma 4 by case analysis, based on the number of vertices on the boundary of  $\mathcal{CH}_1(S)$ .

**Case 1:** There are 5 vertices on the boundary of  $\mathcal{CH}_1(S)$ . We consider two subcases: (a) Suppose that  $i \leq 3$ . Then by Observation A.2  $A$  sees  $p_1$  or  $B$  sees  $p_{i-1}$ . Assume, w.l.o.g., that  $B$  sees  $p_{i-1}$ . Then we draw the desired polygon as in Figure 5(a). (b) Suppose that  $i = 4$ . Then  $p_l \neq p_0$  or  $p_l \neq p_i$ . Assume, w.l.o.g., that  $p_l \neq p_i$ . Then by Observation A.1  $A$  sees  $p_{l+1}$ . The polygon  $A p_{l+1} \dots p_i C p_0 \dots p_l B A$  is the desired polygon (see Figure 5(b)).

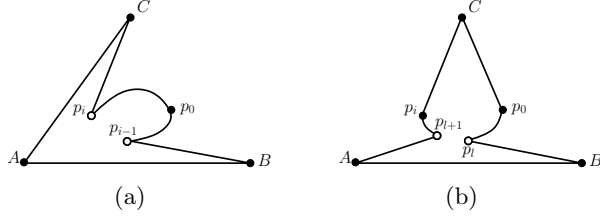


Figure 5: The case of 5 vertices on the boundary of  $\mathcal{CH}_1(S)$

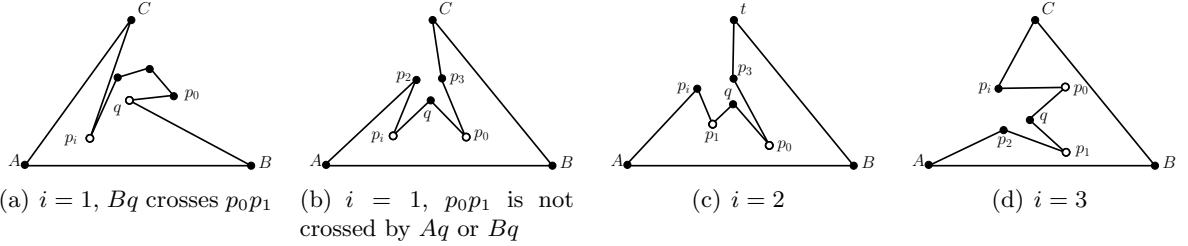


Figure 6: The case of 4 vertices on the boundary of  $\mathcal{CH}_1(S)$

**Case 2:** There are 4 vertices on the boundary of  $\mathcal{CH}_1(S)$ . Let  $q$  be the single vertex on the boundary of  $\mathcal{CH}_2(S)$ . We consider the different subcases, based on the value of  $i$ . (a) Suppose  $i = 1$ . If  $Aq$  or  $Bq$  cross  $p_0p_1$ , then we can draw the desired polygon as in Figure 6(a). Otherwise,  $A$  sees the vertex  $p_2$  of the boundary of  $\mathcal{CH}_1(S)$  (and  $B$  sees  $p_3$ ), and we can draw the polygon as in Figure 6(b). (b) Suppose  $i = 2$ . The vertex  $p_1$  of the pentagon  $ABp_0p_1p_2$  is reflex. Thus at most one of the vertices  $p_0$  and  $p_2$  of this pentagon is reflex (a pentagon has at most two reflex vertices). Assume, w.l.o.g.,  $\angle p_1p_2A$  is less than  $\pi$ , then we can draw the polygon as in Figure 6(c). (c) Suppose  $i = 3$ . Then at most one of vertices  $p_1$  and  $p_2$  of the quadrangle  $Bp_1p_2A$  is reflex. Assume, w.l.o.g., that  $\angle p_1p_2A$  is less than  $\pi$ . Then we draw the polygon as in Figure 6(d). (Note that if  $A$  does not see  $p_2$ , then  $p_3$  is below the segment  $Ap_2$  and therefore  $\angle p_1p_2A$  is greater than  $\pi$ .)

**Case 3:** There are 3 vertices on the boundary of  $\mathcal{CH}_1(S)$ . If  $i = 2$  then by Observation A.2  $A$  or  $B$  sees  $p_1$ . If  $i = 1$ , then  $A$  and  $B$  sees  $p_0$  or  $p_1$ . Hence,  $A$  sees  $p_{i-1}$  or  $B$  sees  $p_1$ . We assume, w.l.o.g., that  $A$  sees  $p_{i-1}$ . Then we have the chain  $p_iCBAP_{i-1}$ . It remains to connect  $p_{i-1}$  to  $p_i$  through  $p_{i+1}$  (addition is modulo 3) and the two vertices  $q_1, q_2$  of the boundary of  $\mathcal{CH}_2(S)$ . Let  $q_1$  be the point such that  $\angle p_{i-1}p_{i+1}q_1 < \angle p_{i-1}p_{i+1}q_2$ . Then the chain  $p_{i-1}q_1p_{i+1}q_2p_i$  completes the desired polygon (see Figure 7).  $\square$

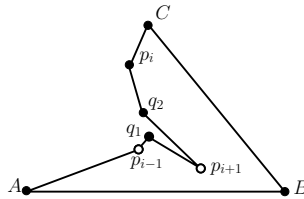


Figure 7: The case of 3 vertices on the boundary of  $\mathcal{CH}_1(S)$