Real Functions and Measures, BSM, Fall 2014

Assignment 8

1. Let $\mathcal{B}$ denote the Borel σ-algebra of $\mathbb{R}$. Construct a non-trivial positive measure μ on $(\mathbb{R}, \mathcal{B})$ that is concentrated on the Cantor set and has no atoms. Find the Lebesque decomposition of μ with respect to the Lebesgue measure.

2. Let μ be a signed measure. In class we defined its total variation measure in two different ways. Show that the two definitions coincide, that is,

$$\mu^+(E) + \mu^-(E) = \sup \left\{ \sum_{n=1}^{\infty} |\mu(E_n)| : E_n \text{ is a partition of } E \right\}$$

holds for any measurable set E. (Hint: use the Hahn decomposition $X = P \cup N$.)

3. Let μ_1 and μ_2 be finite positive measures on $(X, \mathcal{M})$.
Characterize the pairs (μ_1, μ_2) for which

$$(\mu_1 - \mu_2)^+ = \mu_1 \text{ and } (\mu_1 - \mu_2)^- = \mu_2.$$

4. a) Let λ be a complex measure and μ a positive measure on a measurable space $(X, \mathcal{M})$. Prove the following equivalence:

$$\lambda \ll \mu \iff (\forall \varepsilon > 0 \; \exists \delta > 0 : \mu(E) < \delta \Rightarrow |\lambda(E)| < \varepsilon).$$

(This explains why the notion is called *absolute continuity*.)
b) Does the same equivalence hold if λ is a signed measure that takes infinite values?

5. Let $(X, \mathcal{M}, \mu)$ be a measure space and $f \in L_1(\mu)$. Consider the complex measure ν defined by

$$\nu(E) = \int_E f \, d\mu \quad (E \in \mathcal{M}).$$

Prove that its total variation measure is

$$|\nu|(E) = \int_E |f| \, d\mu \quad (E \in \mathcal{M}).$$

(Hint: use the fact that the total variation measure is the smallest positive measure that "dominates" ν.)