Real Functions and Measures, BSM, Fall 2014

Assignment 2

1. Construct simple Borel measurable functions $f_n\colon \mathbb{R} \to \mathbb{R}$ such that they converge increasingly to the function

$$f(x) = \begin{cases} 1/|x| & \text{if } x \notin \mathbb{Q} \\ \infty & \text{if } x \in \mathbb{Q}. \end{cases}$$

Is f Borel measurable?

2. Let X be an uncountable set and let $\mathcal{M}$ be the σ-algebra consisting of the finite and countably infinite subsets of X and their complements. Let $\mu(E) = 0$ if $E \subset X$ is finite or countably infinite, and let $\mu(E) = 1$ if the complement of E is finite or countably infinite.
a) Prove that μ is a measure on the measurable space $(X, \mathcal{M})$.
b) Describe the measurable $X \to \mathbb{R}$ functions.
c) Determine the integrals of such measurable functions w.r.t. μ.

3. A sequence $(a_n)_{n=1}^{\infty}$ is said to be *eventually monotone* if there exists a positive integer n_0 such that $(a_n)_{n \geq n_0}$ is monotone. Let $(X, \mathcal{M})$ be a measurable space and let f_n be measurable $X \to \mathbb{R}$ functions. Prove that the set of points x for which the sequence $f_n(x)$ is eventually monotone is a measurable set.

4. Let f be the following $[0, 1) \to \mathbb{R}$ function. For any $x \in [0, 1)$ consider the binary representation of x: $0.a_1a_2a_3\ldots$ with each a_i being either 0 or 1. Then let $f(x)$ be the real number with decimal representation $0.a_1a_2a_3\ldots$, that is,

$$f(x) = \sum_{n=1}^{\infty} \frac{a_n}{10^n}.$$

Prove that f is Borel measurable.
(Hint: express f as the limit of Borel measurable simple functions.)

5. Let f be the following $[0, 1) \to \mathbb{R}$ function. For any $x \in [0, 1)$ consider the binary representation of x: $0.a_1a_2a_3\ldots$ with each a_i being either 0 or 1. Let $f(0) = 0$ and for $x > 0$ let $f(x)$ be the smallest positive integer k for which $a_k = 1$. Show that f is Borel measurable.

6. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f\colon X \to [0, \infty]$ be measurable. Suppose that for some measurable set $E \in \mathcal{M}$ with positive measure $\mu(E) > 0$ we have $f(x) > 0$ for all $x \in E$. Prove that

$$\int_E f \, d\mu > 0.$$