Real Functions and Measures, BSM, Fall 2014

Exercise sheet: Topological spaces and σ-algebras

1. Let (X, τ) be a topological space. For a non-empty subset $H \subset X$ we define $\tau|_H = \{G \cap H : G \in \tau\}$. Show that $(H, \tau|_H)$ is also a topological space.
$(H, \tau|_H)$ is called the *subspace* of (X, τ), and the elements of $\tau|_H$ are the *relative open sets* in H.

2. Find all possible homeomorphic pairs among the following subspaces of $\mathbb{R}$:

- $(0, 1)$;
- $[2, 5]$;
- $[-1, 3)$;
- $(1, 4]$;
- $\mathbb{R}$;
- $[0, \infty)$;
- $(-\infty, 1)$;
- the set of rational numbers;
- the set of irrational numbers;
- the set of integers;
- the set of positive integers;
- the set of positive rationals;
- the set of non-negative rationals.

3. Show that symbols $\perp$ and $<$ (as subspaces of $\mathbb{R}^2$) are not homeomorphic to each other.

4. Find all possible homeomorphic pairs among the following subspaces of $\mathbb{R}^2$:

- open disk: the set of all points of distance less than 1 from the origin;
- $(0, 1) \times (0, 1)$;
- $[0, 1] \times [0, 1]$;
- closed disk: the set of all points of distance at most 1 from the origin;
- circle: the set of all points of distance 1 from the origin;
- open linear segment;
- two intersecting lines;
- punctured circle: the set of all points of distance 1 from the origin except the point $(1, 0)$.

5.* Prove that the subspaces $\mathbb{Q} \subset \mathbb{R}$ and $\mathbb{Q}^2 \subset \mathbb{R}^2$ are homeomorphic topological spaces.

6. Let τ_e denote the usual (Euclidean) topology on $\mathbb{R}$, and τ_s the topology generated by the half closed intervals $[a, b)$; $a, b \in \mathbb{R}$. (τ_s is called the *Sorgenfrey* topology.)
a) Show that $(\mathbb{R}, \tau_e)$ and $(\mathbb{R}, \tau_s)$ are not homeomorphic spaces.
b) Show that the Borel σ-algebras are the same in the two spaces.

7. Let $\mathcal{M}$ denote the σ-algebra containing the countable subsets of $\mathbb{R}$ and their complements (the so-called *co-countable subsets*). Is $\sin(x)$ a measurable function with respect to $\mathcal{M}$?

8. Let $\mathcal{B}$ denote σ-algebra generated by the open intervals in $\mathbb{R}$. Describe the σ-algebra generated by
a) closed intervals;
b) half-open intervals $[a, b)$; $a, b \in \mathbb{R}$;
c) closed intervals with rational endpoints.

9. Describe the σ-algebra $\mathcal{M}$ generated by the one-element subsets $\{q\}$; $q \in \mathbb{Q}$.
a) Is there a function that is $\mathcal{B}$-measurable but not $\mathcal{M}$-measurable?
b) Is there a function that is $\mathcal{M}$-measurable but not $\mathcal{B}$-measurable?

10. What is the smallest σ-algebra $\mathcal{M}$ on $\mathbb{R}$ such that every monotone real function is measurable w.r.t. $\mathcal{M}$?