Invariant processes on infinite trees

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- Invariant processes over the d-regular infinite tree
- Gaussian Wave Functions
- Local convergence of graphs
- Random regular graphs
- Randomized local algorithms and Factor of IID processes
- Entropy inequalities

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Gaussian Wave Function:

a Gaussian process that satisfies the eigenvector equation

$$\sum_{u \in N(v)} X_u = \lambda X_v$$

for some eigenvalue λ. Such a process exists for any $\lambda \in [-d, d]$.

Local convergence of graphs

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Example: random regular graphs

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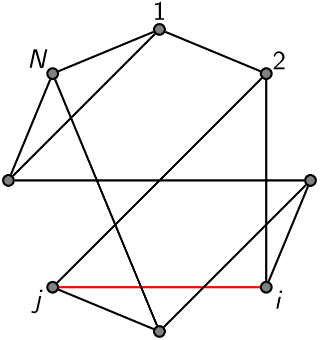

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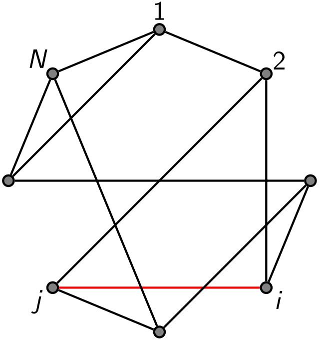

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Spectral properties of the adjacency matrix are closely related to various graph parameters.

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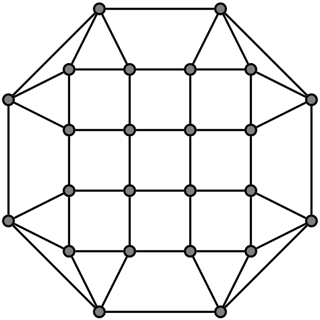
Questions

- Independence ratio for small d?
- Eigenvectors of the adjacency matrix? Delocalization?

Randomized local algorithms on finite graphs



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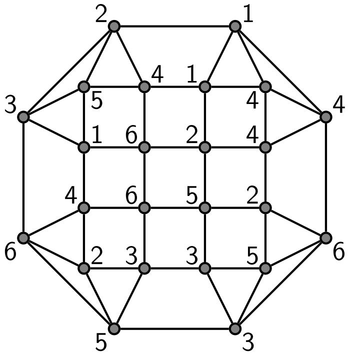

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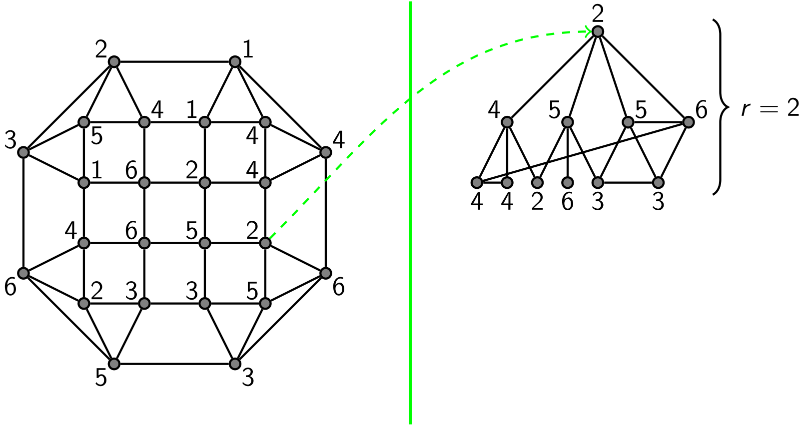

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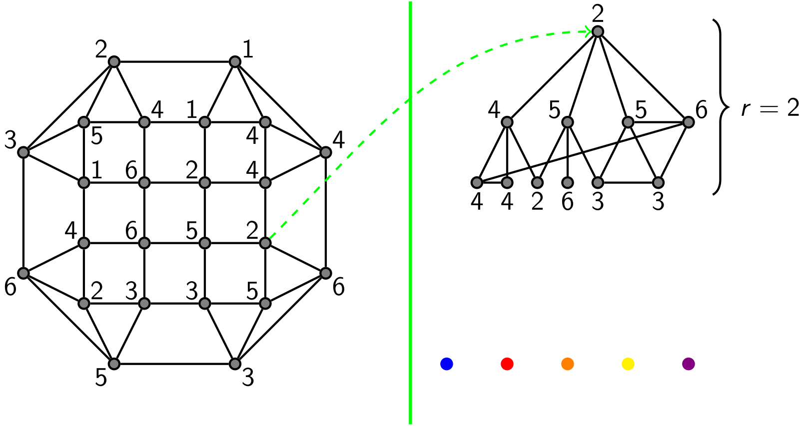

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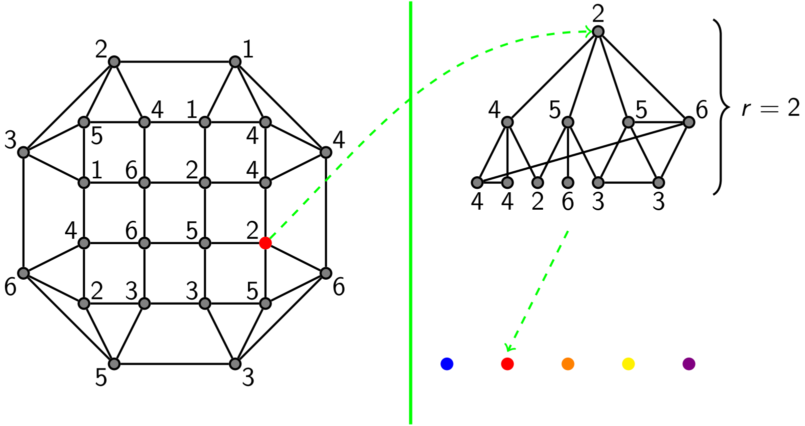

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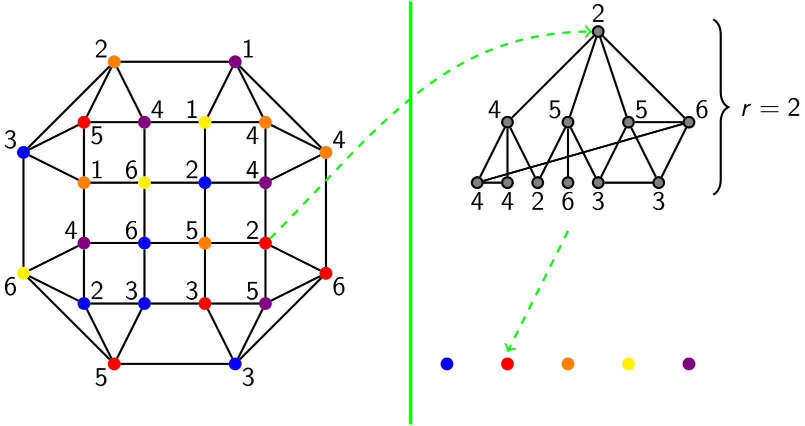

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Randomized local algorithms on large-girth graphs can be described by factor of IID processes. For the previous example: $M = \{\text{red}, \text{black}\}$ and

$$X_v = \begin{cases} \text{red} & \text{if } Z_v > Z_u \text{ for each neighbor } u \text{ of } v \\ \text{black} & \text{otherwise} \end{cases}$$

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Theorem (Harangi, Virág): Factor of IID processes are **not closed** under weak limits.

Independence ratio

Using the fact that the Gaussian Wave Function with eigenvalue $\lambda = -2\sqrt{d-1}$ is a weak limit of factor of IID processes one can construct factor of IID independent sets with large density:

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Theorem(Harangi, Virág)

There is a lower bound for the independence ratio of a vertex-transitive graph in terms of the smallest eigenvalue $\lambda_{\min}$ of its adjacency matrix.

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- "edge-vertex entropy inequality":

$$H\left(\overset{\bullet}{\underset{\bullet}{|}}\right) \geq \frac{2(d-1)}{d} H\left(\bullet\right)$$

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Theorem (Backhausz, Szegedy)

The eigenvectors of the adjacency matrix of the random d-regular graph G_N converge locally (in some sense) to Gaussian Wave Functions on T_d.
Consequence: The "local statistics" of the eigenvectors of a random regular graph are nearly Gaussian.

Key tool: **Entropy inequalities** for factors of IID and other invariant processes.

- "edge-vertex entropy inequality":

$$H\left(\bullet\!-\!\bullet\right) \geq \frac{2(d-1)}{d} H\left(\bullet\right)$$

- "star-edge entropy inequality":

$$H\left(\bullet\!<_{d}\right) \geq \frac{d}{2} H\left(\bullet\!-\!\bullet\right)$$

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The special considered in this talk

$\Gamma = \mathrm{Aut}(T_d)$ and $S = V(T_d)$.