Tube-null sets

Viktor Harangi

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That is, for all $\varepsilon > 0$ there exist strips $T_1, T_2, \ldots$ such that

$$E \subset \bigcup_{i=1}^{\infty} T_i \quad \text{and} \quad \sum_{i=1}^{\infty} w(T_i) < \varepsilon.$$

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The study of tube-null sets was initiated by Carbery, Soria és Vargas in connection with the divergence sets of the localisation problem.

Simple observations

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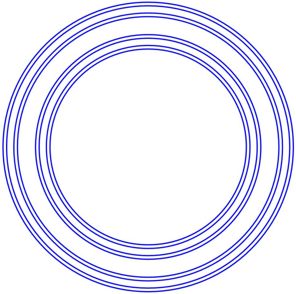
  In particular: purely unrectifiable 1-sets are tube-null.
- Every set of σ-finite $\mathcal{H}^1$-measure is tube-null.
- For a set $H \subset [1,2]$ let $E_H = \{x \in \mathbb{R}^2 : |x| \in H\}$.



  - $\dim(H) < 1/2 \Rightarrow E_H$ is tube-null.
  - $\dim(H) > 1/2 \Rightarrow E_H$ is **not** tube-null.

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One possible method

To construct a measure μ concentrated on E such that

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Question: is there such a measure for any set E that is not tube-null?

Fractal percolations

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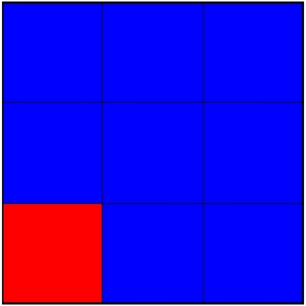

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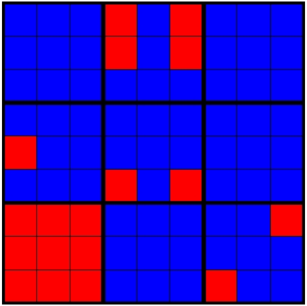

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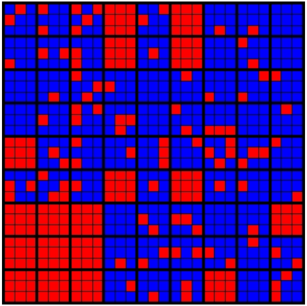

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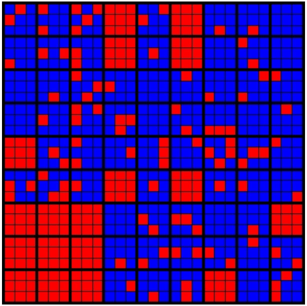


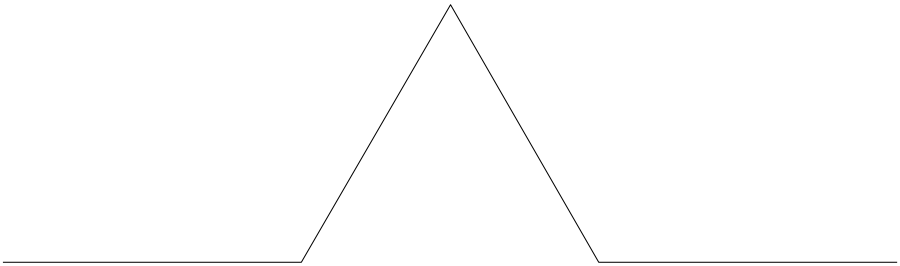
Shmerkin, Suomala

There exist sets of Hausdorff-dimension 1 that are not tube-null.

The Koch snowflake curve

For many concrete fractals it is hard to tell if they are tube-null.

The Koch snowflake curve



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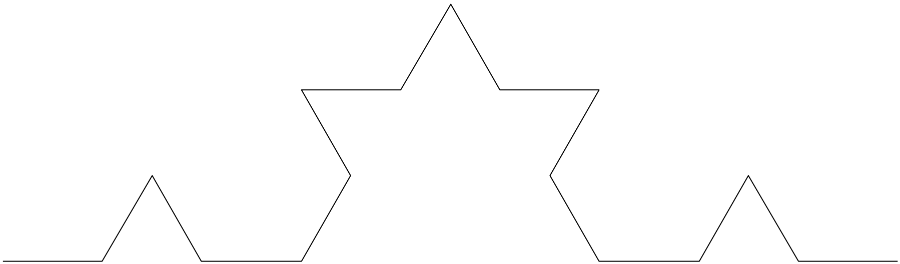

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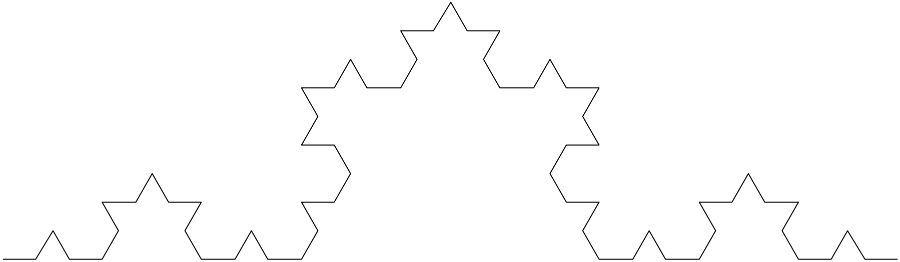

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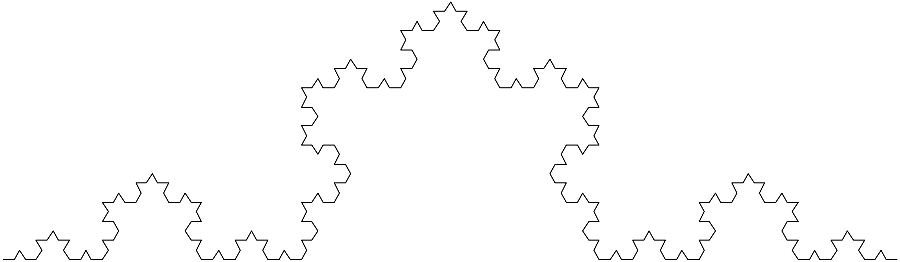

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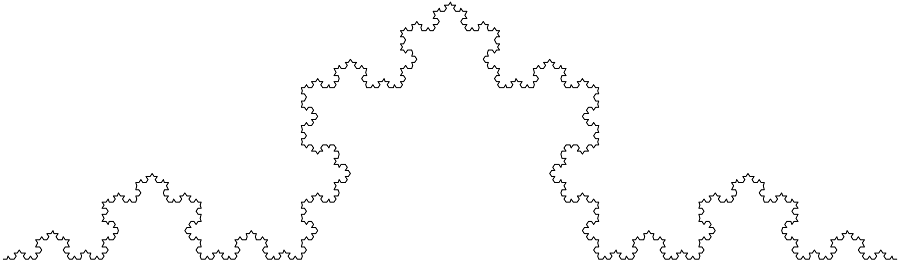

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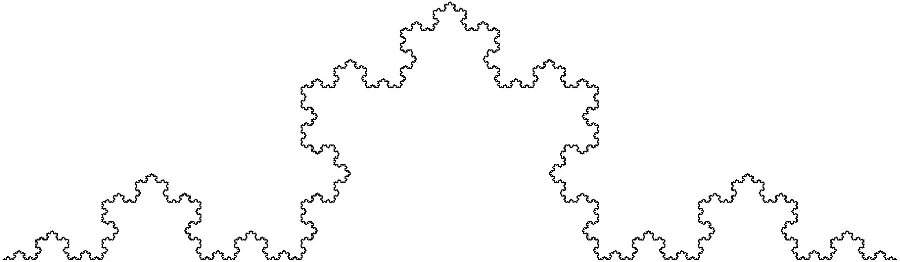

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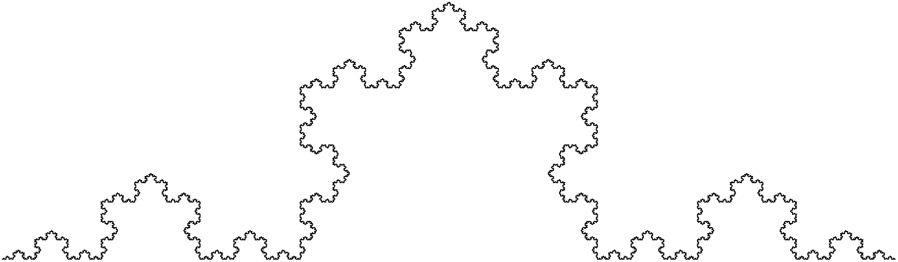

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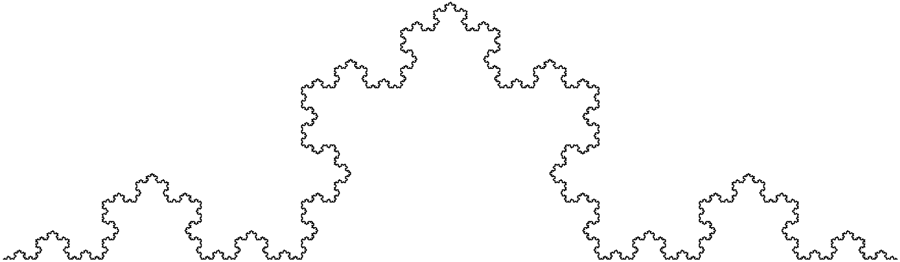

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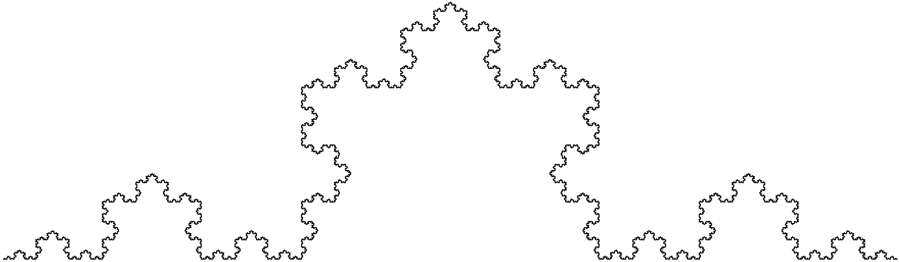

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The snowflake curve is tube-null, that is, it can be covered by strips of arbitrarily small total width.
Moreover, the snowflake curve K has a decomposition $K = K_0 \cup K_1 \cup K_2$ with corresponding projections π_0, π_1, π_2 such that the Hausdorff dimension of $\pi_i(K_i)$ is less than 1 for each $i = 0, 1, 2$.

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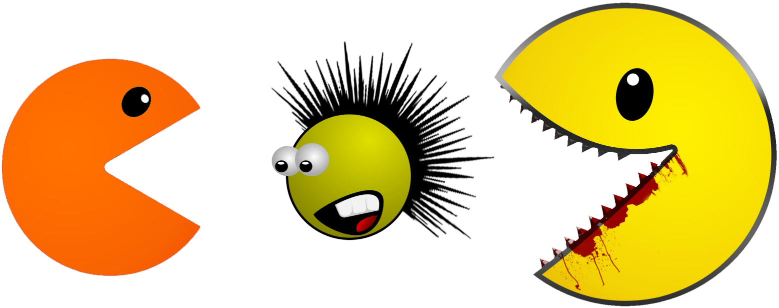

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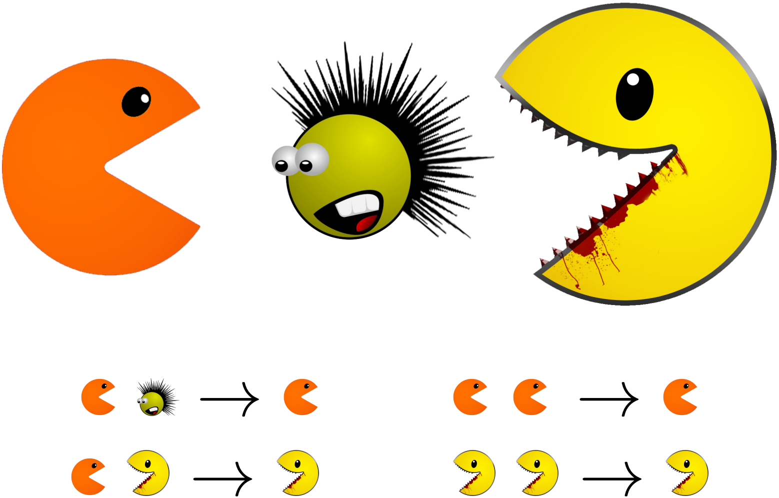

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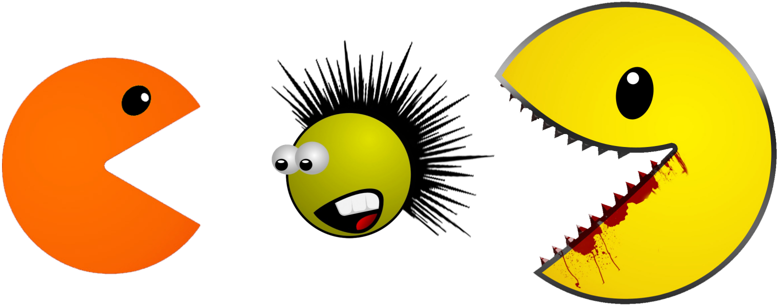

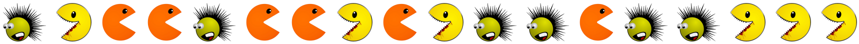

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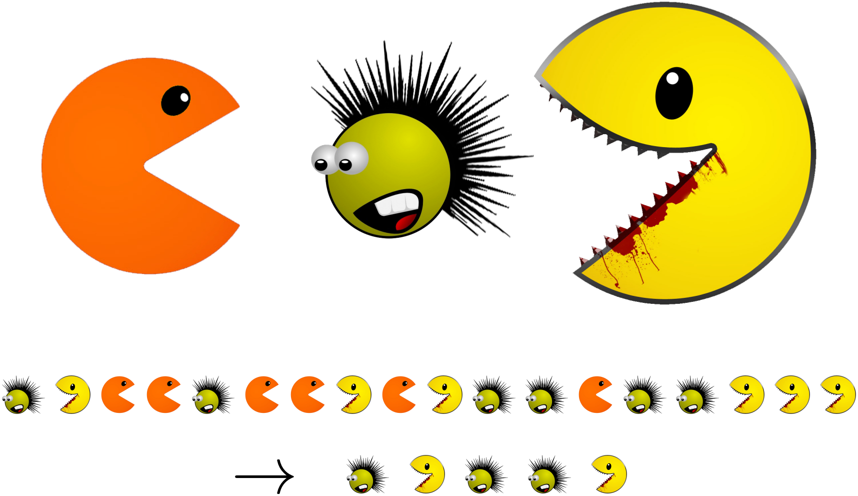

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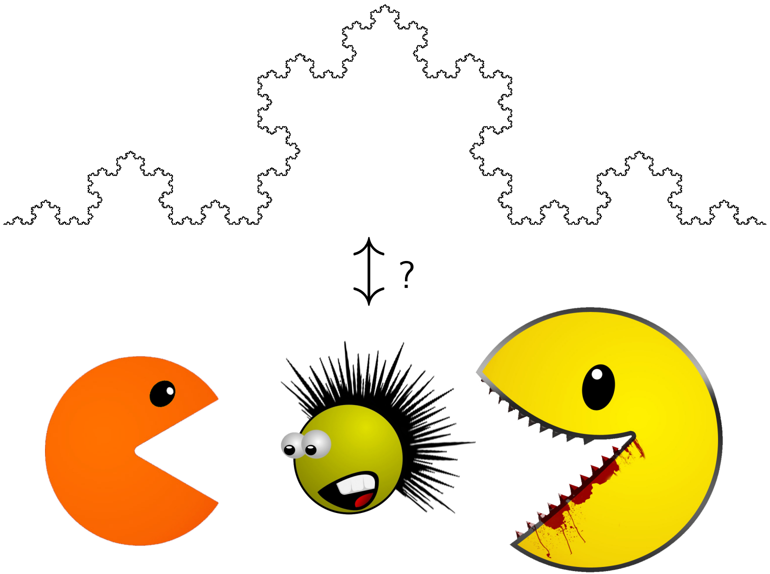

Covering numbers

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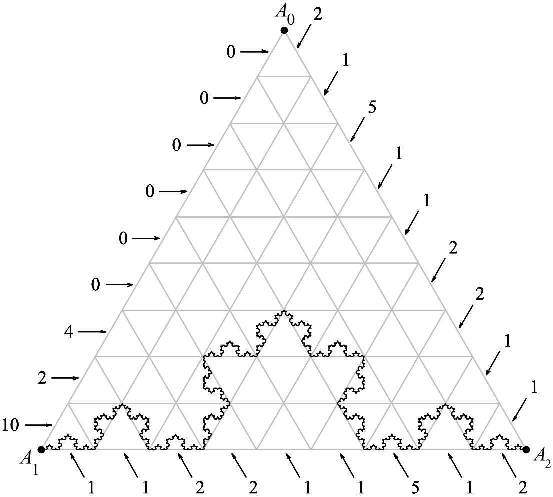

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Corollary

It holds for every level-n piece that one of the three level-n strips *going through it* has covering number at least $2^{n/3}$.

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- BUT: how can we determine these covering numbers?

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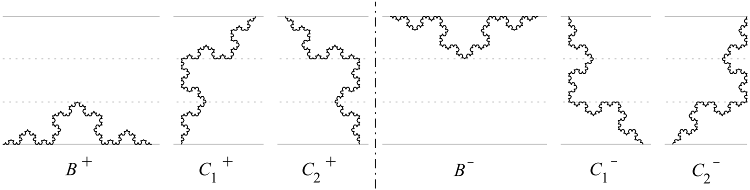

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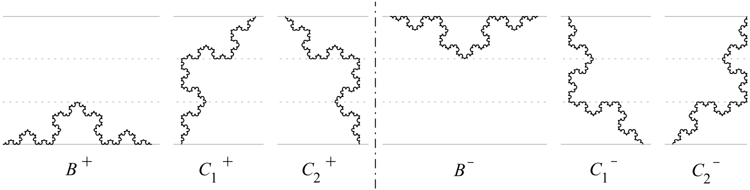

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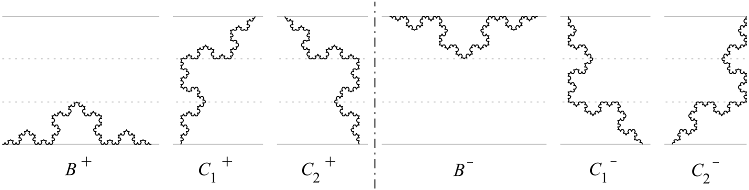

- Two *orientations*.
- *Crossing* and *border* pieces.

Covering vectors

To every piece we associate a *covering vector* (v_1, v_2):

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That is: the next-level covering vectors can be obtained by multiplying by the following 2×2 matrices from the right:

$$A = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}; \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}; \quad C = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

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- Then the covering numbers of the level-n strips:

$$\mathbf{v} M_1 M_2 \cdots M_n \begin{pmatrix} 1 & 1 \end{pmatrix}^{\mathrm{T}},$$

where $M_i \in \{A, B, C\}; i = 1, 2, \ldots, n$.

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- After all possible cancellations we have:

$$(C)A^{k_1}CA^{k_2}C\cdots CA^{k_r}(B \text{ or } C).$$

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- Covering number: $\leq 2^{c_0 + reduced\_length}$.

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- The reduced length of the sequence is the number of letters after the cancellations.

New problem

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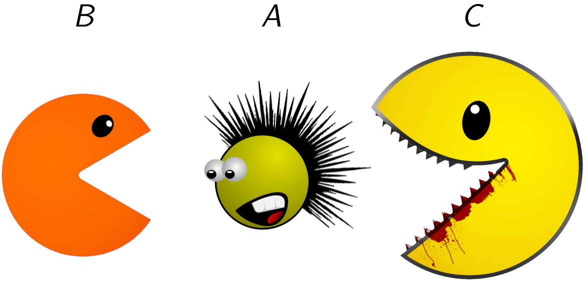
Proposition

There exists a constant $a < 1$ such that

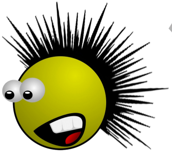
$$\mathbf{P}(\text{the reduced length is at least } n/3 - c_0) < a^n.$$

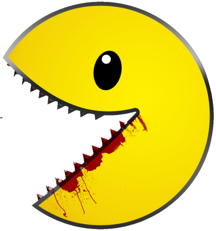
The probabilities of survival



The probabilities of survival

B A C

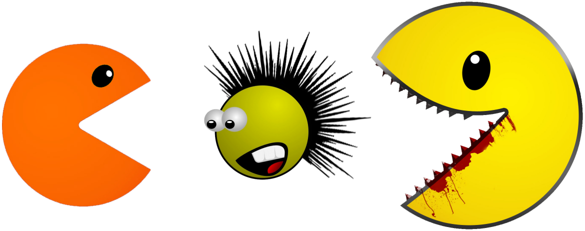




0 $\frac{1}{2}$ $\frac{1}{3}$

The probabilities of survival

B A C



0 $\frac{1}{2}$ $\frac{1}{3}$

The probability of survival: $\frac{1}{3} \cdot \left(0 + \frac{1}{2} + \frac{1}{3}\right) = \frac{5}{18} < \frac{1}{3}$

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