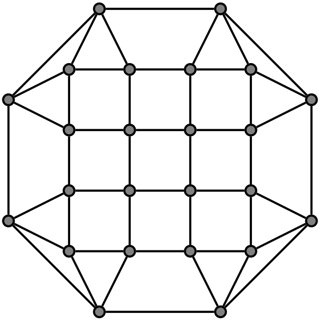
Correlation bounds and one-ended tail triviality for factors of IID on trees

Viktor Harangi

joint work with Ágnes Backhausz, Balázs Gerencsér, and Máté Vizer

21st April 2016

Motivation: randomized local algorithms on graphs



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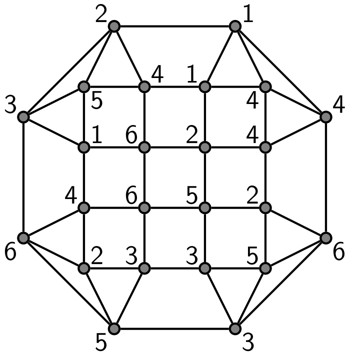

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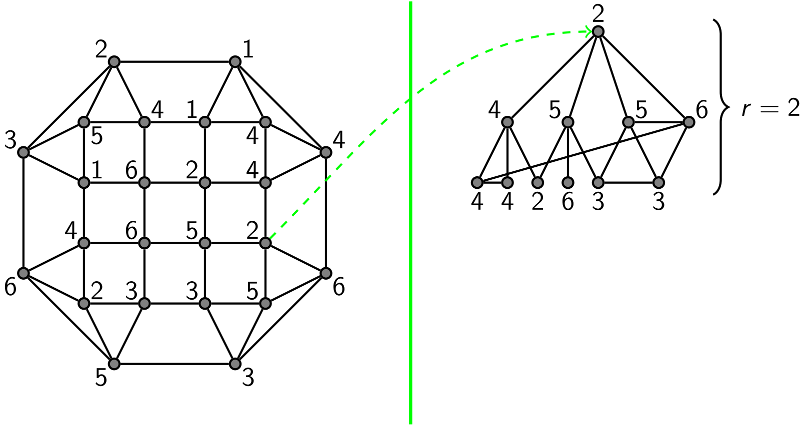

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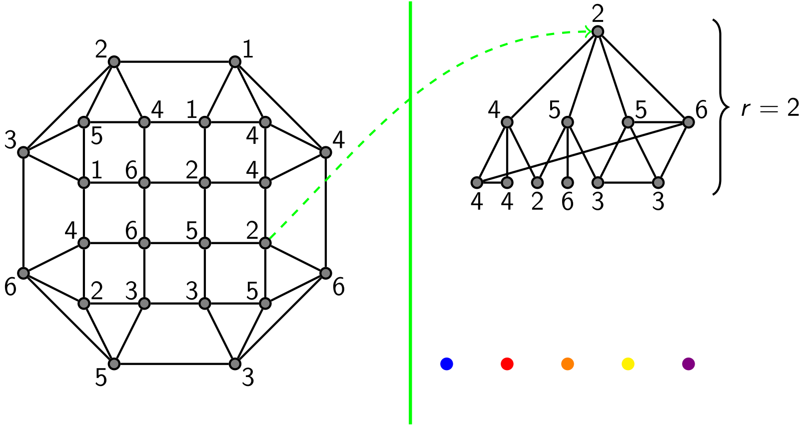

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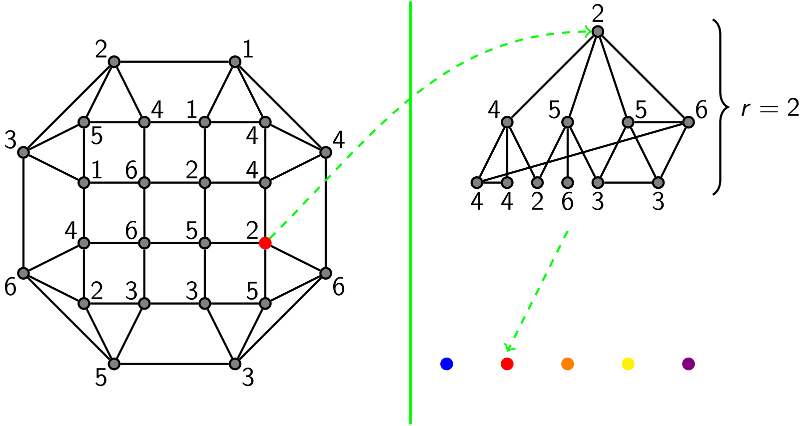

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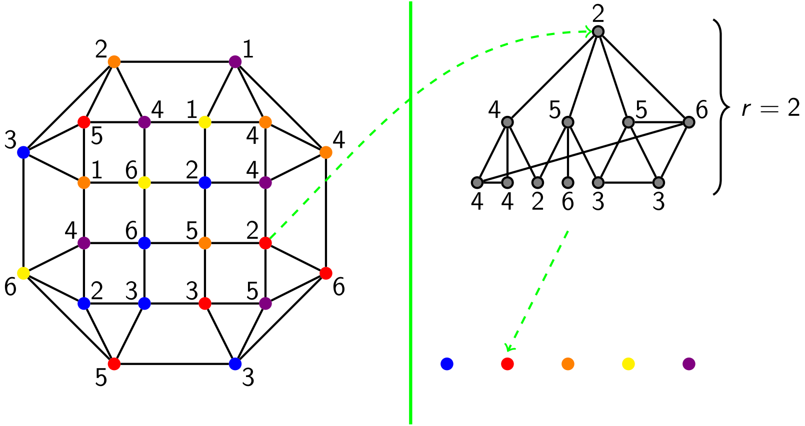

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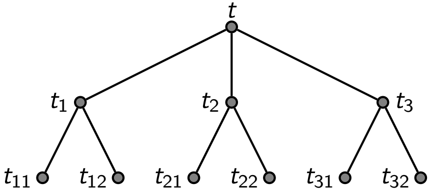

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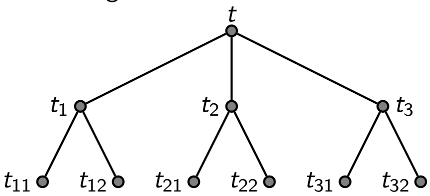


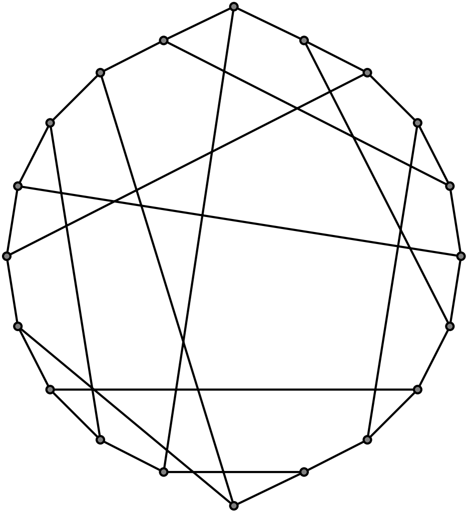
Apply the following local rule

- red if all three inequalities below are satisfied:

$$t - t_1 - t_2 - t_3 > t_1 - t - t_{11} - t_{12} \text{ and}$$
$$t - t_1 - t_2 - t_3 > t_2 - t - t_{21} - t_{22} \text{ and}$$
$$t - t_1 - t_2 - t_3 > t_3 - t - t_{31} - t_{32};$$

- black, otherwise.

Let's see what happens



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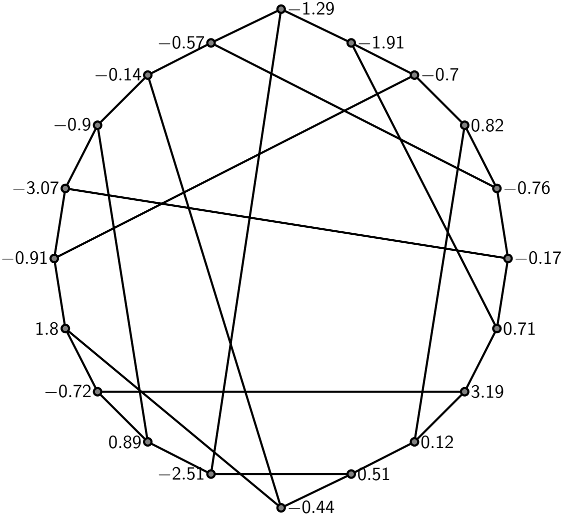

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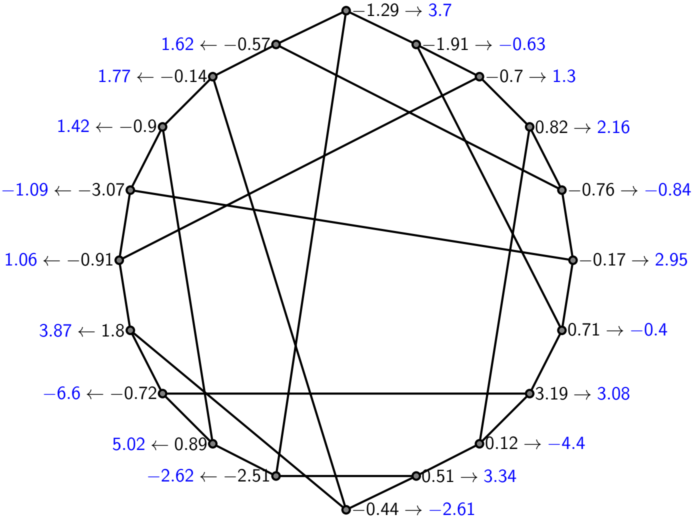

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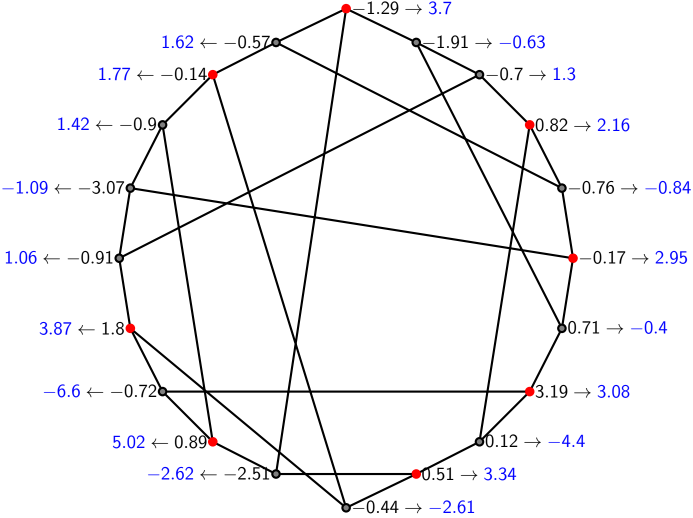

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For this one needs to study processes on the limit graph...

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The special considered in this talk

$\Gamma = \mathrm{Aut}(T_d)$ and $S = V(T_d)$.

Tail σ-algebras

Let $\Omega = M^{V(T_d)}$ and for $v \in V(T_d)$ let $\pi_v \colon M^{V(T_d)} \to M$ denote the natural coordinate projection. For $V \subseteq V(T_d)$ let $\sigma(V)$ be the σ-algebra generated by the maps π_v, $v \in V$.

Tail

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The *1-ended tail σ-algebra* corresponding to an infinite simple path $(v_0, v_1, v_2, \ldots)$ is $\bigcap_n \sigma(D_n)$, where D_n is the set of vertices closer to v_n than to v_{n-1}.

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A σ-algebra is said to be *trivial* w.r.t. a probability measure if it contains only sets of measure 0 or 1.
It is easy to see that for an $\mathrm{Aut}(T_d)$-invariant measure μ on $M^{V(T_d)}$ the 1-ended tails are all trivial or none are trivial.

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- Szegedy’s “sparse tail” for discrete M?

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It is said to be *ergodic* (or Γ-*ergodic*) if for any measurable, Γ-invariant $A \subset \Omega$ it holds that $\mu(A) = 0$ or $\mu(A) = 1$.
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Let $V(T_d) = V_0 \cup V_1$ be the partition of $V(T_d)$ into *even and odd vertices*, and consider the process that is $\mathbb{1}_{V_0}$ with probability $1/2$ and $\mathbb{1}_{V_1}$ also with probability $1/2$.

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Instead: consider the group $\mathrm{Aut}_+(T_d)$ of *parity-preserving* automorphisms of T_d; this is a subgroup of $\mathrm{Aut}(T_d)$ of index 2.

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That is, any measurable, Γ-invariant $\Omega \to \mathbb{R}$ function is μ-a.e. constant.

Question: Does $\mathrm{Aut}(T_d)$-ergodicity imply 1-ended tail triviality? Not quite!
Let $V(T_d) = V_0 \cup V_1$ be the partition of $V(T_d)$ into *even and odd vertices*, and consider the process that is $\mathbb{1}_{V_0}$ with probability $1/2$ and $\mathbb{1}_{V_1}$ also with probability $1/2$.
Instead: consider the group $\mathrm{Aut}_+(T_d)$ of *parity-preserving* automorphisms of T_d; this is a subgroup of $\mathrm{Aut}(T_d)$ of index 2.
Any $\mathrm{Aut}(T_d)$-ergodic process is the *equal mixture* of two $\mathrm{Aut}_+(T_d)$-ergodic processes.

A result of Pemantle

Theorem(Pemantle, 1992)

Let $\Gamma_+ = \mathrm{Aut}_+(T_d)$ and let μ be an Γ_+-invariant process on $\Omega = M^{V(T_d)}$.
If μ is Γ_+-ergodic, then it is 1-ended tail trivial.

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$$\mathbf{E}_\mu \left(\mathbb{1}_B | \sigma(\alpha)\right) \text{ is } \mu\text{-a.e. constant.}$$

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For the sake of simplicity, we will only consider events that depend only on one coordinate:

$$B = \{\omega : \omega_v \in A\} \text{ for some vertex } v \in V(T_d) \text{ and a measurable set } A \subset M.$$

Definitions

The boundary of T_d

- the boundary ∂T_d is defined as the set of ends in T_d;
- there is a natural topology (and the corresponding Borel σ-algebra) on T_d;
- for any vertex $v \in V(T_d)$ a probability measure m_v can be defined on ∂T_d; essentially the "uniform measure" as seen from v;
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Horocycles

The level sets of h_α are called the *horocycles* in direction α.
The horocycle through v in direction α is denoted by $C_\alpha(v)$.

Sketch of the proof

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Γ_+-ergodicity + Claim $\implies$ Theorem

We get two Γ_+-invariant functions: $g_{\mathrm{even}} \colon \Omega \to \mathbb{R}$ and $g_{\mathrm{odd}} \colon \Omega \to \mathbb{R}$. If μ is ergodic, they must be μ-a.e. constant.

Proof of the claim, I

Fix v and let $v_0, v_1, v_2 \ldots$ be a non-backtracking random walk started at $v_0 = v$, so the corresponding (random) end α has distribution m_v.

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$$G(\omega, v, v_k) := \mathbf{E}_{\mathrm{rw}}\left(g(\omega, v, \alpha) | v_0, v_1, \ldots, v_k\right) \text{ is a bounded martingale,}$$

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Thus for any fixed ω and $\varepsilon > 0$:

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Now we fix $N \geq N(\varepsilon)$ and consider two such random walks $v_0, v_1, \dots$ and $u_0, u_1, \dots$ (with corresponding ends α and β) coupled in a way that $v_k = u_k$ holds if and only if $k \leq N$. Then

$$\begin{aligned} \mathbf{P}_{\mu,\mathrm{rw}}&\left(|g(\omega, v, \alpha) - g(\omega, v, \beta)| > 2\varepsilon\right) \\ &\leq \mathbf{P}_{\mu,\mathrm{rw}}\left(|G(\omega, v, v_N) - g(\omega, v, \alpha)| > \varepsilon\right) \\ &\quad + \mathbf{P}_{\mu,\mathrm{rw}}\left(|G(\omega, v, v_N) - g(\omega, v, \beta)| > \varepsilon\right) < \varepsilon + \varepsilon = 2\varepsilon. \end{aligned}$$

Proof of the claim, II

So we saw that $\mathbf{P}_{\mu,\mathrm{rw}}\left(|g(\omega, v, \alpha) - g(\omega, v, \beta)| > 2\varepsilon\right) < 2\varepsilon,$

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If we consider g as a $\Omega \times \{\text{horocycles}\} \to \mathbb{R}$ function, then for horocycles C_1 and C_2 with sufficiently large intersection $g(\omega, C_1)$ and $g(\omega, C_2)$ are arbitrarily likely to be arbitrarily close to each other.

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However, it is easy to see that for any two horocycles C and C' of the same parity, there is an integer m such that for any N there exists a chain of horocycles $C_0 = C, C_1, \ldots, C_m = C'$ with the property that C_{i-1} and C_i share at least $(d-2)(d-1)^{N-1}$ vertices.

Correlation bounds

Let $X = (X_v)_{v \in V(T_d)}$ be a factor of IID process on $\mathbb{R}^{V(T_d)}$. A natural question is "how independent" the random variables X_v are.

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$$|\mathrm{corr}(X_u, X_v)| \le \left(k + 1 - \frac{2k}{d}\right)\left(\frac{1}{\sqrt{d-1}}\right)^k, \text{ where } k = \mathrm{dist}(u, v),$$

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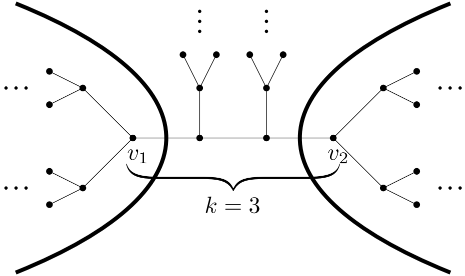
Do we really need connectedness? Yes!

Outline of the proof

Let $\tilde{T}_{d-1}$ denote the rooted $(d-1)$-ary tree.

- Step 1: we might assume that V_1 and V_2 are both isomorphic to $\tilde{T}_{d-1}$ and their roots have distance k.



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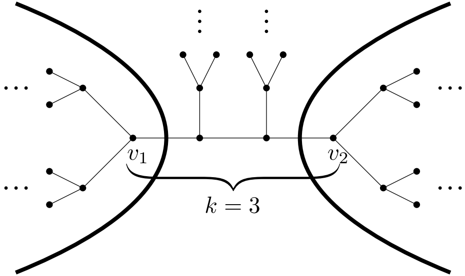

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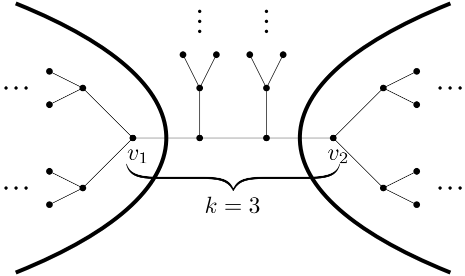

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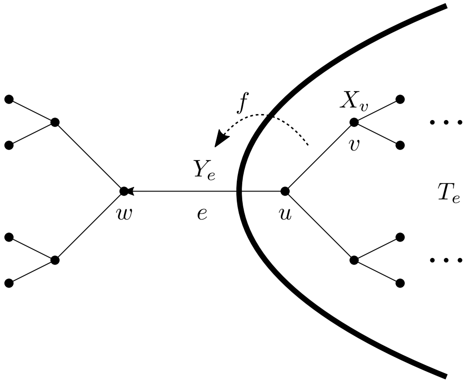
- Step 3: in fact, they should "come from" the same $\mathrm{Aut}(\tilde{T}_{d-1})$-invariant measurable function $f\colon M^{V(\tilde{T}_{d-1})} \to \mathbb{R}$.

Outline of the proof

- Step 4: given such a function f, for any directed edge e of T_d, apply f to the (labelled) subgraph "behind" e and write its value on e. The process we obtain on the directed edge set $E(T_d)$ will be a factor of IID process.



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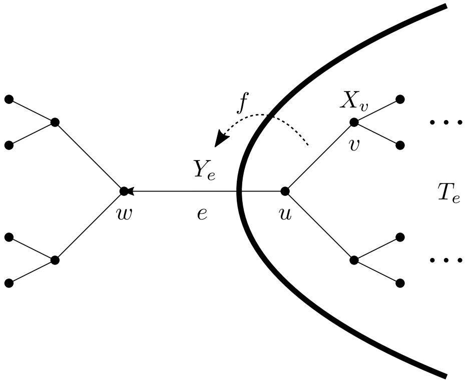

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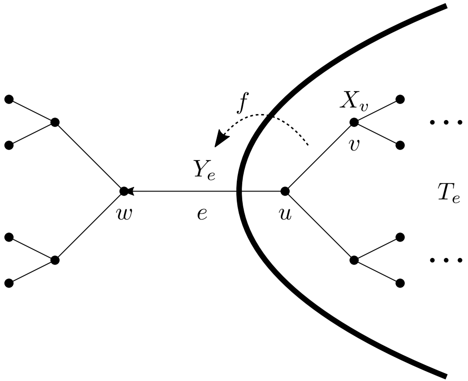

- Step 5: we prove a a correlation decay result similar to Backhausz-Szegedy-Virág but for directed edges instead of vertices.
- Step 6: to finish Step 5 we need to estimate the norms of the powers of the non-backtracking operator.

Step 3: a lemma

Let $(A, \mathcal{F})$ be an arbitrary measurable space. Suppose that

- X_1, X_2 are $(A, \mathcal{F})$-valued random variables;
- X_1 and X_2 are exchangeable,
  that is, (X_1, X_2) and (X_2, X_1) have the same joint distribution;
- there exists a constant $\alpha \geq 0$ with the property that for any measurable $f : A \to \mathbb{R}$ we have

  $$\left|\mathrm{corr}\left(f(X_1), f(X_2)\right)\right| \leq \alpha \qquad (*)$$

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Then for any measurable functions $f_1, f_2 \colon A \to \mathbb{R}$

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provided that $f_1(X_1)$ and $f_2(X_2)$ have finite variances.

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- It follows that

$$\mathrm{corr}\left(f_1(X_1), f_2(X_2)\right) = \mathrm{cov}\left(f_1(X_1), f_2(X_2)\right)$$
$$= \frac{1}{4}\Big(\mathrm{cov}\left((f_1+f_2)(X_1), (f_1+f_2)(X_2)\right) - \mathrm{cov}\left((f_1-f_2)(X_1), (f_1-f_2)(X_2)\right)\Big).$$

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- It follows that

$$\begin{aligned} &\mathrm{corr}\left(f_1(X_1), f_2(X_2)\right) = \mathrm{cov}\left(f_1(X_1), f_2(X_2)\right) \\ &= \frac{1}{4}\Big(\mathrm{cov}\left((f_1+f_2)(X_1), (f_1+f_2)(X_2)\right) - \mathrm{cov}\left((f_1-f_2)(X_1), (f_1-f_2)(X_2)\right)\Big). \end{aligned}$$

- Using the triangle inequality and applying $(*)$ to the function $f = f_1 + f_2$ and to $f = f_1 - f_2$ we obtain that

$$\begin{aligned} \left|\mathrm{corr}\left(f_1(X_1), f_2(X_2)\right)\right| &\leq \frac{\alpha}{4}\Big(\mathrm{var}\left((f_1+f_2)(X_1)\right) + \mathrm{var}\left((f_1-f_2)(X_1)\right)\Big) \\ &= \frac{\alpha}{4}\Big(2\,\mathrm{var}\left(f_1(X_1)\right) + 2\,\mathrm{var}\left(f_2(X_1)\right)\Big) = \alpha. \end{aligned}$$