Functional Analysis, BSM, Spring 2012

Extra problems

1. Suppose that $x \in \ell_{p_0}$ for some $1 < p_0 < \infty$. Prove that $\|x\|_p \to \|x\|_\infty$ as $p \to \infty$.

2. Prove that any $\Lambda \in \ell_\infty^*$ can be uniquely written as

$$\Lambda = \Lambda_y + \Lambda',$$

where $y \in \ell_1$ and $\Lambda' \in \ell_\infty^*$ vanishes on c_0 (that is, $\Lambda' x = 0$ for any $x \in c_0$).

3. a) Let $(X, \|\cdot\|)$ be a normed space, $Y \leq X$ a linear subspace. The distance of $x \in X$ from Y is defined as

$$d(x, Y) \stackrel{\text{def}}{=} \inf_{y \in Y} \|x - y\|.$$

Let x_0 be in $X \setminus Y$ with $d(x_0, Y) > 0$ (in other words, $x_0 \notin \text{cl}(Y)$). Prove that there exists $\Lambda \in X^*$ such that $\Lambda y = 0$ for any $y \in Y$, $\Lambda x_0 = d(x_0, Y)$ and $\|\Lambda\| = 1$.

b) Prove that if X^* is separable, then so is X. Show an example where the converse is not true.

4. Let $(X, \|\cdot\|)$ be a normed space.

a) Show that if X is finite dimensional, then so is X^*.

b) Show that if X^* is finite dimensional, then so is X.

5. Prove that a Banach space X is reflexive if and only if X^* is reflexive.

6. Let X and Y be Banach spaces and $T \in B(X, Y)$ a bounded operator from X to Y. The *Banach space adjoint* of T is the operator T^* from Y^* to X^* defined by

$$(T^* \Lambda)x = \Lambda(Tx) \text{ for any } \Lambda \in Y^*, x \in X.$$

In other words, for $\Lambda \in Y^*$ let $T^*\Lambda$ be the functional that maps x to $\Lambda(Tx)$.

Show that $T^* \in B(Y^*, X^*)$, that is, T^* is bounded. Also show that $\|T^*\| = \|T\|$.

7. Let $X = Y = \ell_1$ and let T be the right shift operator

$$T(\alpha_1, \alpha_2, \ldots) = (0, \alpha_1, \alpha_2, \ldots).$$

We identified ℓ_1^* with ℓ_∞. So T^* can be viewed as an $\ell_\infty \to \ell_\infty$ operator. Which is it?

8. Find continuumly many vectors in ℓ_p that are linearly independent.

9. a) Let $p, q > 1$ be real numbers with $1/p + 1/q = 1$. Prove that for any positive real numbers a, b:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Hint: since log is a concave function, $\log(ta^p + (1-t)b^q) \geq t\log(a^p) + (1-t)\log(b^q)$ for $t \in (0,1)$.

b) Prove the Hölder inequality: if $(\alpha_1, \alpha_2, \ldots) \in \ell_p$ and $(\beta_1, \beta_2, \ldots) \in \ell_q$, then

$$\sum_{i=1}^{\infty} |\alpha_i \beta_i| \leq \sqrt[p]{\sum_{i=1}^{\infty} |\alpha_i|^p} \cdot \sqrt[q]{\sum_{i=1}^{\infty} |\beta_i|^q}.$$

10. Let $p, q, r > 1$ with $1/p + 1/q + 1/r = 1$. Suppose that $(\alpha_1, \alpha_2, \ldots) \in \ell_p$, $(\beta_1, \beta_2, \ldots) \in \ell_q$ and $(\gamma_1, \gamma_2, \ldots) \in \ell_r$. Prove that

$$\sum_{i=1}^{\infty} |\alpha_i \beta_i \gamma_i| \leq \sqrt[p]{\sum_{i=1}^{\infty} |\alpha_i|^p} \cdot \sqrt[q]{\sum_{i=1}^{\infty} |\beta_i|^q} \cdot \sqrt[r]{\sum_{i=1}^{\infty} |\gamma_i|^r}.$$

11. Let X denote the space of continuous functions on $[0,1]$; X is a Banach space with the supremum norm:

$$\|f\| = \sup_{t\in[0,1]} |f(t)| = \max_{t\in[0,1]} |f(t)|.$$

Consider the following subset of X:

$$Y = \left\{ g : g(0) = 0 \text{ and } \int_0^1 g(t)\,\mathrm{d}t = 0 \right\}.$$

Show that Y is a closed linear subspace of X. Let $f \in X$ be the function for which $f(t) = t$; $0 \le t \le 1$. Determine $d(f, Y)$. Show that $d(f,g) > d(f,Y)$ for any $g \in Y$.

12. a) We say that two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on the same vector space X are *equivalent* if there exist $c, C > 0$ such that $c\|x\|_1 \le \|x\|_2 \le C\|x\|_1$ for any $x \in X$. Prove that any two norms on a finite dimensional vector space are equivalent.
b) Prove that every finite dimensional normed space is complete.

13. Let X be a Banach space, $T \in B(X)$ and $T^* \in B(X^*)$ the Banach space adjoint of T.
a) Prove that $\operatorname{ran} T$ is not dense if and only if T^* is not injective.
b) Prove that if $\lambda \in \sigma_r(T)$, then $\lambda \in \sigma_p(T^*)$.
c) Prove that if T is not injective, then $\operatorname{ran} T^*$ is not dense.
d) Prove that if $\lambda \in \sigma_p(T)$, then $\lambda \in \sigma_p(T^*) \cup \sigma_r(T^*)$.

14. a) Let X be a Banach space, $S, T \in B(X)$. Show that $(ST)^* = T^* S^*$.
b) Prove that $\sigma(T^*) \subset \sigma(T)$. (In fact, $\sigma(T^*) = \sigma(T)$.)

15. Let X be a non-trivial normed space. Suppose that $ST - TS = I$ for some linear operators $S, T : X \to X$. Prove that at least one of the operators S, T is unbounded.

16. Let

$$U = \{(\alpha_1, \alpha_2, \ldots) : \alpha_i \in \mathbb{C} \text{ and } \alpha_i = 0 \text{ for all but finitely many } i\text{'s}\}.$$

This is clearly a linear subspace of ℓ_1. Then U has an algebraic complement V: another linear subspace with $U + V = \ell_1$ and $U \cap V = \{0\}$. (Actually, there are a lot of such V; we pick an arbitrary one.) We define a linear functional Λ on ℓ_1: any $z \in \ell_1$ can be uniquely written as $z = u + v$, where $u = (\alpha_1, \alpha_2, \ldots) \in U$ and $v \in V$. Let

$$\Lambda z = \sum_{i=1}^{\infty} \alpha_i.$$

Prove that Λ is not bounded.

17. Let X be a separable normed space and $\Lambda_n \in X^*$. Suppose that there exists C such that $\|\Lambda_n\| \le C$ for all n. Prove that there exist a subsequence Λ_{n_k} and a $\Lambda \in X^*$ such that for all $x \in X$

$$\Lambda_{n_k} x \to \Lambda x \text{ as } k \to \infty.$$

Is this true for non-separable spaces?

18. Let $(X, \|\cdot\|)$ be a normed space and $M \le X$ a linear subspace. In linear algebra we defined the quotient space X/M: it is a vector space, the elements of which are equivalence classes of X with respect to the relation $x \sim y \Leftrightarrow x - y \in M$. The equivalence class of x is usually denoted by $[x]$.
a) Suppose that M is a closed linear subspace of X. Prove that

$$\|[x]\| \stackrel{\text{def}}{=} d(x, M)$$

is well-defined and is a norm on the quotient space X/M.
b) Prove that if X is a Banach space and $M \le X$ is closed, then X/M is a Banach space with the above norm.

19. Let M° denote the following subspace of X^*:

$$M^\circ \stackrel{\text{def}}{=} \{\Lambda \in X^* : \Lambda x = 0 \text{ for all } x \in M\}.$$

Prove that if $M \le X$ is closed, then M^* is isometrically isomorphic to X^*/M°. Also prove that $(X/M)^*$ is isometrically isomorphic to M°.

20. Let X be a Banach space. Show that X is finite dimensional if and only if the closed unit ball $\bar{B}_1(0) = \{x \in X : \|x\| \le 1\}$ is compact.