Functional Analysis, BSM, Spring 2012

Midterm exam, March 26

Solutions

1. Let $x = (\alpha_1, \alpha_2, \ldots)$; $\alpha_i \in \mathbb{C}$. Since

$$\left|\frac{\alpha_1 + \cdots + \alpha_n}{n}\right| \le \frac{|\alpha_1| + \cdots + |\alpha_n|}{n} \le \frac{n \cdot \|x\|_\infty}{n} = \|x\|_\infty,$$

it follows that $\|Tx\|_\infty \le \|x\|_\infty$, thus $\|T\| \le 1$. For $x = (1, 1, 1, \ldots)$ we have $Tx = x$, so $\|T\| = 1$.

We claim that T is injective. Suppose that $Tx = 0$. Clearly, $\alpha_1 = 0$. Then $(\alpha_1 + \alpha_2)/2 = \alpha_2/2 = 0$. Then $(\alpha_1 + \alpha_2 + \alpha_3)/3 = \alpha_3/3 = 0$. We get by induction that $\alpha_n = 0$ for all n.

We claim that $y = (1, 0, 1, 0, 1, 0, \ldots) \notin \operatorname{ran} T$, thus T is not surjective. Assume that there exists $x \in \ell_\infty$ such that $Tx = y$. Then $\alpha_1 = 1$, $\alpha_2 = -1$, $\alpha_3 = 3$, $\alpha_4 = -3$, $\alpha_5 = 5$, $\alpha_6 = -5$, and so on. By induction: $\alpha_{2k+1} = 2k + 1$ and $\alpha_{2k+2} = -(2k + 1)$, which contradicts $x \in \ell_\infty$.

2. Suppose that $x = (\alpha_1, \alpha_2, \ldots) \in \ell_p$. It means that

$$\sum_{i=1}^{\infty} |\alpha_i|^p < \infty.$$

This implies that $|\alpha_i|^p \to 0$, so there exists N such that $|\alpha_i| \le 1$ for $i > N$. Thus for $i > N$ we have $|\alpha_i|^q \le |\alpha_i|^p$. It follows that

$$\sum_{i=1}^{\infty} |\alpha_i|^q = \sum_{i=1}^{N} |\alpha_i|^q + \sum_{i=N+1}^{\infty} |\alpha_i|^q \le \sum_{i=1}^{N} |\alpha_i|^q + \sum_{i=N+1}^{\infty} |\alpha_i|^p \le \sum_{i=1}^{N} |\alpha_i|^q + \sum_{i=1}^{\infty} |\alpha_i|^p.$$

The first term on the right-hand side is a finite sum, so it is finite. The second term is fintie by our assumption $x \in \ell_p$. We conclude that $x \in \ell_q$. Therefore $\ell_p \subseteq \ell_q$. It remains to show that there exists $x \in \ell_q \setminus \ell_p$. Let

$$\alpha_n = \frac{1}{n^{1/p}}.$$

Then

$$\sum_{i=1}^{\infty} |\alpha_i|^p = \sum_{i=1}^{\infty} \frac{1}{n} = \infty,$$

while

$$\sum_{i=1}^{\infty} |\alpha_i|^q = \sum_{i=1}^{\infty} \frac{1}{n^{q/p}} < \infty,$$

since $q/p > 1$.

3. Let $(x_k)_{k=1}^\infty$ be an arbitrary sequence in X. We know that X can be covered by finitely many 1-balls. One of these balls must contain an infinite subsequence of $(x_k)_{k=1}^\infty$: $\left(x_k^1\right)_{k=1}^\infty$. The space X can also be covered by finitely many 1/2-balls. One of these balls must contain an infinite subsequence of $\left(x_k^1\right)_{k=1}^\infty$: $\left(x_k^2\right)_{k=1}^\infty$. If we continue this process, then at step n we get a subsequence $(x_k^n)_{k=1}^\infty$ with the property that all the elements of the sequence lie in the same ball of radius $1/n$. Now let $a_k = x_k^k$. Then (a_k) is a subsequence of (x_k) and it is Cauchy, because if $m > n$, then both a_n and a_m are elements of the sequence $(x_k^n)_{k=1}^\infty$, which yields that they are contained by the same $1/n$-ball, thus $d(a_n, a_m) < 2/n$.

Second solution: Consider the completion $(\widetilde{X}, \tilde{d})$ of the metric space (X, d). Since X is totally bounded and dense in $\widetilde{X}$, it easily follows that $\widetilde{X}$ is also totally bounded. Therefore $\widetilde{X}$ is compact (complete and totally bounded). Now let (x_k) be any sequence in X, which can also be viewed as a sequence in $\widetilde{X}$. Since $\widetilde{X}$ is compact, (x_k) has a subsequence that is convergent in $\widetilde{X}$. In particular, this subsequence is Cauchy.

4. Assume that there exists $0 \neq x \in X$ such that

$$x \in \bigcap_{\Lambda \in S} \ker \Lambda.$$

It means that $\Lambda x = 0$ for any $\Lambda \in S$. However, S is a basis of X^*, so any $\Lambda \in X^*$ can be expressed as the finite linear combination of functionals in S. It follows that $\Lambda x = 0$ for any $\Lambda \in X^*$, which is a contradiction, since we proved (using the Hahn-Banach theorem) that for any $x \in X$ there exists $\Lambda \in X^*$ with $\Lambda x = \|x\|$.

5. Every one-point set $\{x\}$ of a metric space is closed. (We need to show that its complement $X \setminus \{x\}$ is open. This is clear, because for any $y \neq x$ we have $B_r(y) \subset X \setminus \{x\}$ for $r = d(x, y) > 0$.)

Assume for the sake of contradiction that X is countable. Then

$$X = \bigcup_{x \in X} \{x\}$$

is a finite or countably infinite union of closed sets. Since X is complete, Baire category theorem tells us that one of the sets $\{x\}$ contains an open ball. Consequently, there exist $x \in X$ and $r > 0$ such that $B_r(x) = \{x\}$, that is, x is an isolated point, contradiction.

6. We know that

$$((\lambda I - T)f)(x) = (\lambda - x)f(x).$$

We claim that $\lambda I - T$ is injective for any $\lambda \in \mathbb{R}$. Suppose that $f \in \ker(\lambda I - T)$. Then $f(x) = 0$ for any $x \in [0,1] \setminus \{\lambda\}$. Since f is continuous, it follows that $f(x) = 0$ for any $x \in [0,1]$. This means that the point spectrum $\sigma_p(T)$ is empty.

If $\lambda \notin [0,1]$, then $\lambda I - T$ is surjective. For $g \in C[0,1]$ let $f(x) = g(x)/(\lambda - x)$. Clearly, f is continuous and $(\lambda I - T)f = g$.

If $\lambda \in [0,1]$, then $\operatorname{ran}(\lambda I - T)$ is not even dense. For any $f \in C[0,1]$ we have $((\lambda I - T)f)(\lambda) = 0$. Thus

$$\operatorname{ran}(\lambda I - T) \subset \{g \in C[0,1] : g(\lambda) = 0\}.$$

The set on the right-hand side is a closed proper subspace of $C[0,1]$, so it contains even the closure of the range.

It follows that $\sigma_r(T) = \sigma(T) = [0,1]$.

Extra problems:

7. It is easy to see that

$$\sigma_p(T) = \{1, 1/2, 1/3, 1/4, \ldots\}.$$

It is not easy to see that

$$\sigma(T) = \{\lambda \in \mathbb{C} : |\lambda - 1/2| \leq 1/2\}.$$

I am not sure what the residual spectrum is:

$$\sigma_r(T) \supset \{\lambda \in \mathbb{C} : |\lambda - 1/2| < 1/2\} \setminus \{1, 1/2, 1/3, 1/4, \ldots\},$$

but I don't know which points of the boundary belong to $\sigma_r(T)$.

Since the spectrum is uncountable, T is certainly not compact.

8. Assume for the sake of contradiction that both S and T are bounded. If we replace S and T by αS and $\alpha^{-1}T$, then the condition $ST - TS = I$ still holds. So we may assume that $\|T\| = 1$. It is easy to see by induction that $ST^{n+1} - T^{n+1}S = (n+1)T^n$. However, the left-hand side has operator norm at most $2\|S\|$, while the right-hand side has operator norm $n+1$, contradiction.