Functional Analysis, BSM, Spring 2012

Final exam, May 21

Solutions

1. We have $\ker(ST) \supset \ker T$ for any $S, T \in B(H)$. Therefore $\ker(T^*T) \supset \ker T$. So it remains to show that $\ker(T^*T) \subset \ker T$. Let $x \in \ker(T^*T)$, that is, $T^*Tx = 0$. Then

$$\|Tx\|^2 = (Tx, Tx) = (x, T^*Tx) = (x, 0) = 0.$$

Thus $x \in \ker T$; we are done.

2. We need to show that the complement of $\sigma_{ap}(T)$ is open. The complement consists of those complex numbers λ for which

$$\inf_{\|x\|=1} \|(\lambda I - T)x\| > 0.$$

Suppose that $\lambda \notin \sigma_{ap}(T)$ and let

$$\delta \stackrel{\text{def}}{=} \inf_{\|x\|=1} \|(\lambda I - T)x\| > 0.$$

It suffices to show that if $|\lambda' - \lambda| < \delta$, then $\lambda' \notin \sigma_{ap}(T)$. Since

$$\|(\lambda' I - T)x\| \geq \|(\lambda I - T)x\| - \|(\lambda' - \lambda)x\|,$$

it follows that

$$\inf_{\|x\|=1} \|(\lambda' I - T)x\| \geq \inf_{\|x\|=1} \|(\lambda I - T)x\| - |\lambda' - \lambda| = \delta - |\lambda' - \lambda| > 0,$$

thus $\lambda' \notin \sigma_{ap}(T)$ as claimed.

3. Let $Y = \operatorname{ran} T$ and let $v \in Y \setminus \{0\}$. Since Y is one-dimensional, $Y = \{\alpha v : \alpha \in \mathbb{C}\}$. Consider the following bounded linear functional Λ on Y:

$$\Lambda(\alpha v) = \alpha.$$

Then ΛT is a bounded linear functional on H. Riesz representation theorem tells us that there exists $u \in H$ such that

$$\Lambda T x = (x, u) \text{ for all } x \in H.$$

It follows that $Tx = (x, u)v$. It remains to show that $\|T\| = \|u\| \cdot \|v\|$. Since

$$\|Tx\| = \|(x, u)v\| = |(x, u)| \cdot \|v\| \leq \|x\| \cdot \|u\| \cdot \|v\|,$$

we get that $\|T\| \leq \|u\| \cdot \|v\|$. On the other hand, for $x = u/\|u\|$ we have $\|x\| = 1$ and $\|Tx\| = \|u\| \cdot \|v\|$.

4. The assumption is equivalent to $y \in \operatorname{cl}(\operatorname{ran} T^*)$, while the conclusion is equivalent to $y \in \operatorname{cl}(\operatorname{ran}(T^*T))$. However, using $\ker(T^*T) = \ker T$ (see Problem 1) and $\operatorname{cl}(\operatorname{ran} S^*) = (\ker S)^\perp$ for $S = T$ and $S = T^*T$ we get

$$\operatorname{cl}(\operatorname{ran} T^*) = (\ker T)^\perp = (\ker(T^*T))^\perp = \operatorname{cl}(\operatorname{ran}(T^*T)).$$

5. Since $T \in B(H)$ is self-adjoint, we have $\sigma(T) \subset \mathbb{R}$, that is, $\alpha I - T$ is invertible if $\operatorname{Im} \alpha \neq 0$. Using this for $-\alpha$ we get that $-\alpha I - T$ is invertible, thus so is $\alpha I + T$.

We claim that $(\alpha I + T)^{-1}$ commutes with $\beta I + T$ for any $\beta \in \mathbb{C}$. This is clear, because it commutes with both $\alpha I + T$ and $(\beta - \alpha)I$, so it must commute with their sum

$$(\alpha I + T) + (\beta - \alpha)I = \beta I + T.$$

It follows that the operators $\overline{\alpha} I + T$ and $(\alpha I + T)^{-1}$ commute. We need to show that $UU^* = U^*U = I$. Using that $T^* = T$:

$$UU^* = (\overline{\alpha} I + T)(\alpha I + T)^{-1} \left((\alpha I + T)^{-1}\right)^* (\overline{\alpha} I + T)^* = (\alpha I + T)^{-1}(\overline{\alpha} I + T) \left((\alpha I + T)^*\right)^{-1} (\alpha I + T) =$$
$$(\alpha I + T)^{-1}(\overline{\alpha} I + T)(\overline{\alpha} I + T)^{-1}(\alpha I + T) = (\alpha I + T)^{-1} I (\alpha I + T) = I.$$

Proving $U^*U = I$ is similar.

6. Let $B_n(0)$ denote the open ball in X with radius n and center 0. Then

$$\operatorname{ran} T = \bigcup_{n=1}^{\infty} T\left(B_n(0)\right) = \bigcup_{n=1}^{\infty} n \cdot T\left(B_1(0)\right).$$

Since T is compact, $T\left(B_1(0)\right)$ is totally bounded, so it has a finite ε-lattice S_ε for any $\varepsilon > 0$. Let

$$M = \bigcup_{n=1}^{\infty} n \cdot S_{1/n^2}.$$

Since M is countable, it is enough to show that M is dense in $\operatorname{ran} T$. Let $y \in \operatorname{ran} T$ be arbitrary. Then $y \in T\left(B_n(0)\right) = n \cdot T\left(B_1(0)\right)$ if n is large enough. Therefore $y/n \in T\left(B_1(0)\right)$. It follows that there exists $s_n \in S_{1/n^2}$ such that

$$\|y/n - s_n\| < \frac{1}{n^2},$$

thus

$$\|y - n \cdot s_n\| < \frac{1}{n}.$$

Since $n \cdot s_n \in M$ for all n, we get that $y \in \operatorname{cl} M$.

Extra problems:

7. We prove by contradction; we assume that $\exists T \in B(\ell_2)$ such that $T^2 = L$. Let Y be the one-dimensional subspace spanned by $(1, 0, 0, \ldots)$. Since $\ker T \subset \ker T^2 = \ker L = Y$, we either have $\ker T = \{0\}$ or $\ker T = Y$. However, $\ker T = \{0\}$ would imply $\ker L = \ker(T^2) = \{0\}$. So only the second case is possible: $\ker T = Y$. Now let $x = (0, 1, 0, 0, \ldots)$. Then

$$LTx = T^3 x = TLx = T(1, 0, 0, \ldots) = 0.$$

Therefore $Tx \in \ker L = Y = \ker T$, thus $TTx = 0$. However, $TTx = Lx = (1, 0, 0, \ldots)$, contradiction.

If the right shift R had some square root T, then $L = R^* = (T^2)^* = (T^*)^2$, so L would have a square root, too.

8. Since

$$(Tx, y) = (x, u)(v, y) = \left(x, \overline{(v, y)} u\right) = (x, (y, v)u),$$

it follows that $T^* y = (y, v)u$.

We claim that the spectrum consists of 0 and (v, u). Since T has finite rank, it is compact. So every nonzero element λ of the spectrum is an eigenvalue. So there exists $x \in H$ such that $Tx = \lambda x$. Since Tx is in the range $\operatorname{ran} T$, so is x. Therefore we can assume that $x = v$ and $\lambda = (v, u)$ follows.

9. Let P_1 be the orthogonal projection to $\ker(I - T)$ and P_2 the orthogonal projection to $\ker(-I - T) = \ker(I + T)$. We aim to show that $T = P_1 - P_2$.

If T is self-adjoint ($T = T^*$) and unitary ($TT^* = T^*T = I$), then $T^2 = I$, thus $0 = I - T^2 = (I - T)(I + T)$. It follows that $\ker(I - T) \supset \operatorname{ran}(I + T)$. Since $\ker(I - T)$ is closed, we even have

$$\ker(I - T) \supset \operatorname{cl}(\operatorname{ran}(I + T)) = (\ker(I + T^*))^{\perp} = (\ker(I + T))^{\perp}.$$

We also know that $\ker(I - T) \cap \ker(I + T) = \{0\}$, therefore $\ker(I - T)$ and $\ker(I + T)$ must be orthogonal complements. So for any $x \in H$ we have $x = P_1 x + P_2 x$ and consequently $Tx = TP_1 x + TP_2 x = P_1 x - P_2 x$ as claimed.

10. We need to show that for any $y \in H$ and $\varepsilon > 0$ there exists a polynomial q such that

$$\|y - q(T^*)x\| < \varepsilon.$$

Since x is cyclic for T, it suffices to prove this when $y = T^n x$ for some nonnegative integer n. So for any n and $\varepsilon > 0$ we need to find a polynomial q such that

$$\|T^n x - q(T^*)x\| < \varepsilon.$$

However, $T^n - q(T^*)$ is normal, so

$$\|\left(T^n - q(T^*)\right) x\| = \|\left(T^n - q(T^*)\right)^* x\| = \|\left((T^*)^n - \overline{q}(T)\right) x\| = \|(T^*)^n x - \overline{q}(T)x\|,$$

which can be arbitaririly small, because x is cyclic for T.