Functional Analysis, BSM, Spring 2012

Final exam, May 21

1. (15 points)
Let H be a Hilbert space and $T \in B(H)$. Prove that $\ker(T^*T) = \ker T$.

2. (15 points)
Prove that the approximate point spectrum $\sigma_{ap}(T)$ is closed for any bounded operator $T \in B(H)$.

3. (15 points)
Let H be a Hilbert space. Suppose that $T \in B(H)$ has rank 1, that is, the dimension of $\operatorname{ran} T$ is 1. Show that there exist nonzero vectors $u, v \in H$ such that

$$Tx = (x, u)v \text{ for all } x \in H.$$

Also prove that $\|T\| = \|u\| \cdot \|v\|$.

4. (15 points)
Let H be a Hilbert space and $T \in B(H)$. Suppose that there exists a sequence (x_n) in H such that $T^*x_n \to y$ for some $y \in H$. Prove that there exists a sequence (y_n) in H such that $T^*Ty_n \to y$.

5. (20 points)
Let H be a Hilbert space, $T \in B(H)$ self-adjoint and $\alpha \in \mathbb{C}$ with $\operatorname{Im} \alpha \neq 0$. Prove that the operator

$$U = (\overline{\alpha} I + T)(\alpha I + T)^{-1}$$

is unitary.

6. (20 points)
Let X, Y be Banach spaces and $T : X \to Y$ a compact operator. Prove that $\operatorname{ran} T$ is separable.

Extra problems:

7. Show that the left shift operator L has no square root, that is, $\nexists T \in B(\ell_2)$ such that $T^2 = L$. Does the right shift operator R have a square root?

8. Let T be as in Problem 3. What is the adjoint of T? What is the spectrum of T?

9. Suppose that $T \in B(H)$ is self-adjoint and unitary. Prove that there exist orthogonal projections P_1, P_2 such that $T = P_1 - P_2$.

10. Let $T \in B(H)$ be normal. Prove that if $x \in H$ is a cyclic vector for T, then it is also cyclic for T^*.
(We say that x is cyclic for T if $\operatorname{cl}(\operatorname{span}\{x, Tx, T^2x, \ldots\}) = H$.)