Functional Analysis, BSM, Spring 2012

Exercise sheet: spectrum; polar decomposition

Let X be a Banach space and $T \in B(X)$. Recall:

- *Spectral mapping theorem:* if $p(z) = \sum_{i=0}^{m} \alpha_i z^i$ is a polynomial, then for $p(T) \stackrel{\text{def}}{=} \sum_{i=0}^{m} \alpha_i T^i \in B(X)$ we have $\sigma(p(T)) = \{p(\lambda) : \lambda \in \sigma(T)\}$.
- If T is invertible, then $\sigma(T^{-1}) = \{\lambda^{-1} : \lambda \in \sigma(T)\}$.

Definition: we say that λ is an *approximate eigenvalue* of T if $\lambda I - T$ is not bounded below, that is,

$$\inf_{\|x\|=1} \|(\lambda I - T)x\| = 0.$$

In other words, there exist $x_n \in X$ such that $\|x_n\| = 1$ for all n and $\|Tx_n - \lambda x_n\| \to 0$.
The set of approximate eigenvalues is called the *approximate point spectrum* and denoted by $\sigma_{ap}(T)$.

Let H be a complex Hilbert space.
Square root lemma: If $A \in B(H)$ is positive, then there exists a unique $B \in B(H)$ such that B is positive and $B^2 = A$. Furthermore, if $AS = SA$ for some $S \in B(H)$, then $BS = SB$.
Notation: B is denoted by $\sqrt{A}$.
Definition: $|T| \stackrel{\text{def}}{=} \sqrt{T^*T}$.
Polar decomposition: If $T \in B(H)$, then there exists a partial isometry U such that $T = U|T|$.

1. Let $T \in B(H)$ be arbitrary. Prove that

$$\sigma(T^*) = \left\{\overline{\lambda} : \lambda \in \sigma(T)\right\}.$$

2. W11P5. (5 points) Let H be a complex Hilbert space and $T \in B(H)$. Prove the following statements.
a) If T is positive, then $\sigma(T) \subset [0, \infty)$.
b) If T is self-adjoint, then $\sigma(T) \subset \mathbb{R}$.
(Hint: recall that if T is normal, then $\sigma(T) = \sigma_{ap}(T)$.)

3. W11P6. (6 points) Let H be a complex Hilbert space and $T \in B(H)$. Prove that if T is unitary, then $\sigma(T) \subset \{\lambda : |\lambda| = 1\}$.

4. Let $T \in B(H)$ be arbitrary. Prove that

$$\lambda \in \sigma_p(T) \Leftrightarrow \operatorname{ran}(\overline{\lambda} I - T^*) \text{ is not dense.}$$

5. Show that the residual spectrum of a normal operator $T \in B(H)$ is always empty.

6. Let $T \in B(H)$ be arbitrary. Prove that if $\lambda \in \sigma(T)$, then either λ is an approximate eigenvalue of T or $\overline{\lambda}$ is an eigenvalue of T^*.

7. a) Find the polar decomposition of the right shift operator.
b) Find the polar decomposition of the left shift operator.

8.* W11P7. (12 points) Let H be a complex Hilbert space and $A_1, A_2 \in B(H)$ positive operators. Prove that $A_1 A_2$ is positive if and only if A_1 and A_2 commute, that is, $A_1 A_2 = A_2 A_1$.

9. Suppose that $A_n, A \in B(H)$ are positive operators. Prove that if $\|A_n - A\| \to 0$, then $\|\sqrt{A_n} - \sqrt{A}\| \to 0$.

10. Let $T_n, T \in B(H)$. Prove that if $\|T_n - T\| \to 0$, then $\||T_n| - |T|\| \to 0$.

11.* Let H be a Hilbert space and $T \in B(H)$. Prove that the boundary of $\sigma(T)$ is always contained in the approximate point spectrum $\sigma_{ap}(T)$.

Solutions can be found on: `www.renyi.hu/~harangi/bsm/`