Functional Analysis, BSM, Spring 2012

Exercise sheet: special operators

Definition: we say that an operator $T \in B(H)$ is

- *unitary* if $T^*T = TT^* = I$;
- *positive* if $(Tx, x) \geq 0$ for all $x \in H$;
- *an isometry* if $\|Tx\| = \|x\|$ for all $x \in H$;
- *a partial isometry* if the restriction of T to $(\ker T)^\perp$ is an isometry, that is, $\|Tx\| = \|x\|$ for all $x \in (\ker T)^\perp$.

Theorem: let H be a complex Hilbert space. Suppose that $T \in B(H)$ is normal and compact. Then there exists an orthonormal system $s_1, s_2, \ldots$ and $\lambda_1, \lambda_2, \ldots \in \mathbb{C}$ such that $\lambda_n \to 0$ and

$$Tx = \sum_n \lambda_n (x, s_n) s_n \text{ for all } x \in H.$$

The converse is also true: any such operator is normal and compact.

1. a) Show that for any $T \in B(H)$ the operators TT^* and T^*T are both positive.
b) Show that if T is self-adjoint, then T^2 is positive.

2. W11P1. (4 points)
a) Find an $\ell_2 \to \ell_2$ isometry that is not surjective.
b) Find an $\ell_2 \to \ell_2$ partial isometry that is not injective.
c) Find an $\ell_2 \to \ell_2$ partial isometry that is neither injective, nor surjective.

3. Let $T \in B(H)$. Prove that the following are equivalent.

(i) T is an isometry.

(ii) $(Tx, Ty) = (x, y)$ for all $x, y \in H$.

(iii) $T^*T = I$.

4. W11P2. (5 points) Prove that the adjoint of an isometry is a surjective partial isometry.

5. Let $T \in B(H)$. Prove that T is unitary if and only if it is a surjective isometry.

6. W11P3. (8 points) Let $T \in B(H)$ be normal. Prove the following statements.
a) $\|T^2\| = \|T\|^2$. (Hint: recall that $\|S^*S\| = \|S\|^2$. Use this for $S = T^2$, $S = T^*T$ and $S = T$.)
b) $\|T^{2^k}\| = \|T\|^{2^k}$ for every positive integer k.
c) $r(T) = \|T\|$.
d) $\|T^n\| = \|T\|^n$ for every positive integer n.

7.* Let H be a Hilbert space. Prove that an operator $T \in B(H)$ is compact if and only if it is the limit of finite rank operators (in the operator norm), that is, there exist $T_n \in B(H)$ such that $\dim(\operatorname{ran} T_n) < \infty$ for all n and $\|T - T_n\| \to 0$.

8. W11P4. (8 points) Show that the adjoint of a compact operator $T \in B(H)$ is compact.

9. Let H be a complex Hilbert space. Suppose that $T \in B(H)$ is normal and compact. Prove that T is positive if and only if each eigenvalue of T is a nonnegative real number.

10. Let H be a complex Hilbert space. Suppose that $T \in B(H)$ is positive and compact. Prove that there exists a positive compact operator $S \in B(H)$ such that $S^2 = T$.

11.* Let L be the left shift operator on ℓ_2.
a) Prove that $\|L - U\| = 2$ for every unitary operator $U \in B(\ell_2)$.
b) Prove that $\|L - T\| \geq 1$ for every compact operator $T \in B(\ell_2)$.

12.* Let H be a complex Hilbert space and $T \in B(H)$. Prove that $w(T^2) \leq w(T)^2$, where w denotes the numerical radius:

$$w(T) \stackrel{\text{def}}{=} \sup_{\|x\|=1} |(Tx, x)|.$$

13. Let H be a complex Hilbert space. Prove that if $T \in B(H)$ is normal, then $w(T) = \|T\|$.
(Hint: use $\|S\| \leq 2w(S)$ with $S = T^{2^k}$.)

Solutions can be found on: `www.renyi.hu/~harangi/bsm/`