Functional Analysis, BSM, Spring 2012

Exercise sheet: Normal operators

Definition: an operator $T \in B(H)$ is called *normal* if $TT^* = T^*T$ (or equivalently, if $\|Tx\| = \|T^*x\|$ for all $x \in H$).
Recall: if X is a Banach space and $T \in B(X)$, then T is invertible if and only if T is bounded below and $\operatorname{ran} T$ is dense.

1. Is the left shift operator on ℓ_2 normal?

2. Prove the following statements.
a) $\ker T^* = (\operatorname{ran} T)^\perp$;
b) $(\operatorname{ran} T^*)^\perp = \ker T$;
c) $(\ker T^*)^\perp = \operatorname{cl}(\operatorname{ran} T)$.

3. Show that $\ker T = \ker T^*$ if $T \in B(H)$ is normal.

4. a) Let $T \in B(H)$. Show that T is invertible if and only if it is bounded below and $\ker T^* = \{0\}$.
b) Let $T \in B(H)$ be normal. Show that T is invertible if and only if it is bounded below.
Consequently, if T is normal, then $\lambda \in \sigma(T)$ if and only if $\lambda I - T$ is not bounded below, that is,

$$\inf_{\|x\|=1} \|(\lambda I - T)x\| = 0. \tag{1}$$

(When (1) holds, we say that λ is in the *approximate point spectrum* of T. So for normal operators the approximate point spectrum is the same as the spectrum.)

5. W10P5. (5 points) Let $T \in B(H)$ be normal, $x \in H$ and $\lambda \in \mathbb{C}$. Show that if $Tx = \lambda x$, then $T^*x = \overline{\lambda}x$.

6. W10P6. (6 points) Let $T \in B(H)$ be normal, $x, y \in H$ and $\lambda, \mu \in \mathbb{C}$. Suppose that $Tx = \lambda x$ and $Ty = \mu y$. Prove that if $\lambda \neq \mu$, then $x \perp y$.

7. W10P7. (8 points) Let $T \in B(H)$ be normal.
a) Show that $\ker T$ is T^*-invariant, that is, for any $x \in \ker T$ we also have $T^*x \in \ker T$.
b) Show that $(\ker T)^\perp$ is T-invariant.
c) Prove that $\ker T = \ker T^2$.
d) Prove that $\ker T = \ker T^k$ for any positive integer k.

Solutions can be found on: `www.renyi.hu/~harangi/bsm/`